# STATIONARY DISTRIBUTIONS
# OF A NOISY LOGISTIC PROCESS

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Stationary solutions to a Fokker–Planck equation corresponding to a noisy logistic equation with correlated Gaussian white noises are constructed. Stationary distributions exist even if the corresponding deterministic system displays an unlimited growth. Positive correlations between the noises can lead to a minimum of the variance of the process and to the stochastic resonance if the system is additionally driven by a periodic signal.

## 1. Introduction

The logistic equation

$$\dot{x} = ax(1 - x)\,, \ a > 0\,, \ x \geqslant 0\,, \tag{1}$$

is one of the best-known and most popular models in population dynamics. Perturbing this equation by a multiplicative noise is an obvious generalization of the deterministic theory, aiming at describing populations that live in an ever-changing environment. The logistic equation with a fluctuating growth rate

$$\dot{x} = (a + p\,\xi(t))x(1 - x)\,, \tag{2}$$

has been first discussed by Leung in Ref. [1] and later by many other authors. Recently in Ref. [2] we have discussed a further generalization of (2) in which both the growth rate and the limiting population level fluctuate, and these fluctuations are correlated in time:

$$\dot{x} = (a + p\,\xi_{\mathrm{m}}(t))x - (b + q\,\xi_{\mathrm{a}}(t))x^2\,. \tag{3}$$

Here $\xi_{\mathrm{m,a}}$ are two Gaussian white noises (GWNs) that satisfy $\langle \xi_i(t) \rangle = 0$, $\langle \xi_i(t_1)\xi_i(t_2) \rangle = \delta(t_1 - t_2)$, $i = \mathrm{m}, \mathrm{a}$, $\langle \xi_{\mathrm{m}}(t_1)\xi_{\mathrm{a}}(t_2) \rangle = c\,\delta(t_1 - t_2)$, $p$, $q$ are the amplitudes of the two noises and the correlation coefficient $c \in [-1, 1]$. For the sake of terminology, we will sometimes call the noise $\xi_{\mathrm{m}}(t)$ "multiplicative" and the noise $\xi_{\mathrm{a}}(t)$ "additive"; see Eq. (5) below for a rationale behind these names. Please note, though, that on the level of Eq. (3) both these noises are coupled multiplicatively to the process $x(t)$. Note also that if $b > 0$, the corresponding deterministic equation converges to a stable fixed point; accordingly, we will call a system with a positive $b$ "convergent". If $b \leqslant 0$, the corresponding deterministic system displays unlimited growth. We will call a system with a negative $b$ "exploding". Recently Mao *et al.* have shown in Ref. [3] that in the absence of $\xi_m$, the system (2) remains positive and bounded even in the "exploding" case. In Ref. [4] we have discussed certain difficulties that may arise in numerical simulations of such a system. This paper generalizes the result of Mao *et al.* to the case of both noises present.

In Ref. [2] we have shown how the dynamics (3) is related to the problem of a linear stochastic resonance. We have mapped the nonlinear equation (3) into a linear Langevin equation with two correlated noises and used the solutions of the latter to heuristically explain the behaviour of the noisy logistic equation. Specifically, the substitution

$$y = \frac{1}{x} \tag{4}$$

converts Eq. (3) into a linear equation

$$\dot{y} = -(a + p\,\xi_{\mathrm{m}}(t))y + b + q\,\xi_{\mathrm{a}}(t) \tag{5}$$

which can be solved exactly for realizations of the process $y(t)$. The process (5) has a convergent mean if

$$a - \tfrac{1}{2}p^2 > 0 \tag{6a}$$

and a convergent variance if a stronger condition

$$a - p^2 > 0 \tag{6b}$$

holds. Using the properties of the process $y(t)$, useful prediction can be made about the noisy logistic process. Since in the presence of correlations, for certain values of parameters the variance of $y(t)$ first shrinks and then grows as a function of the "multiplicative" noise strength, $q$, one expects a similar behaviour for the logistic process $x(t)$ as well. These predictions have been corroborated numerically in Ref. [2]. In particular, if $c = \pm 1$,

$bp \mp aq = 0$ and the condition (6b) is satisfied, the variance of the process $y(t)$ vanishes. The relation between the processes $y(t)$ and $x(t) = 1/y(t)$ *intuitively* means that if almost all realizations of the former asymptotically reach the same constant value, so do almost all realizations of the latter. However, as a *formal* relation between moments of these processes is not trivial, the predictions based on the properties of $y(t)$ have only a heuristic value.

In the following we will construct mathematically exact stationary solutions to the Fokker–Planck equation corresponding to Eq. (3) and re-examine the above results from the point of view of these stationary solutions. Furthermore, we will show numerically that if the parameters undergo periodic (for example, circaannual or seasonal) oscillations, a positive correlation between the noises leads to a stochastic resonance.

In the Appendix we extend to the case of two correlated noises the proof originally proposed by Mao *et al.* in Ref. [3] that solutions to Eq. (3), when started from a positive initial condition, never become negative almost surely.

## 2. The Fokker–Planck equation

The problem of constructing a Fokker–Planck equation corresponding to a process driven by two correlated Gaussian white noises has been first discussed in Ref. [5], where the two noises have been decomposed into two independent processes. The same result has been later re-derived in [6], where the authors have attempted to avoid an explicit decomposition of the noises but eventually resorted to a disguised form of the decomposition. The general Langevin equation

$$\dot{x} = h(x) + g_1(x)\xi_{\mathrm{m}}(t) + g_2(x)\xi_{\mathrm{a}}(t), \tag{7}$$

where $x(t)$ is a one-dimensional process and $\xi_{\mathrm{m,a}}$ are as in Eq. (3), leads to the following Fokker–Planck equation in the Ito interpretation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}h(x)P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}B(x)P(x,t), \tag{8a}$$

where

$$B(x) = [g_1(x)]^2 + c\,g_1(x)g_2(x) + [g_2(x)]^2. \tag{8b}$$

In the case of Eq. (3) the corresponding Fokker–Planck equation therefore reads

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[(a - bx)xP(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[x^2(p^2 - 2cpqx + q^2x^2)P(x,t)\right]. \tag{9}$$

It is apparent that the absolute signs of the two noise amplitudes do not influence the solutions to the above equations, only their relative sign, $\text{sgn}(pq)$, does. In the following we will assume that $\text{sgn}(pq) = +1$. This comes at no loss to the generality as Eq. (9) is invariant under a simultaneous change of signs of $pq$ and the correlation coefficient, $c$.

Stationary solutions to Eq. (9) are the normalizable solutions to [7,8]

$$x^2(p^2-2cpqx+q^2x^2)\frac{dP_{\text{st}}(x)}{dx}+2x\left(2q^2x^2 + (b-3cpq)x - (a-p^2)\right)P_{\text{st}}(x) = 0\,.$$
(10)

A slight modification of the argument presented originally in Ref. [3] shows that all solutions to Eq. (3) which start from a positive initial condition remain positive almost surely; see the Appendix for a proof. Physically speaking, this results from a presence of an absorbing barrier at $x = 0$ in Eq. (3): should the population suddenly drop to zero, it would stay there forever. The formal result of Ref. [3], extended here to the case of two noises present, ensures that the population never actually becomes nonpositive although in certain cases (see below) it may dynamically cluster in a close proximity of $x = 0^+$. Therefore, we can divide both sides of (10) by $x$ and obtain

$$x(p^2-2cpqx+q^2x^2)\frac{dP_{\text{st}}(x)}{dx}+2\left(2q^2x^2 + (b - 3cpq)x - (a - p^2)\right)P_{\text{st}}(x) = 0\,,$$
(11)

provided $P_{\text{st}}(x)$ is normalizable over the $x > 0$ half-axis.

One may be tempted to try to immediately solve Eq. (11) by standard methods, but a word of caution is needed here: should the coefficient at $dP_{\text{st}}/dt$ vanish, special care must be taken.

### 2.1. The case of only one noise present

Before proceeding to the general case, we will discuss the special cases where only one of the amplitudes $p$, $q$ does not vanish.

*Purely "multiplicative" noise.* If $q = 0$, the Langevin equation (3) reduces to

$$\dot{x} = (a + p\,\xi_{\text{m}}(t))x - bx^2$$
(12)

and we find for $P_{\text{st}}(x)$

$$P_{\text{st}}(x) = \mathcal{N}x^{2(a-p^2)/p^2}\exp\left(-\frac{2bx}{p^2}\right),$$
(13)

where $\mathcal{N}$ is a normalization constant. This function is normalizable for $x > 0$ if $b > 0$, or if the system is convergent, and

$$a - \tfrac{1}{2}p^2 > 0\,,$$
(14a)

which determines the behaviour of the distribution around $x = 0$. Moreover, if

$$a - p^2 > 0 , \tag{14b}$$

$P_{\text{st}}(x)$ goes to zero as $x \to 0^+$ and has a maximum at $x = (a - p^2)/b$. Note that the conditions (14a), (14b) coincide with the conditions (6a), (6b) determining the properties of the linear process (5). For $p^2 > a > \frac{1}{2}p^2$, the distribution is mildly divergent at $x = 0$ and decreases monotonically with an increasing $x$.

*Purely "additive" noise.* If $p = 0$, the Langevin equation takes the form

$$\dot{x} = (a - bx)x + qx^2\xi_{\text{a}}(t) \tag{15}$$

and we obtain for the stationary distribution

$$P_{\text{st}}(x) = \frac{\mathcal{N}}{x^4} \exp\left(-\frac{a - 2bx}{q^2 x^2}\right) \tag{16}$$

which is normalizable whenever $a > 0$. Note that no bounds on $b$ are imposed: The stationary distribution exists for both convergent and exploding systems. This special case falls into a broad category discussed recently by Mao *et al.* in Ref. [3], without further generalizations provided by the present paper. The distribution (16) has a maximum at the positive root of

$$2q^2 x^2 + bx - a = 0 . \tag{17}$$

### 2.2. The general case

If both noises are present and are not maximally correlated, $|c| \neq 1$, the solution to (11) reads

$$P_{\text{st}}(x) = \frac{\mathcal{N} x^{2(a-p^2)/p^2}}{(p^2 - 2cpqx + q^2 x^2)^{(a+p^2)/p^2}} \exp\left[-\frac{2(bp - acq)\arctan\left(\frac{qx - cp}{\sqrt{1 - c^2}\,p}\right)}{\sqrt{1 - c^2}\,p^2 q}\right] . \tag{18}$$

Since the exponential term is limited, the convergence (normalization) properties of (18) are determined by those of the fractional term. The denominator is always strictly positive. For $x \to \infty$, $P_{\text{st}}(x) \sim x^{-4}$ for all possible values of parameters. $P_{\text{st}}(x)$ is therefore normalizable if it does not diverge too rapidly at $x \to 0^+$, or again if the condition (14a) holds. Either in this case, no bounds on $b$ are imposed. If the condition (14b) holds as well, the distribution (18) approaches zero as $x \to 0^+$, but this condition is no

longer associated with the presence of a maximum. The maximum of $P_{st}(x)$ coincides with the positive root of

$$2q^2x^2 + (b - 3cpq)x - (a - p^2) = 0\,, \qquad (19)$$

*cf.* Eq. (11), provided such a root exists. It certainly does for $a - p^2 > 0$, but it can appear also for $p^2 > a > \frac{1}{2}p^2$, where the distribution is mildly divergent. Example stationary distributions in a strongly correlated case are presented in Fig. 1. For large values of the additive noise strength, $q$, the distributions are highly skewed and squeezed against the $x = 0$ axis. For comparison, in Fig. 2 we show example stationary distributions for the uncorrelated and a strongly anticorrelated cases. The distributions presented are skewed and much wider than the distribution from Fig. 1 with the same value of $q = 0.5$. These distributions also get squeezed as the additive noise strength becomes large. Note that the distribution corresponding to the anticorrelated noises is already more squeezed than the distribution for the uncorrelated case.
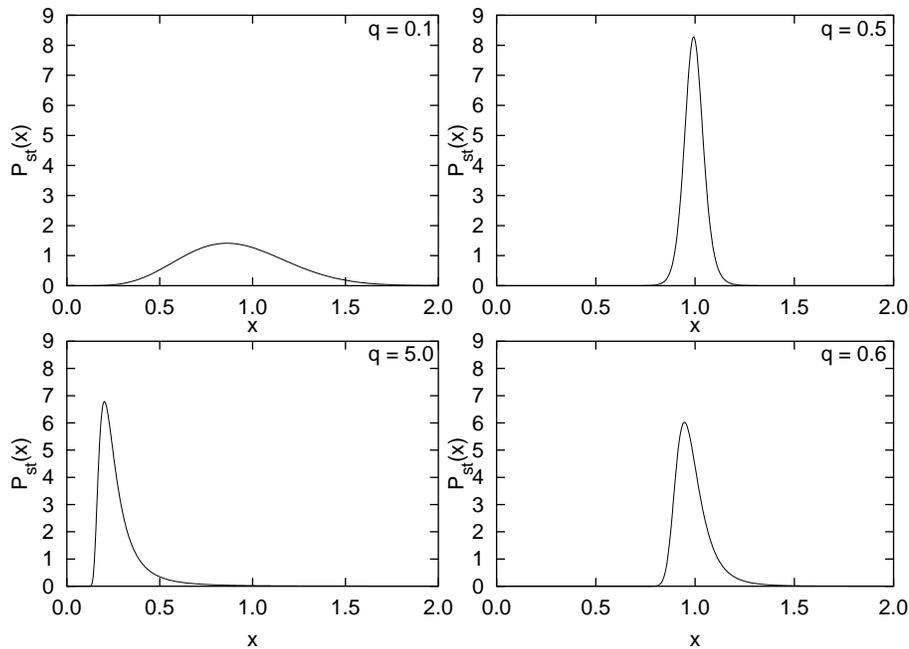


Fig. 1. Stationary distributions (18) in a strongly correlated case, $c = 0.99$. Clockwise, from top-left $q = 0.1$, $q = 0.5$, $q = 0.6$, and $q = 5.0$. Other parameters, common for all panels, are $p = 0.5$, $a = b = 1$.
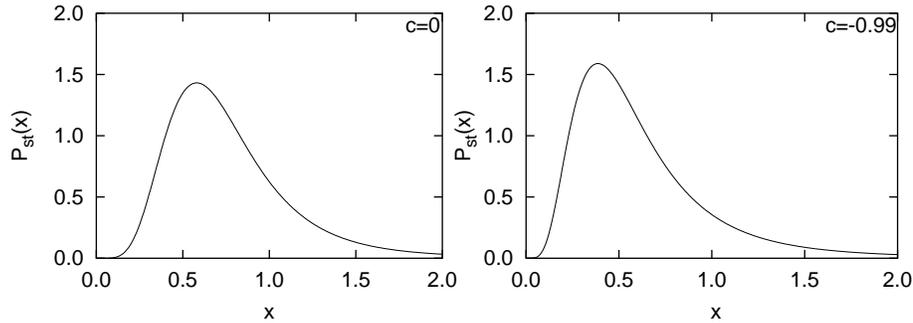
Fig. 2. Stationary distributions for the uncorrelated ($c = 0$, left panel) and a strongly anticorrelated ($c = -0.99$, right panel) cases. Other parameters are $a = b = 1$, $p = q = 0.5$.

If the distribution (18) is normalizable, it has a convergent mean and a variance. Its higher moments are divergent.

### 2.3. The maximally correlated case

If the two noises are maximally (anti)correlated, $c = \pm 1$, Eq. (11) takes the form

$$x(p \mp qx)^2 \frac{dP_{\text{st}}}{dx} + 2\left[(p \mp qx)^2 + q^2 x^2 + (b \mp pq)x - a\right] P_{\text{st}} = 0\,, \qquad (20)$$

leading to the following candidate solution:

$$P_{\text{st}}(x) = \mathcal{N} \frac{x^{2(a-p^2)/p^2}}{(p \mp qx)^{2(a+p^2)/p^2}} \exp\left[\mp \frac{2(bp \mp aq)}{pq(p \mp qx)}\right]\,. \qquad (21)$$

With our sign convention adopted, $\text{sgn}(pq) = +1$, this solution is normalizable if $c = -1$ and the condition (14a) holds. The distribution (21) with the "$+$" sign can be obtained from Eq. (18) by taking the limit $c \to -1$. This distribution decreases as $x^{-4}$ with $x \to \infty$. The maximum of (21), if it exists, coincides with the positive root of Eq. (19) with $c = -1$.

The case of $c = +1$ is more challenging. First, if the resonant condition

$$bp - aq = 0 \qquad (22)$$

holds, Eq. (20) is solved by

$$P_{\text{st}}(x) = \delta\left(x - \frac{p}{q}\right) \qquad (23)$$

regardless of the value of $p$. This result is stronger than that reported in Ref. [2] where we could predict a $\delta$-shaped distribution only in the $a - p^2 > 0$ case, or when the variance of the corresponding linear system was convergent, as otherwise any predictions based on the linear system failed. Note that with the condition (22) statisfied, Eq. (3) reduces to a rescaled form of Eq. (2).

If $c = +1$ and the condition (22) does not hold, Eq. (20) does not have a normalizable solution. This observation is slightly surprising, but formally speaking, it results from the fact that the double limit

$$\lim_{\substack{x \to p/q \\ c \to 1}} \exp\left[ -\frac{2(bp - acq)\arctan\left(\frac{qx - cp}{\sqrt{1 - c^2}p}\right)}{\sqrt{1 - c^2}\,p^2 q} \right] \tag{24}$$

does not exits: Its value depends on which route the singularity is approached. The nonexistence of stationary solutions in the fully correlated, non-resonant case is, therefore, related to the essential singularity of the complex exponential at infinity. The fact that with $c = +1$, $bp - aq \neq 0$, the drift and diffusive terms in Eq. (20) both vanish, but at different points, is the physical reason for this apparent oddity: If the population gets located around the point of the vanishing diffusion, it is washed away by the drift, and if it gets located around the point of the vanishing drift, it diffusively leaks from there. Nevertheless, if $a - \frac{1}{2}p^2 > 0$, in numerical simulations the cases of $c = 1$ and $c = 1 - \varepsilon$ with $0 < \varepsilon \ll 1$ are undistinguishable. In the latter case, the distribution (18) is perfectly normalizable.

## 3. Resonant effects and the shape of the stationary distribution

Perhaps the most important prediction based on the analysis of the linear equation (5) and discussed in Ref. [2] is that, for certain values of parameters, the variance of the process $x(t)$ should, in the asymptotic regime, first shrink, reach a minimum, and then grow as a function of the additive noise strength, $q$. As we have mentioned before, these are heuristic, intuitive conclusions based on the behaviour of the linear system associated with the logistic process, but because of the complicated relation between the moments of these two processes, they do not amount to a formal proof. In Ref. [2] we have confirmed these predictions numerically for a certain range of the additive noise strengths. As we have seen above, in the fully correlated and resonant case, the stationary distribution becomes $\delta$-shaped and its variance indeed vanishes, much as predicted by the linear system. Since we now know the mathematically exact stationary distributions, we can test the behaviour of the variance in the general case directly.
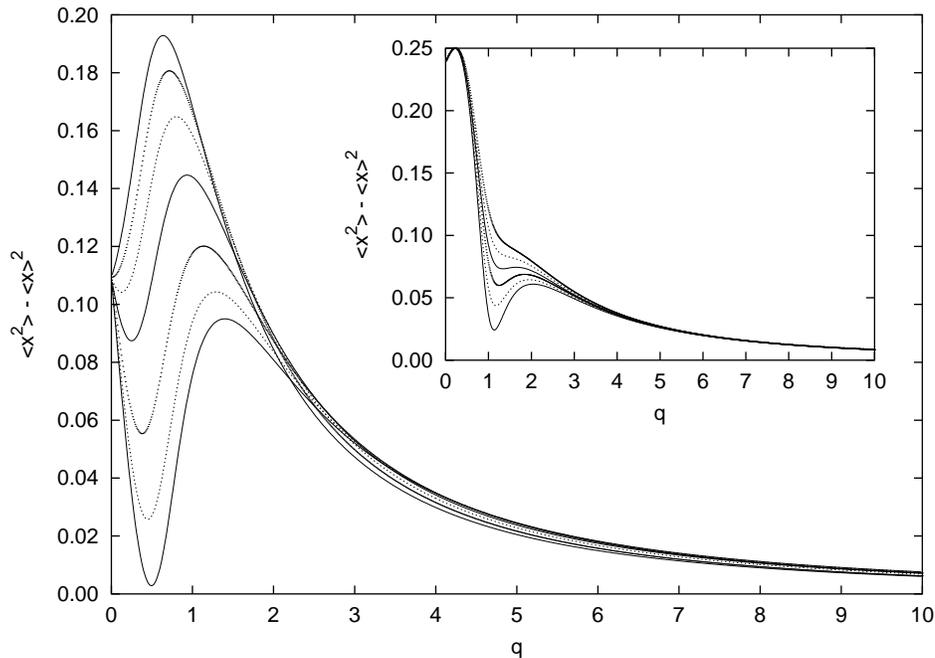
Fig. 3. The variance $\langle x^2 \rangle - \langle x \rangle^2$ determined form the distribution (18) as a function of the additive noise strength, $q$. Main panel: $p = 0.5$, the curves, from bottom to top, correspond to $c = 0.99$, $c = 0.90$, $c = 0.75$, $c = 0.50$, $c = 0.25$, $c = 0$, and $c = -0.25$, respectively. Inset: $p = 1.1$, the curves correspond, from bottom to top, to $c = 0.99$, $c = 0.98$, $c = 0.97$, $c = 0.96$, $c = 0.95$, and $c = 0.94$, respectively. Other parameters, common for all curves presented, are $a = b = 1$.

Recall that the distribution (18) has the two first moments convergent whenever it is normalizable. Unfortunately, analytical expressions for these moments cannot be obtained, mainly due to the presence of the complicated exponential term. Therefore, we have calculated the moments by numerically integrating over the distribution (18). Results are presented in Fig. 3. If the distribution approaches zero as $x \to 0^+$, or when the condition (14b) is satisfied, and if the two noises are positively correlated, $0 < c < 1$, the variance $\langle x^2 \rangle - \langle x \rangle^2$ displays a clear minimum as a function of the additive noise strength, $q$. The minimum becomes shallower as the correlations decrease towards zero, where it eventually disappears. It is not present for the negative correlations or when one of the amplitudes vanishes. These effects agree with predictions based on the linear system (5). The presence of the minimum of the variance is a clear and beneficial effect of positive

correlations between the two noises. However, for larger values of $q$ a new phenomenon appears: The variance starts decreasing again. This is because for large values of the additive noise, the stationary distribution gets squeezed against the $x = 0$ axis, *cf.* Fig. 1 above. This effect cannot be predicted within the linear approach — note that the process described by Eq. (5) has a support that formally spreads over the entire real axis and, moreover, is Gaussian whenever the condition (14b) holds, while the noisy logistic process is restricted to the positive half-axis.

If the stationary distribution mildly diverges at zero, or if $p^2 > a > \frac{1}{2}p^2$, a distinct minimum in the variance of $x$ also appears but it is present only for fairly large (and positive) values of the correlation coefficient, *cf.* the inset in Fig. 3. Note that this effect cannot possibly be predicted by analysing the linear system (5) as in this regime the variance of the linear process diverges and any predictions break.

## 4. Stochastic resonance

We now assume that parameters of the logistic process are not only subjected to noise, but also to periodic, deterministic perturbations, resulting for example from seasonal changes in the environment. Specifically, we consider

$$\dot{x} = (a + p\,\xi_{\mathrm{m}}(t))x - (b + A\sin(\Omega t + \varphi) + q\,\xi_{\mathrm{a}}(t))x^2\,. \qquad (25)$$

We have shown analytically in Ref. [2] that the linear system associated with Eq. (25) displays a stochastic resonance (SR) if the noises are positively correlated. SR is one of the most spectacular examples of a constructive role of noise — see Ref. [9] for a review. Because we do not know exact solutions of a time-dependent Fokker–Planck equation corresponding to Eq. (25), we will demonstrate the SR phenomenon numerically. We will use the Signal-To-Noise Ratio (SNR) as a measure of the SR:

$$\mathrm{SNR} = 10\log_{10}\frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}(\omega = \Omega)}\,, \qquad (26)$$

where $P_{\mathrm{signal}}$ is the height of the peak in the power spectrum at the driving frequency and $P_{\mathrm{noise}}$ is the noise-induced background.

We have solved the equation (25) numerically with the Euler–Maryuama algorithm and a timestep equal $2^{-16}$. The GWNs have been generated by the Marsaglia algorithm [10] and the famous Mersenne Twister [11] has been used as the underlying uniform generator. We have let the system to equilibrate, run the simulations for $2^{25}$ steps and collected the results of every $2^9$-th step, calculated the power spectrum, calculated the SNR and averaged the results over 128 realizations of the stochastic processes and the initial phases, $\varphi$. Selected results are presented in Fig. 4.
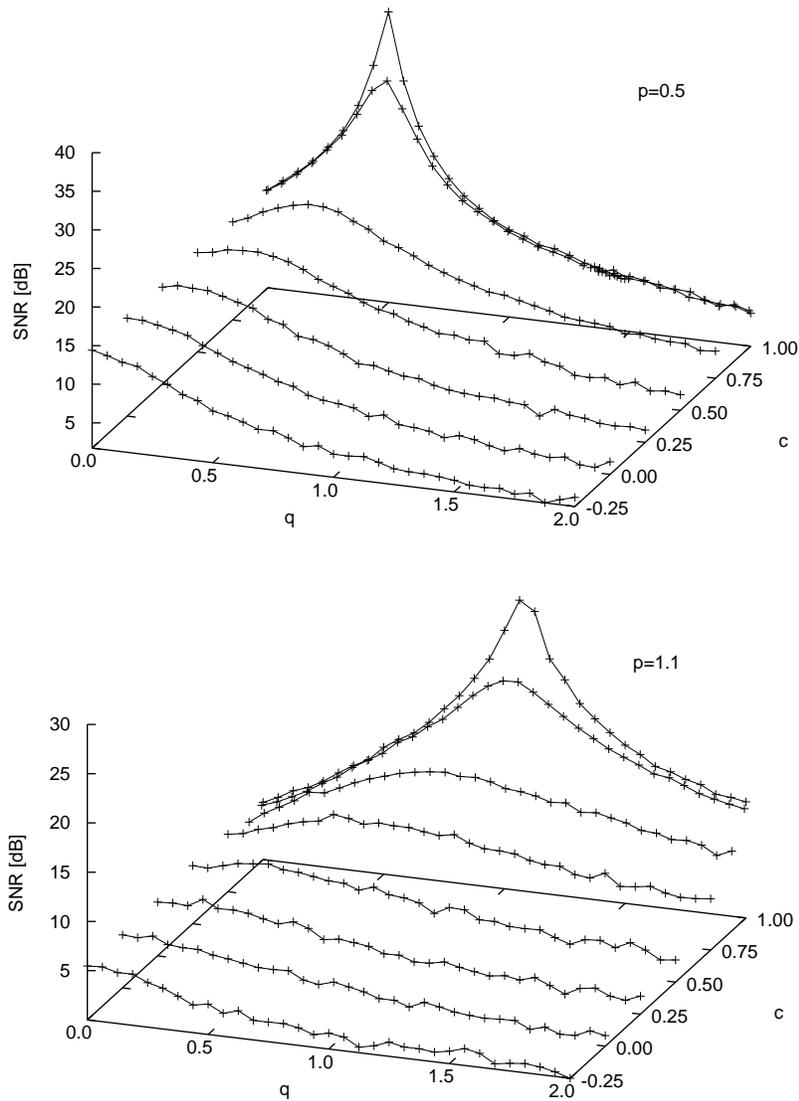
Fig. 4. Stochastic resonance in the system (25). The upper panel — the condition (14b) is satisfied, $p = 0.5$. The lower panel — the condition (14b) is not satisfied, $p = 1.1$. Other parameters, common for the two panels, are $a = b = 1$, $A = 0.5$, $\Omega = 2\pi$. Curves presented correspond, back to front, to $c = 1.0$, 0.99, 0.9 (lower panel only), 0.75, 0.5, 0.25, 0.0, and $-0.25$, respectively.

The upper panel corresponds to the situation when the condition (14b) holds, or when the noisy logistic process without the periodic signal has a convergent variance. We can clearly see that the system (25) displays a SR for positive correlations between the two noises: For positive correlations between the noises, there is a certain level of the "additive" noise that maximizes the impact that seasonal changes in the environment have on the population. As we have shown in the preceding Section, this range of parameters corresponds to the presence of the minimum in the variance of (18). One may be tempted to conclude that the SR and the minimum of the variance are two facets of the same phenomenon, much as in the linear case. However, the lower panel of Fig. 4, corresponding to the situation when the condition (14b) is not satisfied, shows that this is not the case. The SR, albeit much weaker than in the previous case, is clearly present even when the signal-free system no longer displays a minimum of the variance. A minimum of the variance and the stochastic resonance are two different constructive effects of positive correlations between the noises.

## 5. Conclusions

In this paper we have constructed stationary distributions corresponding to a noisy logistic process driven by two correlated GWNs. These distributions are restricted to the positive semiaxis and if they are normalizable, they have two convergent moments. In particular, if the noises are maximally correlated and a certain resonant condition holds, the stationary distribution is $\delta$-shaped, which has been reported previously as a result of many numerical simulations. Surprisingly, if the noises are maximally correlated but the resonant condition does not hold, the process does not have a stationary distribution, even though it can numerically manifest itself as if it had one.

Positive correlations between the noises lead to a minimum of the variance of the noisy logistic process and to a stochastic resonance if the parameters of the system undergo additional periodic changes. As we have numerically demonstrated, these are two different effects. By constructing the exact stationary distributions, we have extended our previous analysis of the system performed mainly by formally converting the system into a linear one. Several features of the system, and "squeezing" of the stationary distribution in case of a strong "additive" noise in particular, cannot be described by analysing the linear process. This is because the linear process (5) is Gaussian if it has two convergent moments but the nonlinear logistic process is not. Nevertheless, there are nice parallels between the properties of the logistic process and its formal linearization: If $a - \frac{1}{2}p^2 > 0$, the linear process has a convergent mean and the logistic process has a normalizable stationary distribution that decreases for large $x$ as $x^{-4}$. If $a - p^2 > 0$, the

stationary distribution of the logistic process approaches zero as $x \to 0^+$ and the linear process has a convergent variance. If $\frac{1}{2}p^2 < a < p^2$, the stationary distribution of the noisy logistic process mildly diverges at $x = 0^+$ and the population dynamically clusters around that point. Note that for $a > p^2$, the stationary distribution has a maximum and the population is actually pushed away from $x = 0^+$. Thus, the level of the multiplicative noise $p^2 = a$ marks a qualitative change in the population described by the noisy logistic equation.

It is, perhaps, surprising that a stationary distribution of the noisy logistic process may exist even if the corresponding deterministic process is clearly divergent. The fact that noise can prevent a population from exploding has been recently reported by Mao *et al.* in Ref. [3] for a more restricted, in a sense, class of systems. The present work is an extension of this research to a class that includes two correlated sources of the noise.

## Appendix A

To show that solutions to Eq. (3) remain positive almost surely when started from a positive initial condition, we first decompose the correlated noises $\xi_\mathrm{m}$, $\xi_\mathrm{a}$ into two independent processes:

$$\xi_\mathrm{m}(t) = \xi(t)\,, \tag{A.1a}$$

$$\xi_\mathrm{a}(t) = c\xi(t) + \sqrt{1 - c^2}\,\eta(t)\,, \tag{A.1b}$$

where $\eta(t)$, $\xi(t)$ are two identical, uncorrelated GWNs. The decomposition (A.1) is a variant of the method originally used in Ref. [5]. We now cast Eq. (3) in a form customarily used by mathematicians:

$$dx = (ax - bx^2)\, dt + (px - qcx^2)\, du - q\sqrt{1 - c^2}\, x^2\, dw\,, \tag{A.2}$$

where $du$, $dw$ are differentials over two identical, independent Wiener processes. Incidentally, observe that the Fokker–Planck equation (9) follows immediately from Eq. (A.2).

**Theorem 1.** *If the initial condition $x_0 > 0$, for any $q \neq 0$ the solution to Eq. (A.2) remains positive for all $t > 0$ almost surely.*

*Proof.* The proof of this theorem follows closely that of Theorem 2.1 from the work of Mao *et al.* [3] and we encourage readers interested in mathematical details to familiarize with that proof first; to save the space, we will show only this part in which the proof of Theorem 1 differs from that of Mao *et al.*

First, the authors of Ref. [3] consider a multispecies (multidimensional) system, while we restrict ourselves to a simpler single-species case.

Second, the proof is based on properties of the function

$$V(s) = \sqrt{s} - 1 - \tfrac{1}{2} \ln s. \tag{A.3}$$

This function is nonnegative for any $s > 0$. We calculate $V(x(t))$ along the trajectory generated by Eq. (A.2) with an initial condition $x_0 > 0$ and calculate the stochastic differential of $V(x(t))$ using Ito formula. Because Eq. (A.2) differs from that considered by Mao *et al.*, we obtain a slightly different expression. Specifically, if $x(t) > 0$,

$$
\begin{aligned}
dV(x(t)) &= \tfrac{1}{2}\left(x^{-1/2} - x^{-1}\right)\left[(ax - bx^2)dt + (px - qcx^2)du - q\sqrt{1-c^2}\,x^2 dw\right] \\
&\quad + \tfrac{1}{4}\left(x^{-2} - \tfrac{1}{2}x^{-3/2}\right)\left[(px - qcx^2)^2 + q^2(1-c^2)x^4\right]dt.
\end{aligned} \tag{A.4}
$$

The second term in (A.4) would be absent if the noises were interpreted in the Stratonovich sense. After a simple algebra,

$$
\begin{aligned}
dV(x(t)) &= \left[\tfrac{1}{2}\left(x^{1/2} - 1\right)(a - bx) + \tfrac{1}{4}\left(1 - \tfrac{1}{2}x^{1/2}\right)(p^2 - 2pqcx + q^2 x^2)\right]dt \\
&\quad - \tfrac{1}{2}\left(x^{1/2} - 1\right)(p - qcx)\,du - \tfrac{1}{2}\left(x^{1/2} - 1\right)q\sqrt{1-c^2}\,x\,dw.
\end{aligned} \tag{A.5}
$$

If $q = 0$ and $b < 0$, or when the corresponding deterministic system explodes, the coefficient at $dt$ in (A.5) may assume arbitrarily large values. On the contrary, for any $q \neq 0$ and regardless of the sign of $b$, this coefficient is bounded from above by a certain positive number $K$. Thus[1]

$$
\begin{aligned}
\int_0^T dV(x(t)) \leqslant KT &- \int_0^T \tfrac{1}{2}\left(x^{1/2} - 1\right)(p - qcx)\,du \\
&- \int_0^T \tfrac{1}{2}\left(x^{1/2} - 1\right)q\sqrt{1-c^2}\,x\,dw,
\end{aligned} \tag{A.6}
$$

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[1] Some subtleties of the notation are omitted here, see [3] for a fully rigorous treatment.

where $T$ is a time such that $x(t)$ is positive for $0 \leqslant t < T$ almost surely. By taking the expectation values, we obtain

$$\langle V(x(T)) \rangle \leqslant V(x_0) + KT \, . \qquad (A.7)$$

The rest of the proof now proceeds exactly as in Ref. [3] to show that $T = \infty$. $\qquad\qquad\square$

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