ON THE THERMODYNAMICS OF COSMIC DUST

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The horizon hypothesis of quantum cosmology was put forward previously to justify the derivation of the Wheeler–DeWitt equation for the wave function of the Universe Ψ in the mini-superspace, assuming the Friedmann line element $ds^2 = dt^2 - a^2(t)d\boldsymbol{x}^2$, when the three-space $d\boldsymbol{x}^2$ is flat, by imposing a cut-off on spatial integrals at the causal horizon $\xi^{-1}(t) \equiv \{d\,[\ln a(t)]\,/dt\}^{-1}$. If the theory is defined by a Lagrangian L which includes higher-derivative gravitational terms $\mathcal{R}^2$, the resulting Schrödinger equation contains a potential that is constant in the case of cosmic dust, and after Wick rotation of the comoving time coordinate, $t \to -i\tau$, the solution for Ψ takes the form of a Boltzmann distribution, from which it is possible to ascribe a finite temperature $T \equiv \lambda/\tau$ to the dust, where the parameter λ subsumes the details of the cut-off, and which can be understood from the Heisenberg time-energy indeterminacy principle $\Delta t \Delta E \sim \hbar$ applied to the observable Universe. Recently, Cai *et al.* have shown that the apparent horizon $\tilde{r}_{\rm a}$ in Friedmann cosmology possesses thermal characteristics, associated with a temperature $T = 1/2\pi\tilde{r}_{\rm a}$ when measured by a Kodama observer comoving with the horizon. This result thus vindicates the horizon hypothesis, and leads to the exact value $\lambda = 3\pi/2$. The infra-red cut-off at the apparent horizon applies to all quantum field theoretical phenomena, and the inclusion of length scales that lie only *inside* the horizon is equivalent to the inclusion only of length scales lying *outside* their Schwarzschild radius, as conjectured by Cohen *et al.*, following earlier speculation by 't Hooft. Analogies with the theory of superfluidity are also discussed.

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1. Introduction

The classical solution to the spherically symmetric vacuum Einstein equations for a source of mass M is the Schwarzschild metric, expressed in coordinates $(\tilde{t}, r, \theta, \phi)$ by the line element

$$ds^2 = \left(1 - 2Mr^{-1}\right) d\tilde{t}^2 - \left(1 - 2Mr^{-1}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \quad (1)$$

Units are chosen such that $c = \hbar = k_{\rm B} = G_{\rm N} = 1$. Next, let us transform to the Vaidya coordinate system $(t^* \equiv \tilde{t} - r + r^*, r, \theta, \phi)$, where $r^* \equiv \int dr/(1 - 2Mr^{-1})$ is the tortoise radial coordinate, the line element being

$$\begin{aligned} ds^2 \;=\; & \left(1 - 2Mr^{-1}\right) dt^{*2} - 4Mr^{-1}dt^*dr - \left(1 + 2Mr^{-1}\right) dr^2 \\ & - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \end{aligned} \quad (2)$$

The application of quantum mechanics on the apparent horizon $r = 2M$, where all the metric components g_{ij} are now finite, including an additional scalar matter field ζ, yields the Wheeler–DeWitt equation [1, 2] for the wave function Ψ in the form of a Schrödinger equation, obtained by Tomimatsu [3] (see also Refs. [4–6]).

From this equation, it is possible to derive the Hawking temperature [7]

$$T_{\rm H} = 1/(8\pi M) , \quad (3)$$

after Euclideanization of the time coordinate, as described in Refs. [5,6].

Macroscopic quantum mechanics can also be applied to the whole Universe, yielding the Wheeler–DeWitt equation [8–13] for the cosmological wave function Ψ. If the Lagrangian L defining the theory includes a contribution from higher-derivative gravitational terms $\mathcal{R}^2$, this again takes the form of a Schrödinger equation, in the mini-superspace (Friedmann space-time) approximation. Expressly, starting from the ten-dimensional effective action for the heterotic superstring theory of Gross *et al.* [14–16], let us consider the reduced $\mathcal{D} \equiv (M + 1)$-dimensional Lagrangian

$$L = -\tfrac{1}{2}\kappa_{\mathcal{D}}^{-2} R + B_N \left(R^2 - R_{ij}R^{ij}\right) + \ldots . \quad (4)$$

Here $\kappa_{\mathcal{D}}^2 \equiv 8\pi G_{\mathcal{D}}$ is the $\mathcal{D}$-dimensional gravitational coupling, $G_{\mathcal{D}}$ being the $\mathcal{D}$-dimensional Newton constant, the coefficient of $\mathcal{R}^2$ is given by the integral over the internal $N \equiv (10 - \mathcal{D})$-dimensional space $\bar{g}_{\mu\nu}$,

$$B_N = \frac{1}{512}\zeta(3)\frac{A_{\rm r}\alpha'^3}{B_{\rm r}^2\kappa_{\mathcal{D}}^2} \int d^N y\sqrt{\bar{g}}\bar{R}_{\mu\nu\xi o}\bar{R}^{\mu\nu\xi o}/d^N y\sqrt{\bar{g}} , \quad (5)$$

$\zeta(3) = 1.202$ is the Riemann zeta-function, α' is the Regge slope parameter, $A_{\rm r}$ and $B_{\rm r}$ are moduli defined from the ten-metric as

$$\hat{g}_{AB} = {\rm diag}\left(A_{\rm r}^{-1} g_{ij}, B_{\rm r}\bar{g}_{\mu\nu}\right)\,, \tag{6}$$

$g = \det g_{ij}$, $\bar{g} = \det \bar{g}_{\mu\nu}$, R_{ij} is the Ricci tensor and R the Ricci scalar.

Thus, we assume the Friedmann line element, which is expressible in the alternative forms

$$ds^2 = dt^2 - a^2(t)d\boldsymbol{x}^2 \equiv h_{ab}dx^a dx^b - \tilde{r}^2 d\Omega^2_{\mathcal{D}-2}\,, \tag{7}$$

where t is comoving time and $a(t) \equiv a_0 e^{\alpha(t)}$ is the radius function of the M-space $d\boldsymbol{x}^2$ of curvature K, $h_{ab} = {\rm diag}[1, -a^2/(1-Kr^2)]$, $x^a = (x^0, x^1) \equiv (t, r)$ and $\tilde{r} = ar$. From the first of expressions (7), the canonical coordinates are $(\alpha, \xi \equiv \dot{\alpha})$, where $\dot{} \equiv d/dt$, and Ψ satisfies the Schrödinger equation (Eq. (78) of Ref. [8])

$$i\,\frac{\partial\Psi}{\partial t} \equiv i\,\xi\frac{\partial\Psi}{\partial\alpha} = \left(-A_M a^{-M}\frac{\partial^2}{\partial\xi^2} + \mathcal{V}(\alpha,\xi) + \mathcal{H}_{\rm m}\right)\Psi\,, \tag{8}$$

where

$$A_M = [4M(3M-1)B_N]^{-1}\,. \tag{9}$$

The (densitized) potential, assuming the N-space to be Ricci-flat, is given by

$$\mathcal{V}(\alpha,\xi) = \Big\{\tfrac{1}{2}\kappa_{\mathcal{D}}^{-2}M(M-1)\left(\xi^2 - Ka^{-2}\right) + B_N\Big[\tfrac{1}{3}M^3(M-3)\xi^4 + 2KM(M-1)\left(M^2-5M+2\right)\xi^2a^{-2} - K^2M(M-1)^3a^{-4}\Big]\Big\}a^M \tag{10}$$

and $\mathcal{H}_{\rm m}$ is the matter Hamiltonian density.

Note that the potential (10) is positive semi-definite only if the M-space is flat and of dimensionality no less than three, that is

$$K = 0\,, \qquad M \geq 3\,. \tag{11}$$

To lowest order in α', we can relate the geometry to the matter semi-classically through the Einstein equations. For a perfect-fluid source characterized by energy density ρ and pressure p, linked through the adiabatic index γ by the equation of state

$$p = (\gamma - 1)\rho\,, \tag{12}$$

the Friedmann and Raychaudhuri equations read

$$\xi^2 = \frac{2}{M(M-1)}\kappa_{\mathcal{D}}^2\rho - \frac{K}{a^2} \tag{13}$$

and

$$\dot{\xi} = -\frac{1}{M-1}\,\gamma\kappa_{\mathcal{D}}^2\rho + \frac{K}{a^2}\,, \tag{14}$$

respectively, the solution to which, when $K = 0$, is

$$a = a_0\, t^{2/M\gamma}\,. \tag{15}$$

For reasons that will become clear below, and have been discussed in detail in Ref. [4], we are especially interested in the case when $\mathcal{V}(\alpha, \xi)$ is constant. By inspection of Eqs. (10), (13) and (15), we find that this condition is satisfied if, and only if,

$$M = 3\,, \qquad \gamma = 1\,, \tag{16}$$

that is for dust in a flat three-dimensional space, where the contribution to $\mathcal{V}(\alpha, \xi)$ quartic in ξ vanishes, so that $\mathcal{H}_{\rm m} = \mathcal{V}$ and the pseudo-Hamiltonian is

$$\widetilde{\mathcal{H}} \equiv \mathcal{H}_{\rm m} + \mathcal{V} = 2\mathcal{V}\,. \tag{17}$$

Ignoring the kinetic term in $a^{-3}\partial^2/\partial\xi^2$, this enables us to rewrite Eq. (8) in the standard form

$$i\,\frac{\partial\Psi}{\partial t} \approx \widetilde{\mathcal{H}}\Psi\,, \tag{18}$$

which is Eq. (35) of Ref. [13] and can be integrated to yield

$$\Psi \simeq \Psi_0 \exp\left(-i\widetilde{\mathcal{H}}t\right). \tag{19}$$

After Euclideanization of the time via the Wick rotation

$$t \to -i\tau\,, \tag{20}$$

Eq. (19) assumes the form of a Boltzmann distribution,

$$\Psi \approx \Psi_0 \exp\left(-\widetilde{\mathcal{H}}\tau\right), \tag{21}$$

and as first noted by Bloch [17], τ can be reinterpreted as inverse temperature — see also Feynman [18] (and Ref. [19]).

2. Unitarity

It has long been known that a wave function obeying the Schrödinger equation, which contains only a single time derivative $\partial/\partial t$, evolves without violation of unitarity, as discussed by von Neumann [20, 21], implying no loss of information, and in the case of radiating black holes, constancy of total entropy. Therefore, by analogy we expect this feature to manifest itself in cosmology as well — that is, when $\mathcal{V}$, and hence $\widetilde{\mathcal{H}}$, are constant, the evolution should proceed isentropically.

From the discussion of Sec. 1, we see that this is exactly what happens, for constancy of $\mathcal{V}$ requires $\gamma = 1$, corresponding to a dust Universe with zero pressure, which expands without doing work against the boundary of $a(t)$. The total mass of the Universe also remains constant during the expansion — in the present normalization of a, whereby the fundamental, or fiducial, three-volume is set equal to unity,

$$V_3 \equiv \int \sqrt{-g}\, d^3x = 1\,, \tag{22}$$

we have Eq. (23) of Ref. [12],

$$M_{\rm U} = \rho a^3\,. \tag{23}$$

The first law of thermodynamics,

$$dU = Td\mathcal{S} - pdV, \tag{24}$$

then implies that the total entropy $\mathcal{S}$ remains constant.

In obtaining this result, we have to clarify the semi-classical approximation of Eq. (17). The derivation of the quantum-cosmological Schrödinger equation (8) from the classical Hamiltonian constraint $H = 0$ proceeds by quantization of the gravitational degrees of freedom (α, ξ) remaining in the mini-superspace, via the operator replacements of the corresponding classical momenta,

$$\pi_\alpha \to -i\partial/\partial\alpha\,, \qquad \pi_\xi \to -i\partial/\partial\xi\,, \tag{25}$$

while the matter source is treated classically — that is the matter fields $(\zeta, \psi, A_i, \ldots)$ have not been quantized.

The justification for this seeming disparity consists simply in the fact that we are dealing with the whole Universe, which is a *macroscopic* quantum-mechanical system, characterized by a very large quantum number, and it is therefore self-consistent not to quantize the matter.

It is interesting to contrast this cosmological case with that of the Schwarzschild black hole [3–6], which involves quantization on the apparent horizon of the dynamical variables, both gravitational and matter, the background space-time being otherwise treated as classical. In the theory of black-hole evaporation [7], it is self-consistent, in the fixed background geometry created by a macroscopic black hole of mass $M_{\rm bh} \gg M_{\rm P}$, instead to quantize the matter, which is now treated *microscopically*. The back reaction of the emitted radiation on the geometry can be taken into account by quantizing not only the matter degrees of freedom, but also the resultant motion of the apparent horizon.

Both situations exemplify the Born–Oppenheimer [22] approximation, in which moving horizons or test particles are treated quantum mechanically as microscopic in a classical background space-time, in the former cosmological instance because the mass of the whole Universe is a macroscopic quantity, in the sense that $M_{\rm U} \gg M_{\rm P}$, and in the latter because the black-hole mass is macroscopic.

3. The horizon hypothesis

The description of a generic four-dimensional gravitational theory in terms of the one-parameter mini-superspace or Friedmann space-time (7) is in the first instance a statement concerning the symmetry of the metric $g_{ij}(x^k)$, which is now expressible through one function $a(t)$, depending only upon the comoving time coordinate t. When this is possible, the three spatial coordinates $x^\alpha (\alpha = 1, 2, 3)$ become redundant and the four-action S_4 can be converted into a one-action S as

$$S_4 = \int d^4x \mathcal{L}_4 \left(x^i\right) \to S = V_3 \int dt \mathcal{L}(t) \,, \tag{26}$$

where V_3 is defined by Eq. (22).

Classically we do not require the absolute magnitude of the action, the equation of motion following in the standard way by variation of S_4 with respect to g^{ij} or S with respect to α. In quantum theory, on the other hand, the magnitude of the action determines the precise expressions for the operator replacements of the canonical momenta π_n by the derivatives of the canonical coordinates $-i\partial/\partial q_n$, assuming the wave function to be of the form

$$\Psi \approx \Psi_0 \exp\left(iS\right). \tag{27}$$

The fundamental three-volume (22) is defined from the Friedmann metric (7) as

$$V_3 = 4\pi \int_0^r \frac{r^2 dr}{\sqrt{1 - Kr^2}} = \begin{cases} 2\pi^2 \\ \frac{4}{3}\pi r^3 \\ 2\pi \left(r\sqrt{1+r^2} - \sinh^{-1} r\right) \end{cases} \quad \text{for} \quad K = \begin{cases} 1 \\ 0 \\ -1\,. \end{cases} \tag{28}$$

Strictly speaking, V_3 is only finite for a closed three-space, when $K = 1$ and $V_3 = 2\pi^2$. Here, however, we assume a flat three-space, in accordance with the quantum-theoretical argument leading to Eqs. (11), and also from the analysis of recent observations of the anisotropy of the cosmic microwave background radiation on angular scales $\gtrsim 10'$ by the WMAP satellite [23] — see Ref. [24] and references therein. Therefore, we have argued [12] that a cut-off should be imposed upon the integral (28) on the grounds of causality, which makes it possible to set $V_3 = 1$ in the case $K = 0$, this being the cosmological horizon hypothesis, discussed in further detail in Ref. [13].

Let us emphasize that the radius function $a(t)$ is *a priori* a completely free parameter, devoid of intrinsic geometrical significance. We can therefore absorb the numerical factor of $(4\pi/3)^{1/3}$ into the definition of $a(t)$, so that $V_3 = 1$ and the volume of the Universe at time t is $V(t) = V_3 a^3(t) = a^3(t)$, which leads to Eq. (23). To be more precise, we should include a numerical form factor λ in the expression for the one-action,

$$S = \lambda \int dt \mathcal{L}(t) \,, \tag{29}$$

whilst the mass of the Universe and the pseudo-Hamiltonian density are given by the unmodified formulae

$$M_{\mathrm{U}} = \rho a^3 \,, \qquad \widetilde{\mathcal{H}} = 2M_{\mathrm{U}} = 2\rho a^3 \,. \tag{30}$$

The parameter λ takes the cut-off into account more accurately, the value unity deriving from the step-function approximation of Refs. [11–13].

From recent developments, λ can now be calculated as follows. Cai *et al.* [25, 26] have shown that the Friedmann space-time exhibits thermal behaviour on the apparent horizon $\tilde{r}_{\mathrm{a}}$, defined by the equation $(\nabla \tilde{r})^2 = 0$, the solution to which is

$$\tilde{r}_{\mathrm{a}} = 1/\sqrt{\xi^2 + K/a^2} \,. \tag{31}$$

Referred to the conserved-energy formalism of Kodama [27], the local temperature as measured by an observer comoving with the apparent horizon is

$$T = 1/2\pi\tilde{r}_{\mathrm{a}} \,. \tag{32}$$

The validity of formula (32) has been confirmed for fermionic particles by Li *et al.* [28].

In the maximally symmetric de Sitter space, setting $K = 0$, $\xi = \sqrt{\Lambda/3} =$ constant, we have

$$T = \xi/2\pi = \sqrt{\Lambda/12\pi^2} \,, \tag{33}$$

which is the formula obtained by Gibbons and Hawking [29, 30].

Returning to the Schrödinger equation (18) and the Ansatz (29), we note that the Friedmann equation is the same for all three curvatures $K = 0, \pm 1$ in the limit $a \to \infty$, since then the term $K/a^2 \to 0$. Therefore, we can use expression (28) for the closed space $K = 1$ to deduce that

$$\lambda = V_3(K=1)/V_3(K=0, r=1) = 3\pi/2\,. \tag{34}$$

After Euclideanization of the time coordinate t via the Wick rotation (20), this results in the temperature

$$T = 1/(2\lambda|t|) = 1/(2\pi\tilde{r}_{\rm a})\,, \tag{35}$$

in exact agreement with Eq. (32), for the flat Universe, and thus vindicating the quantization scheme used to derive Eq. (18) and the horizon hypothesis.

Note, in the denominator of the first of expressions (35), that the factor of 2 is due to the coefficient 2 in the second of Eqs. (30), while the factor of λ derives from the operator replacement for the pseudo-Hamiltonian density resulting from the modified Eq. (29) for the one-action,

$$\mathcal{H}(t) \to \frac{i}{\lambda}\frac{\partial}{\partial t}\,. \tag{36}$$

4. Particle creation in the Friedmann space-time

The calculation leading to Eq. (35) contains a topological element, which is reminiscent of the analysis by Mamaev *et al.* [31] of boson-pair production in the Friedmann Universe (7) with open, closed and flat spatial sections. The method of the Bogolyubov [32] transformation relating creation and annihilation operators was applied to obtain expressions for the energy density ρ and pressure p of created particles in the form of spectral integrals. Massless bosons are created with the radiative equation of state $p = \rho/3$. The most interesting case is that of the closed universe with $K = 1$, in which such massless particles appear in a vacuum background as a thermal Planck distribution characterized by the temperature

$$T = 1/(2\pi a)\,. \tag{37}$$

This effect is attributed to the non-Euclidean topology S^3 of the spatial section, which changes the continuous spectrum of energies and momenta occurring when $K = 0$ into a discrete spectrum when $K = 1$.

Now the background dynamics of the space-time (7) is determined by the Friedmann and Raychaudhuri equations (13) and (14), respectively, for a perfect fluid. In three spatial dimensions, the background gravitational vacuum state is defined by

$$\xi^2 = \dot{\xi} = 0\,, \tag{38}$$

and we find from Eq. (31) that the apparent horizon is only real and finite for $K = 1$, when $\tilde{r}_{\mathrm{a}} = a$ and Eq. (32) reduces precisely to Eq. (37).

There has to be some background matter to curve the three-space, and we immediately see that it has to satisfy the equation of state $\gamma = 2/3$ describing a universe dominated by stringy matter, for which the Nordström [33] energy-density (see also Tolman [34]) vanishes,

$$\rho_{\mathrm{N}} \equiv (3\gamma - 2)\rho = 0\,. \tag{39}$$

The fundamental nature of this vacuum solution was discussed in Ref. [35].

When $K = 0, -1$, on the other hand, it was found in Ref. [31] that no radiation with a thermal spectrum is created. This result can now be understood from the alternative perspective of the apparent horizon, which from Eq. (31) becomes infinite if $K = 0$ or imaginary if $K = -1$, implying from Eq. (32) a temperature which is zero or imaginary, respectively.

From the standpoint of the Schrödinger equation (18), we require a finite pseudo-Hamiltonian in order to be able to define a finite temperature, which again limits consideration to the case $K = 1$, in the vacuum background defined by Eqs. (38).

5. Discussion

We have emphasised in Ref. [13] that the horizon hypothesis, according to which the action integral does not extend beyond the causal horizon, applies not only to quantum cosmology, but to all quantum field theoretical phenomena, Schwinger [36] having shown that such an infra-red cut-off in quantum electrodynamics does not appear in expressions for quantities that are observable experimentally.

More recently, this idea has been put forward from a different point of view by Cohen *et al.* [37], who argued that there is an infra-red cut-off in quantum field theory which excludes all states that lie within their Schwarzschild radius. Applying this argument to the spatially flat Friedmann Universe (7), we first note that the mass contained within a spherical region of physical radius $\tilde{r}$ is

$$M(\tilde{r}) = \tfrac{4}{3}\pi\rho\tilde{r}^{\,3} = \tfrac{1}{2}\xi^2\tilde{r}^{\,3}\,. \tag{40}$$

Therefore, a point at radius $\tilde{r}$ lying *outside* its Schwarzschild radius $r_{\mathrm{S}} \equiv 2M(\tilde{r})$ must lie *inside* its apparent horizon, for from Eq. (40) we have the inequalities

$$\xi^{-1} \geq \tilde{r} \geq 2M(\tilde{r})\,. \tag{41}$$

Thus, we see that the proposed infra-red cut-off occurs at the apparent horizon, which, referred back to the metric (7) expressed in coordinates $(t, \tilde{r}, \theta, \phi)$, is the point where the metric signature changes from $(+ - --)$ to $(- - --)$.

The idea of imposing such an infra-red cut-off in fact originates with the researches of 't Hooft [38], who realized some time ago, even before the formulation of the holographic analogy [39, 40], reviewed in Ref. [41], that in the absence of a cut-off quantum fields would make a divergent contribution to the energy and entropy of a black hole in the vicinity of the horizon. To deal with this phenomenon, he suggested that the wave functions all vanish within some fixed, but unspecified, distance h from the horizon,

$$\phi(\tilde{r}) = 0 \qquad \text{for} \qquad \tilde{r} \leq 2M + h\,. \tag{42}$$

This is in agreement with the hypotheses of both Refs. [13] and [37], if we set $h = 0$.

It remains to understand why dust — which in classical thermodynamics is characterized by vanishing temperature — displays a finite temperature upon the application of quantum theory. As we have discussed previously [42], analogies with the fundamental aspects of superfluidity are sometimes useful in quantum cosmology. Superfluidity in helium-4 was first attributed to zero-point fluctuations by Bennewitz and Simon [43, 44] and London [45], who emphasized that the helium atoms cannot be localized on a spatial lattice, due to the large zero-point energy.

Subsequently, London [46] explained this phenomenon in terms of Bose–Einstein condensation and the Heisenberg [47] indeterminacy principle, at the macroscopic level. If the indeterminacies in the magnitude of the position $\boldsymbol{x}$ and momentum $\boldsymbol{p}$ of a helium atom in the condensate are Δx and Δp, respectively, we have the *configurational* statement that

$$\Delta x \Delta p \sim \hbar\,. \tag{43}$$

In *dynamical* terms, the kinetic and potential energies of an atom of mass m and interaction constant k are $(\Delta p)^2/2m$ and $k(\Delta x)^2/2$, respectively, in the ground state. For light particles and weak interactions, it therefore follows that the total energy is minimized by making Δx as large and Δp as small as possible. If the system is of infinite extent, we can set $\Delta x = \infty$ and $\Delta p \equiv m\Delta v = 0$, which can be interpreted to mean that the particles are all indistinguishable, and hence all have the same velocity, as originally argued by Einstein [48]. For this ideal case, long-range order becomes an exact symmetry.

In practice, of course, the container of helium is of finite, but macroscopic, dimension l. Therefore, by setting $\Delta x = l$ and $\Delta p \sim \hbar/l$, we still ensure indistinguishability of the particles, which consequently persist in a fluid state down to the absolute zero of temperature.

Turning now to cosmology, let us consider the Friedmann space-time (7) generated by dust, which exhibits a number of interesting parallels to the theory of superfluidity. On the one hand, below the critical point the viscosity of liquid helium-4 at zero pressure tends to zero as the temperature tends to zero. In the perfect-fluid idealization in cosmology, on the other hand, the Universe is modelled by a fluid, which, in the dust approximation, has zero pressure and temperature and is inviscid by definition — it is assumed to consist for the most part of particles of dark matter, which interact with one another only via the weak gravitational force.

Again, if we regard the phenomenon of superfluidity as the result of a phase transition from the normal to the superfluid state, as the temperature is lowered, we can ascribe an order parameter to the process, introduced by Landau [49, 50], in terms of which the free energy is defined as a power series, and which is assumed to be independent of the spatial coordinates. Analogously, the free-energy density in Friedmann cosmology is defined by the Hubble parameter ξ, which thus plays the rôle of order parameter, and which vanishes in Minkowski space. In this case, it is important to note that long-range order is imposed as a symmetry at the outset.

Further, the theory of superfluidity and the cosmological Wheeler–DeWitt equation are both examples of macroscopic quantum mechanics. It was Landau [51] who analysed the quantum superfluid as a collective phenomenon, using a single wave function to describe the ensemble of helium atoms. While the cosmological wave function of the Universe Ψ refers to the whole cosmos by initial construction, although this is not actually a Bose–Einstein condensate.

Inspired by these analogies, we are led to apply the indeterminacy principle to cosmology in the same fashion. In place of the container of liquid helium of finite dimension, we are now dealing with the entire content of the Universe within the apparent horizon. For the stationary background analysed in Ref. [31], there is a spatial indeterminacy over the dimension of the Universe, which is only finite in the closed case. The causal nature of the boundary allows us to convert length into time, as a result of which the indeterminacy in energy is $\Delta E \sim \hbar/\Delta x \sim \hbar/a$. If we regard these energy fluctuations as thermal in origin, we obtain the estimate for the corresponding temperature

$$T \approx \hbar/a\,, \tag{44}$$

in agreement with Eq. (37) up to a factor $\sim 2\pi$. By contrast, the open and flat cases yield $\Delta x = \infty$, and consequently $T = 0$, exactly as found in Ref. [31].

In the general time-dependent Universe, there is a finite apparent horizon $\tilde{r}_{\rm a}$ even for the open or flat spatial topology, given by Eq. (31), and accordingly now a finite temperature

$$T \approx \hbar/\tilde{r}_{\rm a}\,, \tag{45}$$

in approximate agreement with Eq. (32), which can therefore also be understood from this point of view.

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