

STABILITY OF A METRIC $f(R)$ GRAVITY THEORY
IMPLIES THE NEWTONIAN LIMIT

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We show that the existence of the Newtonian limit cannot work as a selection rule for choosing the correct gravity theory from the set of all $L = f(R)$ gravity theories. To this end we prove that stability of the ground state solution in arbitrary purely metric $f(R)$ gravity implies the existence of the Newtonian limit of the theory. And the stability is assumed to be the fundamental criterion of viability of any gravity theory. The Newtonian limit is either strict in the mathematical sense if the stable ground state of a theory is flat spacetime, or approximate and valid on length scales much smaller than the cosmological scale if the ground state is de Sitter or anti-de Sitter space. Hence regarding the Newtonian limit a metric $f(R)$ gravity does not differ from general relativity (with arbitrary Λ). That stability implies the existence of the Newtonian limit is exceptional to Lagrangians depending on R and/or the Ricci tensor but not on the Weyl tensor. An independent selection rule is necessary.

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1. Introduction and summary

In recent years the metric nonlinear gravity (NLG) theories have attracted vivid attention as a possible explanation for the acceleration of the universe without invoking the dark energy concept. These theories differ from general relativity only by their Lagrangian $L = f(g_{\mu\nu}, R_{\alpha\beta\mu\nu})$ being any smooth scalar function of the Riemann–Christoffel tensor for the metric $g_{\mu\nu}$. As the cardinality of the set of all analytic functions of one or several variables is higher than continuum, the fundamental problem is to choose one or at worst a narrow class of Lagrangians out of this set. These theories are tested where they are required, *i.e.* in cosmology, and exclusively applying the Robertson–Walker (RW) spacetime (usually in the special case where the

spatial sections are flat, $k = 0$). However this spacetime is “flexible” in that it contains an arbitrary function and thus provides a Friedmannian solution for any NLG theory (except some singular cases) while Minkowski, de Sitter and anti-de Sitter spacetimes are not universal solutions. This implies that there is an infinity of Lagrangians generating solutions exhibiting the accelerated evolution of the world. In fact, roughly one half of the Lagrangians investigated up to now in the literature fairly well fit the astronomical data. Also the solar system tests cannot uniquely put stringent bounds on possible Lagrangians due to inherent ambiguity of NLG theories. In these theories there is infinity of mathematically equivalent dynamical frames out of which only two, Jordan frame and Einstein one are employed in practice, and in different frames the initial and boundary conditions for the full gravitational field are more or less determined by the matter distribution. For example, in Jordan frame the initial and boundary conditions are determined by the local matter distribution while in Einstein frame they are determined only in a part by it [1]. In this sense the solar system observations are to some extent inconclusive.

Instead of attempting to deduce the correct NLG theory from the astronomical data, which are scarce and theoretically ambiguous, one should first verify if the theory (or a class of) under consideration meets the general requirements imposed on any classical field theory. A fundamental and indisputable criterion is that a theory have a stable maximally symmetric ground state. This criterion works effectively and it has been shown that many $L = f(R)$ gravity theories (R being the curvature scalar for the Riemann tensor) which are attractive on cosmological grounds, are actually unstable and thus untenable [2]. Unfortunately still infinite number of Lagrangians is allowed by this criterion and further viability conditions are needed to reduce the set of tenable gravity theories. Undoubtedly the existence of a properly defined Newtonian limit should a priori be such a criterion.

The textbook definition of the Newtonian limit of general relativity or an alternative gravity theory is that it is a static weak-field limit of gravitational interactions corresponding to slow-motion approximation for self-gravitating matter systems whose energy–momentum tensor is dominated by their energy density. Though intuitively clear, the definition is mathematically obscure and incomplete and particularly in the case of NLG theories it gives rise to some confusion. To make it clear why we claim in this work that a large number of metric gravity theories do have a Newtonian limit we first provide a very brief review of what is precisely known about this limit in the case of general relativity.

In order to express in mathematically rigorous terms the notion of Newtonian limit of general relativity (GR) it is necessary to first formulate Newton's gravity theory (NGT) in a four-dimensional spacetime formalism. Only within such a wide framework comprising both relativistic metric gravity theories (for the time being out of all these theories only GR is relevant for our considerations) and NGT proper, where the structure of all these theories has the same mathematical and physical interpretation, can one make a meaningful transition to Newton's theory and recognize for which relativistic theories the 'Newtonian limit' does exist and for which ones does not. The spacetime formulation of NGT is as follows.

The physical four-dimensional spacetime M is foliated by hypersurfaces S_t of simultaneity with respect to the absolute time t and the hypersurfaces are simply connected complete Euclidean spaces. The spacetime is endowed with a spatial metric¹ $s^{\alpha\beta}$ of rank 3 which defines the Euclidean metric equal to δ_{ik} on each S_t and a temporal metric $t_{\alpha\beta}$ of rank 1 measuring temporal intervals. The curvature tensor of M for the symmetric and metric (for both $s^{\alpha\beta}$ and $t_{\alpha\beta}$) connection satisfies the Einstein field equations (with the cosmological constant $\Lambda = 0$) for the matter source being the energy-momentum tensor for a perfect fluid [3]. Actually this theory is more general than NGT and is called Newton-Cartan theory. In order to get Newton's theory proper one must impose global and asymptotic conditions since Newton's gravity is a theory of isolated material systems and only for isolated systems it has been reliably confirmed. The isolatedness is expressed in two conditions:

- (i) on each S_t the support of the fluid energy-momentum tensor is compact;
- (ii) the spacetime is asymptotically spatially flat (two expressions quadratic in the Riemann tensor vanish at spatial infinity) [3, 4].

The first requirement means that we are always concerned with isolated systems and the rest of the world outside them is empty. The second condition implies that in this special case of Newton-Cartan theory there exists a distant parallelism of spatial vectors; in physical terms this means that the axes of neighbouring freely falling gyroscopes do not rotate with respect to each other. The two conditions explicitly exclude the cosmological constant from the field equations. The connection is uniquely determined by a scalar function and the field equations reduce to Poisson's equation for the function which is then interpreted as the gravitational Newtonian potential. The standard integral formula for the general solution of the equation valid in the case of a compact support of the mass density shows that the connection falls off as r^{-2} at the infinity.

¹ The Greek indices run from 0 to 3 and Latin ones from 1 to 3.

Having this formulation of NGT in hand one places it in a general framework of relativistic metric gravity theories. The framework is named Ehlers' frame theory or Cartan–Friedrichs formalism [3]. The starting point is GR whose laws are recasted in terms of an arbitrary parameter μ and Einstein's theory is recovered for $\mu = c^{-2}$, c the light velocity. The Lorentzian space-time metric $g_{\alpha\beta}$ gives rise to a temporal metric $t_{\alpha\beta}(\mu)$ and spatial one $s^{\alpha\beta}(\mu)$, the connection $\Gamma(\mu)$ is symmetric and metric. The field equations are Einstein ones for the connection and the matter source is any perfect fluid. The frame theory is meaningful for any $\mu \geq 0$. For $\mu > 0$ the metrics $t_{\alpha\beta}(\mu)$ and $s^{\alpha\beta}(\mu)$ are of rank 4 while for $\mu = 0$ they degenerate and the pair $(t_{\alpha\beta}, s^{\alpha\beta})$ forms the so-called Galilei metric. All the fields of the frame theory form a set $F(\mu) \equiv \{t_{\alpha\beta}(\mu), s^{\alpha\beta}(\mu), T^{\alpha\beta}(\mu), \dots\}$. If the fields belonging to $F(\mu)$ and their first derivatives converge pointwise to those forming the family $F(0)$, $\lim_{\mu \rightarrow 0} F(\mu) = F(0)$, then $F(0)$ is said to be a Cartan–Friedrichs limit of μ -rescaled solutions of GR. $F(0)$ represents laws and solutions of Newton–Cartan theory [3]. Again to recover Newton's theory proper the following global conditions are imposed in the frame theory:

- (i) the support of $T^{\alpha\beta}(\mu)$ is spatially compact;
- (ii) the spacetime $(M, t_{\alpha\beta}(\mu), s^{\alpha\beta}(\mu), \Gamma(\mu))$ is asymptotically spatially flat [3, 4].

Clearly condition (ii) ensures that there are no contributions to the gravitational field from any sources outside the isolated system. The tidal gravitational forces due to the system vanish at the infinity on each of appropriately chosen spacelike hypersurfaces which foliate the spacetime.

It is conjectured that if the fields $F(\mu)$ for $\mu > 0$ satisfy the conditions (i) and (ii) and the limit $\lim_{\mu \rightarrow 0} F(\mu) = F(0)$ does exist, then $F(0)$ represents the four-dimensional formulation of NGT. There is no general proof of the conjecture, only a number of specific solutions in GR confirm it [3].

The Ehlers' frame theory shows that the mathematical structure of NGT is a degenerate special case of that of GR. This degeneracy may also be expressed in other terms: the spacetime structure of GR is determined by a Lorentz bundle over M while that of Newton's theory is given by a Galilei bundle [5]. These principal fiber bundles are locally determined by their structure groups. The Galilei group is a contraction of the Lorentz group showing the degeneracy of the structure.

It should be emphasized once again that the Newton's theory of gravitation is reliable and the Newtonian limit of GR is mathematically well defined only for isolated matter systems, what requires asymptotic spatial flatness, otherwise in the limit $\mu \rightarrow 0$ one arrives at Newton–Cartan theory. For unbounded mass distribution the inertial frames do not exist and

the structure of NGT is broken. This is the case of cosmology and what is usually named ‘Newtonian cosmology’ is a theory which only in some aspects resembles NGT and has no evolution equations; evolution of a model follows from some symmetry assumptions and should be formulated within the frame theory [6], this means that Newtonian cosmology is not a self-contained theory.

Strictly speaking for $\Lambda \neq 0$ general relativity has no Newtonian limit. However this is a mathematical theorem while in physics one is usually satisfied with a plausible approximation. The problem is that of a scale. A physical system determines its own distance scale. In mathematics the infinity is unique whereas ‘physical infinity’ depends on the scale. For example, in quantum mechanics the wave function must be normalized to unity over the whole space implying that the function must sufficiently quickly fall off at infinity. The scale for the hydrogen atom is 10^{-8} cm and in practice its infinity is at a distance of few meters and the wave function is practically zero there and farther. If the spacetime has a nontrivial topology, is not asymptotically flat or there are event horizons and singularities, the rigorous formulation of quantum mechanics and quantum field theory encounters the well known difficulties, nevertheless for distances small in comparison to the characteristic scale of these features, the quantum phenomena are indistinguishable from those in Minkowski spacetime and in this sense standard quantum theory is approximately valid. The same holds for gravitational interactions. For the gravitational field of the Sun its practical infinity begins not far from the outer edge of the Solar system, *i.e.* at distance of one parsec. For the Milky Way it is even relatively closer, at the distance of order 100 Mpc, just outside the outer edge of the Local Supercluster, where the gravitational field is dominated by other clusters of galaxies. These scales are small when compared with the characteristic scale related to the cosmological constant. The observationally determined upper limit for Λ_0 is $|\Lambda_0| \leq 10^{-52} \text{ m}^{-2}$ and the characteristic length is at least 10^4 Mpc and is of the order of the Hubble radius c/H_0 , where H_0 is the present value of the Hubble constant; the dark energy density corresponding to the upper value of $|\Lambda_0|$ is $7 \times 10^{-30} \text{ g cm}^{-3}$. Thus both in the Solar system and the Milky Way one can safely put $\Lambda = 0$. These simple scale comparisons are confirmed by a detailed calculation: the cosmological constant is undetectable in the Solar system since the effect most sensitive to it, the perihelion shift of Mercury, requires $|\Lambda| \geq 10^{-41} \text{ m}^{-2}$ [7]. If the cosmological constant is sufficiently close to zero, small perturbations of de Sitter or anti-de Sitter spacetime may be fairly well approximated on macroscopic (*i.e.* smaller than cosmological) scales by a Newtonian perturbation of Minkowski spacetime.

The metric nonlinear gravity theories fit the frame theory but their main problem with possessing the Newtonian limit is that they are governed by various fourth order field equations while Ehlers' theory requires to this end the Einstein field equations. It might therefore seem that only few of them, with very specific field equations, would admit the Newtonian limit. However, although for a given NLG theory its field equations are of fourth order in the original Jordan frame and in many other frames², there is infinity of frames wherein the field equations are of second order. Contrary to a wide belief these theories are not inherently dynamically higher order ones. Among the latter frames there is one distinguished by the canonical form of its dynamics: it is Einstein frame. We employ this frame to show that any $L = f(R)$ gravity theory may be recasted in the form of GR plus a scalar field representing a nongeometric spin-zero component of the gravitational field; the scalar acts as a 'matter source' for the metric field in Einstein field equations. Then after 'switching off' this additional gravitational degree of freedom one may directly apply the frame theory to get in vacuum the Newtonian limit of the theory under consideration. The procedure makes sense if the theory is physically viable, *i.e.* if it is stable, what means that its ground vacuum state, being either Minkowski, de Sitter or anti-de Sitter spacetime, is dynamically stable. Our conclusion is: *if a given metric nonlinear gravity theory has a stable vacuum ground state, then it also has a Newtonian limit, either exactly in the sense of the frame theory (if the ground state is the flat spacetime) or approximately on a suitable distance scale (if the background is curved).* Stability implies the Newtonian limit. Hence the existence of the Newtonian limit cannot work as an independent criterion to establish which NLG theory fits the real world. There are still infinity of gravity theories which are in this sense viable and a distinct selection rule is necessary to reduce this collection. Such a rule is at present missing.

In this paper we investigate the Newtonian limit for $L = f(R)$ gravity theories, Sections 2 to 4. For more general Lagrangians a universal method is not available yet and only special cases have been studied; we briefly comment on them in section 5. Since the frame theory, being rigorous, is far from the physical intuition, for the sake of completeness we discuss in Appendix B the physical and geometrical obstacles preventing one from defining a Newtonian limit in general relativity for $\Lambda \neq 0$, that is as a small perturbation of de Sitter or anti-de Sitter spacetime.

We emphasize that our approach and results apply to purely metric gravity theories. If one studies $f(R)$ gravity in the purely affine or metric-affine framework (Palatini formalism) one may get a satisfactory Newtonian limit without invoking a ground state solution [8].

² By a 'frame' we always mean in this context a set of dynamical variables of the theory. These variables may be subject to arbitrary transformations and redefinitions, then the new variables form a new frame.

2. Stability of a ground state

We recall that both general relativity (GR) and all metric gravity theories differ from Lagrangian classical field theory (CFT) in Minkowski spacetime in that the relationship of the notion of energy and of the ground state is reverted. In CFT the notion of the ground state is based on the concept of energy, being the solution of the Lagrange equations of motion corresponding to the lowest energy state; the latter does exist (except peculiar cases as Liouville field theory) because the Hamiltonian is positive definite. This formal definition agrees with an intuitive picture of the ground state in which the field is absent or is covariantly constant and its symmetric energy-momentum tensor (derived by taking a formal metric variation of the Lagrangian) is either zero or Lorentz invariant.

As is well known in GR the Hamiltonian formalism is imperfect and in particular cannot be used for defining the ground state. GR is a geometrical theory and the ground state is defined in geometrical terms: it is the solution of the field equations possessing the maximal 10-parameter (in dimension four) isometry group, *i.e.* admitting 10 independent Killing vector fields. The solution is unique and depending on the value of the cosmological constant Λ it is Minkowski (\mathcal{M}), de Sitter (dS) or anti-de Sitter (AdS) space³. The same holds for $L = f(R)$ gravity theories, the only difference being that for these theories the cosmological constant is not a fundamental one appearing in L , as we shall see below it is merely the curvature scalar of the maximally symmetric solution. And once the primary notion, that of the ground state, has been identified in terms of the isometry group, the only meaningful notion of energy in GR, that of total energy (being effectively a charge) with respect to the ground state, may be introduced employing only Einstein's field equations and without any resort to the gravitational Hamiltonian. This is Arnowitt-Deser-Misner (ADM) energy in the case of $\Lambda = 0$ and Abbott-Deser (AD) energy for $\Lambda \neq 0$.

The fundamental assumption underlying the very notion of the Newtonian limit of any relativistic theory of gravity is that the ground state of the theory is stable. Otherwise any small time dependent perturbation of the Newtonian interaction (which is a specific weak-field solution of the relativistic theory) will unboundedly diverge quickly destroying this interaction. In short: stability of the background is a necessary condition for both viability of the theory and existence of the Newtonian limit.

The first step in the search for the Newtonian limit of a gravity theory consists in determining the ground state of the theory. For an NLG theory with $L = f(R)$ the field equations take on the form (in vacuum)

³ By anti-de Sitter space we always mean the covering AdS space without closed time-like curves and with topology \mathbf{R}^4 .

$$E_{\mu\nu}(g) \equiv f'(R)R_{\mu\nu} + \frac{1}{6}g_{\mu\nu}[f(R) - 2Rf'(R)] - f'''(R)R_{;\mu}R_{;\nu} - f''(R)R_{;\mu\nu} = 0, \quad (1)$$

here $f' \equiv \frac{df}{dR}$ and we have employed that the trace $E_{\mu\nu}g^{\mu\nu} = 0$ gives rise to the equation for the scalar R ,

$$f''(R)\square R + f'''(R)R_{;\alpha}R^{;\alpha} + \frac{1}{3}[Rf'(R) - 2f(R)] = 0, \quad (2)$$

where $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. The ground state spacetime should be maximally symmetric, *i.e.* Minkowski, de Sitter or anti-de Sitter space [2,9]. This state exists if and only if the field equations admit Einstein spaces, $R_{\mu\nu} = \frac{1}{4}\lambda g_{\mu\nu}$ with $R = \lambda = \text{const}$, as a special class of solutions. From (1) or (2) one finds that the curvature scalar λ satisfies the algebraic equation [2,9,10]

$$\lambda f'(\lambda) - 2f(\lambda) = 0. \quad (3)$$

In general Eq. (3) has multiple solutions giving rise to multiple vacua [11]. We exclude from considerations the degenerate Lagrangians for which any value of λ is a solution of (3) (continuous spectrum, $L = R^2$), the only solutions are infinite ($L = 1/R$) or the equation has no solutions at all [2]. Thus Eq. (3) has at least one and at most countable number of finite solutions. Each ground state defines a separate dynamical sector of the theory, *i.e.* a given Lagrangian corresponds to a number of distinct dynamical sectors, each sector being actually a distinct gravity theory (for examples *cf.* [2]). Classically there are no transitions between different sectors for the same Lagrangian; may be distinct vacua are related via quantum tunnelling processes.

Any root of this equation may be interpreted in a restricted sense as a cosmological constant of the theory, $\Lambda \equiv \lambda/4$. In fact, a small perturbation of the corresponding ground state has R close to λ as is the case of general relativity where the ground state curvature is $R = 4\Lambda$. It is therefore necessary to make a comment on the notion of the cosmological constant. In general relativity Λ is both the constant appearing in the Einstein–Hilbert Lagrangian, $\Lambda = -\frac{1}{2}L(0)$, hence it explicitly appears in the fields equations and in all solutions to them, and the curvature of the unique maximally symmetric ground state, $\Lambda = \lambda/4$. Yet in metric NLG theories this notion has a very limited sense. If $f(0)$ is infinite, as is in most Lagrangians employed in cosmological applications, the definition is meaningless. If $f(0) \neq 0$ is finite one may define Λ as $-\frac{1}{2}f(0)$, however this quantity does not appear in the field equations and influences the solutions only in an implicit way via dimensional constants which are unavoidable to ensure the correct dimensionality of the Lagrangian; *e.g.* for $f(R) = \frac{1}{a}e^{aR}$ with $a > 0$ the unique

ground state solution is dS space with $\lambda = 2/a$. Clearly for $f(0) = 0$ this definition gives $\Lambda = 0$ while in general besides Minkowski space ground state ($\lambda = 0$) there are other ground states with $\lambda \neq 0$. For example, for $f(R) = R + aR^2 + \alpha^{-2}R^3$, $\alpha > 0$, there are three ground state solutions with $\lambda_1 = 0$, $\lambda_2 = \alpha$ and $\lambda_3 = -\alpha$, defining three separate dynamical sectors of the theory. Alternatively, Λ may be defined as $\lambda/4$ for each ground state, then it has different values in different sectors of the theory. In what follows we shall always use the notion of Λ only in the sense of the curvature of the (stable) maximally symmetric ground state. It is relevant in that solutions to the field equations may asymptotically tend to the ground state with $R = 4\Lambda$.

A given solution to (3) is a genuine ground state of a gravity theory if it is stable in this theory against purely gravitational excitations (no matter). In the presence of some kind of matter the candidate ground state may be stable or not. If some species of matter causes instability, this is either an indication that this species is merely unphysical or that producing it would be unreasonable and dangerous. Quantum massless fields make both Minkowski and de Sitter space unstable [12] in general relativity and this outcome is not regarded as an argument against validity of Einstein's theory; general relativity may be challenged only on completely different grounds. What is relevant is the stability of pure gravity theory.

In general relativity the flat spacetime \mathcal{M} is globally dynamically stable [13] and de Sitter space ($\lambda = 4\Lambda > 0$) is globally nonlinearly stable too [14]; the case of anti-de Sitter space is distinct and we comment on it in Appendix B.

In an NLG theory with arbitrary $L = f(R)$ one may investigate the stability employing the remarkable fact that general relativity plus a minimally coupled scalar field is a universal Hamiltonian image of any such gravity theory under a suitable Legendre map [1, 2, 15, 16]. One may therefore apply the methods developed in general relativity. The classical method is based on positivity of total ADM energy for both gravitational field and a matter source. The energy is positive provided the energy-momentum tensor for the matter source satisfies the dominant energy condition (DEC) and the latter holds if the interaction potential is nonnegative and attains minimum at the ground state under consideration. Thus investigation of extrema of the potential for the scalar field becomes an effective method for studying the stability in NLG theories.

It should be noted that from the rigorous mathematical approach viewpoint the classical method of proving stability based on the positivity of energy, is of rather little reliability [17]. In proving the dynamical stability (of evolution, meaning that there are no unboundedly growing modes) only the exact field equations are relevant. However in the few cases in the rigorous approach where matter sources are present, DEC does hold. It is

therefore reasonable to conjecture that \mathcal{M} , dS and AdS are globally nonlinearly stable only if any self-gravitating matter (in the present case the scalar component of gravity) does satisfy the condition.

In Jordan frame (JF) the $L = f(R)$ gravity is described by the field $g_{\mu\nu}$ which is a kind of a unifying field mixing the pure spacetime metric (still equal to $g_{\mu\nu}$) and a spin-0 component of gravity since the unifying field carries 3 degrees of freedom. The field is decomposed into the components carrying definite masses and spins in Einstein frame (EF); in the latter frame it is a doublet $\text{EF} = \{\tilde{g}_{\mu\nu}, \phi\}$. The transformation from JF to EF is a Legendre map being in this case a conformal rescaling of the original metric [2, 15, 16]. The scalar component of gravity is defined as $p \equiv \frac{df}{dR}$, then the definition is inverted to give R as a function of the canonical momentum p , $R(g) = r(p)$, *i.e.*,

$$f'(R)|_{R=r(p)} \equiv p.$$

For convenience the scalar is redefined as

$$p \equiv \exp\left(\sqrt{\frac{2}{3}}\kappa\phi\right),$$

where $\kappa^2 = 8\pi G/c^4$ and the Einstein frame metric is $\tilde{g}_{\mu\nu} \equiv pg_{\mu\nu}$. The fourth order field equations (1)–(2) in JF are equivalent in EF to $\tilde{G}_{\mu\nu}(\tilde{g}) = \kappa^2 T_{\mu\nu}(\phi, \tilde{g})$ for a minimally coupled scalar field with a self-interaction potential

$$V(p(\phi)) = \frac{1}{2\kappa^2} \left[\frac{r(p)}{p} - \frac{f(r(p))}{p^2} \right] \quad (4)$$

and the equation of motion

$$\tilde{\square}\phi = \frac{dV}{d\phi} = \sqrt{\frac{2}{3}}\kappa p \frac{dV}{dp}. \quad (5)$$

The Legendre transformation to EF should exist at least in a neighbourhood of a ground state solution with $R = \lambda$; it occurs iff $f'(\lambda) \neq 0$ and $f''(\lambda) \neq 0$. Without loss of generality we assume $f'(\lambda) > 0$ to preserve the metric signature. The two conditions additionally restrict the class of allowable Lagrangians [2].

The following proposition holds [2]: *If*

$$\frac{1}{f''(\lambda)} - \frac{\lambda}{f'(\lambda)} > 0$$

*the maximally symmetric solution of the theory for $R = \lambda$ is stable against gravitational (i.e. metric and the scalar field) perturbations*⁴.

⁴ An almost equivalent stability criterion based on linear perturbation theory of RW spacetimes in a fourth-order theory in Jordan frame has been given in [18]. The only difference is that the strong inequality in this formula is replaced by a weak one there.

Here some comments are in order.

1. The proposition is explicitly formulated in EF and states the stability of the ground state in the framework of general relativity. The inverse Legendre map is simply $g_{\mu\nu} = \frac{1}{f'(\lambda)} \tilde{g}_{\mu\nu}$, hence Minkowski, dS and AdS spaces in EF are mapped onto \mathcal{M} , dS and AdS spaces in JF respectively, preserving the sign of the curvature scalar. Since it is assumed that the Legendre map is regular in a neighbourhood of the ground state in EF, the corresponding ground state solution in JF is stable as well. We stress this mathematically obvious fact since there were some suggestions in the literature that the stability might be spoiled under transformation between different frames.
2. The proposition is of mathematical nature and its validity is independent of the issue of which frame is physical. While considering $f(R)$ gravity one is usually interested in physics in Jordan frame, but in this frame the stability problem is hard. Yet Einstein frame, which is mathematically (though not physically) equivalent to JF, allows to solve the problem in a neat and general way.
3. Long ago a paper by Pechlaner and Sexl [19] made impression that fourth order equations of motion generate instabilities which are revealed whenever a small amount of matter is present [9]. Actually the higher order terms in an equation only signal the presence of additional field degrees of freedom; in the present case this is the scalar component of gravity. It is the precise form of the full Lagrangian (in JF) rather than the mere presence of higher derivatives in the field equations that determines whether the ground state is stable or not.
4. Here stability means the dynamical stability (linear or exact) of the ground state solution of a theory. Yet in the view of the obvious cosmological applications, most research in $f(R)$ gravity have up to now been focused on stability of RW spacetimes or other cosmological models using either the phase space method or a minisuperspace approach (in both the cases the perturbations are spatially homogeneous) [20] or a linear theory of inhomogeneous perturbations [18, 21]. However a cosmological solution only exceptionally coincides with the ground state one⁵ and we stress that it is the stability of the latter that is relevant for the physical viability of the theory and for possible existence

⁵ Recall that a RW spacetime reduces to the flat one only for the flat or open spatial sections and then only provided that the cosmic scale factor is constant or a linear function of the cosmic time respectively; these trivial cases are not studied in these works. Anti-de Sitter space cannot at all be expressed in terms of the RW metric, therefore these three approaches do not apply to it.

of a Newtonian limit. Investigating solely the cosmological solutions is in a sense misleading: as a number of specific examples show, a stable cosmological solution may have interesting features [21] while the ground state is unstable making the underlying theory unphysical [2]. The approach based on positivity of total energy (in both the frames!) or more precisely, on DEC for the scalar field, is universal and in this sense is superior to the other methods.

3. Gravitational vacuum in Einstein frame and the Newtonian limit

Assume that a given Lagrangian $L = f(R)$ admits n different solutions of the ground state equation (3) λ_i , $i = 1, \dots, n$, and each of the corresponding maximally symmetric spacetimes with $R = \lambda_i$ is stable; this means that the Lagrangian describes n physically distinct gravitational sectors of the theory. Consider the field equations in Einstein frame. These are $\tilde{G}_{\mu\nu}(\tilde{g}) = \kappa^2 T_{\mu\nu}(\phi, \tilde{g})$ and the nonlinear wave equation (5) for ϕ . Clearly an arbitrary solution to these equations cannot have the Newtonian limit. In the absence of ordinary matter the scalar gravity acts as a specific matter source for the metric field $\tilde{g}_{\mu\nu}$ and any solution contains contributions from the scalar which are also present in the weak-field limit and perturb the Newtonian interaction. Therefore in the search for a Newtonian limit one needs to study “scalar gravity vacuum” solutions where the spin-0 component of gravity is ‘switched off’, *i.e.* $\phi = \text{const}$. This may occur only for a stationary point of the potential $V(p(\phi))$ in Eq. (5). From the form (4) of the potential one easily finds that $dV/dp = 0$ implies $\frac{2}{p}f(r(p)) - r(p) = 0$. Recalling that $p = f'(r)$ one gets that all stationary points are determined by

$$2f(r(p)) - r(p)f'(r) = 0$$

and this equation viewed as an equation for $r(p)$ coincides with Eq. (3). Hence $\phi = \text{const}$ only for $r(p) \equiv r(p_i) = \lambda_i$ and conversely $p_i = p(r_i) = p(\lambda_i) = f'(\lambda_i)$. The scalar field is in its ground state either in the ground state of the entire gravitational doublet (the spacetime is \mathcal{M} , dS or AdS in both JF and EF, depending on the sign of λ_i) or in a spacetime which in Jordan frame has the same curvature scalar $R = \lambda_i$ as the ground state of the given sector. Since by assumption $f'(\lambda_i) > 0$, in general the scalar gravity does not vanish in its ground state,

$$\phi_i = \sqrt{\frac{3}{2}} \frac{1}{\kappa} \ln f'(\lambda_i).$$

The energy-momentum tensor for ϕ reduces to its potential part, $T_{\mu\nu}(\phi_i, \tilde{g}) = -\tilde{g}_{\mu\nu}V(p_i)$. Also the potential does not vanish and from (4) one gets

$$V(p_i) = \frac{1}{2\kappa^2} \left[\frac{\lambda_i}{f'(\lambda_i)} - \frac{f(\lambda_i)}{(f'(\lambda_i))^2} \right]$$

and applying Eq. (3) for $\lambda = \lambda_i$ and following from it relation $f(\lambda_i)/f'(\lambda_i) = \lambda_i/2$ one finally arrives at

$$V(p_i) = \frac{\lambda_i}{4\kappa^2 f'(\lambda_i)}. \quad (6)$$

The Einstein field equations for $\tilde{g}_{\mu\nu}$ may be written in the case $\phi = \phi_i$ as

$$\tilde{G}_{\mu\nu}(\tilde{g}) + \Lambda_i \tilde{g}_{\mu\nu} = 0, \quad (7)$$

where

$$\Lambda_i \equiv \frac{\lambda_i}{4f'(\lambda_i)} \quad (8)$$

is interpreted as a cosmological constant in the given sector of the theory for this class of solutions. One sees here another difference between the two frames: while in JF the cosmological constant refers only to the curvature of the ground state, in EF it also appears in the field equations. (This constant may be singled out in the field equations in general, *i.e.* when the scalar field is present.)

Clearly the gravity theory (7) has the correct Newtonian limit (exact or approximate). Coming back to Jordan frame one simply rescales the metric by the constant factor, $g_{\mu\nu} = (f'(\lambda_i))^{-1} \tilde{g}_{\mu\nu}$, and the gravitational interaction takes on the same Newtonian form in this frame too.

4. Minkowski space as the ground state

Finally we make some remarks about those $L = f(R)$ gravity theories which have the flat spacetime as the stable ground state solution⁶. For $\lambda = 0$ Eq. (3) implies $f(0) = 0$ excluding a constant from the Lagrangian. Assuming analyticity⁷ and normalizing $f'(0)$ to 1 one has⁸

$$L = f(R) = R + aR^2 + \sum_{n=3}^{\infty} c_n R^n \quad (9)$$

⁶ More precisely, we now consider the $\lambda = 0$ sector of a given theory.

⁷ Actually it is necessary to assume that $f(R)$ is of C^3 class at $R = 0$.

⁸ We use all the conventions of the book [22].

with $a \neq 0$. The existence of the first two terms in the expansion is essential for the equivalence of Jordan and Einstein frames. The conformal factor, *i.e.* the scalar component of gravity, is

$$p = 1 + 2aR + O(R^2)$$

and hence is positive in a neighbourhood of the ground state⁹. The stability condition reduces now to $a > 0$ [16]. The “scalar gravity vacuum” simplifies to $\phi = 0$ or $p = 1$ and implies $T_{\mu\nu} = 0$. Then the theory becomes identical to vacuum general relativity and applying Ehlers’ frame theory to appropriate solutions to $\tilde{G}_{\mu\nu} = 0$ (*e.g.* Schwarzschild one) one gets the desired correct Newtonian limit of the theory. By taking the inverse conformal mapping one finds $g_{\mu\nu} = \tilde{g}_{\mu\nu}$ in JF and thus $G_{\mu\nu}(g) = 0$ and the same solutions give rise in the weak-field limit to the Newtonian interaction. In summary, any theory of the form (9) and $a > 0$ has the flat spacetime as a stable ground state solution and exactly the Newtonian limit interaction described by a potential U satisfying $\Delta U = 0$.

It is difficult to derive this theorem working solely in Jordan frame. Firstly, there is the problem of proving stability of the ground state. Secondly, when one finds out the stability criterion $f''(0) > 0$ employing perturbation theory as in [18], there remains to decouple the massive scalar field contribution from the pure massless gravitation, both contained and mixed in the unifying field $g_{\mu\nu}$. (Recall that the Newtonian interaction is a far distance force and the very presence of the scalar gravity will distort it.) It is impossible to decouple the scalar directly from the field equations (1)–(2), one can only identify its contribution to specific solutions.

Exact solutions (in any frame) for the analytic Lagrangians are not known¹⁰. Of course, the Schwarzschild metric is always a solution and in the sense of Ehlers’ frame theory it guarantees the existence of the Newtonian limit, but the physical interpretation of the metric (absence of the scalar gravity) cannot be recognized on the level of the fourth order equations, *i.e.* in Jordan frame. In the linear approximation a SSS solution for $L = R + aR^2$ was found long ago by Pechlaner and Sexl [19] and Stelle [28]. The general solution to the field equations (1) and (2) for the metric components g_{00} and g_{11} depends on 3 parameters (before imposition of a boundary condition) and for $a > 0$ is of the form (up to signs)

⁹ We notice in passing that, contrary to what is frequently met in the current literature on $f(R)$ cosmology, the condition $R \approx 0$ does not necessarily imply that the gravitational field is weak; as a matter of fact this occurs mainly in the cosmological setting of general relativity. In general one may have $R = 0$ for arbitrarily strong gravity.

¹⁰ Few static spherically symmetric (SSS) solutions different from Schwarzschild’s one are known in non-analytic cases: for L containing \sqrt{R} term [23] and for $L = R^s$, s real [23–25]. If one assumes $R = 0$ the unique SSS solution is Schwarzschild’s metric [26]. It is worth noting that a SSS solution different from Schwarzschild’s one can be found for L being a non-analytic function of the Weyl tensor [27].

$$1 + \frac{c_1}{r} + \frac{c_2}{r}e^{-mr} + \frac{c_3}{r}e^{mr},$$

the sum of the Newtonian and Yukawa potentials. $m = (6a)^{-1/2}$ is easily identified in EF as the mass of the scalar gravity. The analogous result was recently found in the linear approximation for the general Lagrangian (9) [29]. Thus if the scalar degree of freedom is switched off ($c_2 = c_3 = 0$) the Newtonian interaction is a weak-field limit of any SSS solution for each analytic Lagrangian. For $a < 0$ the approximate solution is complex in the Yukawa terms (the mass is imaginary) and its physical interpretation given in [29] is rather obscure or, if only its real part is taken into account, it quickly oscillates [19]. Actually this behaviour just signals the instability (the corresponding time-dependent modes are divergent) which is immediately recognized in Einstein frame.

At first sight the theorem that the stability implies the Newtonian limit is perhaps a little surprising. One might a priori imagine gravity theories whose ground state is stable under, say, radiative mode transmission while they do not admit the Newtonian interaction. This may occur for some specific theories while for $L = f(R)$ Lagrangians it is impossible. As already mentioned in Section 2, this class of gravity theories is distinguished in the entire space of possible relativistic theories of gravitation by the fact that they can be Legendre transformed into GR plus the scalar field. As long as the ordinary matter is not included, these theories merely represent general relativity in disguise.

In summary, those $L = f(R)$ theories where flat spacetime is unstable, are rejected as unphysical and those for which this spacetime is stable contain as a subclass of solutions all the solutions of vacuum general relativity (both in Jordan and Einstein frames). This subclass (and only this one) contains solutions which subject to the specific mathematical procedure give rise to rigorous Newton's gravity theory.

5. More general Lagrangians

The Legendre transformation from Jordan to Einstein frame works for all Lagrangians $L = f(g_{\mu\nu}, R, R_{\alpha\beta})$ (no dependence on the Weyl tensor) and whose Hessian with respect to $R_{\alpha\beta}$ does not vanish [15]. In EF the unifying field $g_{\mu\nu}$ is decomposed into a metric $\tilde{g}_{\mu\nu}$ (which now is not conformally related to $g_{\mu\nu}$), a massive scalar field χ and a massive spin-two field $\phi_{\mu\nu}$ actually being a “ghost” (what is not so disastrous as it might seem, *cf.* [30]). The indefiniteness of the energy-momentum tensor for $\phi_{\mu\nu}$ causes that the general-relativistic method of studying stability of the ground state (based on DEC) does not work. One can only study various Lagrangians case

by case. There are arguments that the most physically interesting case corresponds to the simplest regular Lagrangian in this class,

$$L = R + \frac{1}{3m^2}(R^2 - 3R_{\mu\nu}R^{\mu\nu}). \quad (10)$$

The coefficients are so chosen that the scalar gravity vanishes and the gravitational field is a doublet consisting of two spin-2 fields carrying together seven degrees of freedom [31]; m is the mass of the non-metric component of gravity ($\phi_{\mu\nu}$). The ground state in both the frames is Minkowski spacetime (in EF it is supplemented by $\phi_{\mu\nu} = 0$) and is linearly stable [31], hence the ghost-like nature of the massive gravity does not result in instability. Metrics satisfying $R_{\mu\nu} = 0$ are always solutions to (10) and this is sufficient to conclude that the theory has the correct Newtonian limit. (No exact SSS solutions different from Schwarzschild metric are known.)

Finally we comment on theories explicitly depending on the Weyl tensor. In this case Einstein frame does not exist [32] and all frames obtained via various Legendre transformations from the original Jordan frame give rise to fourth-order equations of motion what makes investigations of these theories rather hard. In the special case of

$$L = R + \sqrt{3}a|C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}|^{1/2}, \quad (11)$$

$a < 1/4$ or $a > 1$, the Lagrangian is a homogeneous function of order 1 of the Riemann tensor, as in general relativity. For this theory all exact SSS solutions have been found [27] and they do not include Schwarzschild metric. A preliminary calculation shows that the maximally symmetric spaces, *i.e.* flat, dS and AdS spaces are not solutions too (though it needs a deeper investigation) and it is unclear whether a ground state may at all be defined. If it cannot it would be a clear indication that the Lagrangian (11) is unphysical.

6. Conclusions

As regards the existence of the Newtonian limit the $L = f(R)$ gravity theories do not differ from general relativity. If their ground state solution, being Minkowski, de Sitter or anti-de Sitter spacetime, is dynamically stable (as is the case of general relativity with arbitrary cosmological constant), then the Newtonian limit does exist. The limit is either rigorously defined in the case $\Lambda = 0$, *i.e.* for the flat ground state solution, or approximate otherwise, the approximation is valid for length scales small compared to the cosmological scale being of the order of $|\Lambda|^{-1/2}$. Stability of the ground state is a necessary and sufficient condition for the Newtonian force to exist, at least on macroscopic scales. This unexpected theorem is due to the fact

that in vacuum this class of gravity theories is dynamically equivalent to general relativity (plus a scalar field).

For more general Lagrangians depending on Ricci and Weyl tensors the situation is more complex. Stability of a ground state spacetime remains the necessary condition for the Newtonian limit to exist also in this case but now there is no universal, effective and simple method for checking the stability and one must resort to perturbation theory. Next, one must show that the fourth-order field equations (in the case of Weyl tensor) of the theory admit exact solutions having the appropriate Newtonian limit.

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Appendix A

Problems with solutions in general relativity possessing the exact Newtonian limit

In the two appendices we discuss some problems with obtaining the Newtonian limit in general relativity for both $\Lambda = 0$ and $\Lambda \neq 0$. Most of the material is by no means new, but for the sake of conceptual completeness and the reader's convenience we present it here. In this appendix we deal with the case $\Lambda = 0$, then there exist exact relativistic solutions giving rise to the rigorous Newtonian limit.

In Ehlers' frame theory the Newtonian limit is defined as a limit for particular classes of solutions (families of spacetimes) of GR depending on some free parameters. As the specific examples in the third reference in [3] show, the method of the frame theory works properly under two assumptions. Firstly, a foliation of the spacetimes by spacelike hypersurfaces must be chosen in such a way that after performing their linearization the linearized solutions become perturbations of the ground state (flat Minkowski spacetime) and the time coordinate labelling the foliation becomes the time coordinate in the global inertial reference frame being the (almost) proper frame for the matter source. Geometrically the latter feature means that the foliating hypersurfaces flatten upon the linearization. This time coordinate of the proper inertial frame in Minkowski spacetime is then identified with the absolute time in Galilei spacetime. Secondly, as the case of the FLRW cosmological spacetime shows [3], a coordinate system compatible with the foliation should be carefully chosen, otherwise the derived Newtonian limit of the relativistic spacetimes does not resemble at all the ordinary Newton's gravity. In fact, in the standard (Lagrangian) Friedmann coordinates

in the Newtonian limit the gravitational field strength is space independent (and time dependent) and the “Newtonian scalar potential” does not exist. (Physically this is reasonable showing that the infinite homogeneous distribution of matter, static or evolving, is beyond the scope of Newton’s gravity theory.)

The first condition is crucial. For example, for Schwarzschild spacetime expressed in the standard coordinates (t, r, θ, ϕ) where t is the parameter on the trajectories of the time-like Killing vector $\partial/\partial t$, one gets in the Newtonian limit the gravitational acceleration $1/r^2$, while after a coordinate transformation

$$t = T \cosh R, \quad r = T \sinh R,$$

the 3-spaces become hyperboloids with the variable curvature scalar. If one takes the Newtonian limit in these coordinates following the Ehlers’ prescription, the temporal and spatial metrics describing Galilei spacetime do not acquire their proper forms and the two conditions (given in [3]) for convergence of a relativistic class of spacetimes to Newtonian gravity are not satisfied; actually the Newtonian limit for this foliation does not exist. This result is easily understood: Schwarzschild spacetime in (T, R) coordinates reduces in the limit of vanishing mass to flat spacetime foliated with the hyperboloids of constant negative curvature (Lobatchevski spaces) and this is a fully relativistic description of the spacetime. Yet Schwarzschild spacetime in Painlevé–Gullstrand coordinates is foliated by flat spaces and the metric is time independent and in the limit of vanishing mass this metric reduces to that in an inertial frame in flat spacetime (*i.e.* the foliating spaces become hyperplanes); for this metric the Ehlers’ method works well giving rise to the correct Newtonian limit. It is then essential to properly identify the appropriate foliation of the spacetime.

Furthermore, there are mathematical subtleties causing that in general a linearized form of an exact solution written in arbitrary parameterization need not be a solution to the linearized field equations. In fact, in general relativity (for $\Lambda = 0$) one linearizes the Einstein field equations around the flat spacetime writing in Cartesian coordinates $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and assuming that both $|h_{\mu\nu}| \ll 1$ and the derivatives $|h_{\mu\nu,\alpha}| \sim |h_{\mu\nu,\alpha\beta}| \sim |h_{\mu\nu}|$. Then in the harmonic gauge for $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha$ one gets the equations

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{A.1}$$

with appropriately linearized $T_{\mu\nu}$. This means that one restricts the class of allowable solutions to those satisfying these conditions. This is the case of radiation fields (plane waves). Yet a linearized exact solution is merely of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and no restrictions on the derivatives. For example, if $h_{\mu\nu} = O(\frac{1}{r})$ then $|h_{\mu\nu,\alpha}| = O(h^2)$ *etc.* and a

weak-field approximation may not be a solution of (A.1). This occurs for the linearized Schwarzschild solution in the standard coordinates,

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 + \frac{2M}{r} \right) dr^2 + r^2 d\Omega^2,$$

for $M/r \ll 1$; the linearized Einstein tensor (with no gauge imposed) does not vanish, $G_{00}^L(h) = 12M/r^3$. On the other hand, once a differential equation has been generated it forgets the conditions under which it was derived and the space of solutions is determined solely by its form. Thus Eq. (A.1) in vacuum has no static solutions which are globally bounded and the conditions for the derivatives cannot hold. Yet for the linearized Schwarzschild metric expressed in the isotropic coordinates,

$$ds^2 = - \left(1 - \frac{2M}{\bar{r}} \right) dt^2 + \left(1 + \frac{2M}{\bar{r}} \right) (d\bar{r}^2 + \bar{r}^2 d\Omega^2),$$

the perturbations satisfy the harmonic gauge condition and are a solution to Eq. (A.1), clearly they give rise to the Newtonian acceleration $1/\bar{r}^2$.

Appendix B

The issue of a Newtonian limit in de Sitter and AdS spaces

Here we discuss physical and geometrical arguments showing that no Newtonian limit does exist in the rigorous sense in de Sitter or anti-de Sitter space: the Newtonian field may be defined only ‘locally’, *i.e.* on length scales much smaller than $|A|^{-1/2}$, it cannot fill the entire spacetime. As it was discussed in Appendix A the heart of the problem lies in geometry the ground state spacetime, whether it admits a foliation and a coordinate system giving rise to a structure which is close to Galilei spacetime.

De Sitter space is globally dynamically stable in general relativity. Yet its geometrical structure does not allow to define the Newtonian limit in the framework of Ehlers’ frame theory. In fact, in the literature there are known eleven families of coordinate systems exhibiting various features of dS geometry and these can be divided into three groups corresponding to three distinct foliations of the spacetime [33].

- (i) Standard cosmological coordinates. The spacetime is sliced with space-like 3-spheres S^3 which are $O(4)$ invariant. The coordinate system is global (covers the entire manifold) and the spaces are almost exponentially expanding (or contracting) in the proper time of the observers at rest. The metric exhibits an everywhere time-like conformal Killing vector. A foliation by 3-spheres can also be done in terms of static

coordinates (the metric is time-independent), which cover only a half of dS (the spaces consist of two hemispheres). The static coordinates make explicit the hypersurface orthogonal Killing vector which is time-like only within the cosmological event horizon.

- (ii) The flat cosmological coordinates which are global if the conformal time is employed. The spaces are just flat euclidean spaces which are $E(3)$ invariant. The spatial metric is conformal-time dependent and this time coordinate defines a conformal time-like Killing field. Another slicing by flat hyperplanes may be introduced in a region of dS manifold using conformally Minkowski coordinates. The metric is also time-dependent and no (conformal) Killing vector is generated by this time coordinate.
- (iii) The open (hyperbolic) coordinates. The foliating spacelike hypersurfaces are isometric to the homogeneous Lobatchevski space H^3 of constant negative curvature which is $O(1, 3)$ invariant. The coordinates cover only a half of dS. The hyperboloids H^3 expand or contract almost exponentially in the time variable which generates a conformal time-like Killing field.

None of these foliations is in general superior to the others and none of them is compatible with Galilei spacetime and thus it is clear that the frame theory cannot provide a satisfactory notion of Newtonian limit for de Sitter space.

The case of anti-de Sitter space is distinct. This manifold has topology \mathbf{R}^4 and is globally static and though is not globally hyperbolic, at first sight it should be more likely to have a structure close to Galilei spacetime. Also this manifold does not admit a foliation by static spacelike hyperplanes, but the main difficulty lies in properties of motion of test particles and metric perturbations. Contrary to the case of dS spacetime, AdS does admit a natural global static decomposition into space and time,

$$ds^2 = a^2[-\cosh^2 r dt^2 + dr^2 + \sinh^2 r(d\theta^2 + \sin^2 \theta d\varphi^2)], \quad (\text{B.1})$$

(the cosmological constant is $\Lambda = -3/a^2$) where the Lobatchevski hyperboloids H^3 given by $t = \text{const}$ are orthogonal to the globally time-like Killing vector $\partial/\partial t$ and the radial coordinate r is distinguished among many radial variables in H^3 by the feature that it directly measures the distance along spatial radial geodesic lines, $s = ar$. However the hyperboloids, being spaces of constant curvature, do not flatten under the linearization ($r \rightarrow 0$ or $r \rightarrow \infty$). Further, test particles behave in a rather bizarre way in this spacetime. A particle with an initial position $r = r_0 > 0$ (there is a coordinate singularity at the centre $r = 0$ while H^3 is a homogeneous space)

cannot escape to the spatial infinity $r = \infty$ and it cannot remain at rest even if its initial three-velocity is zero. The particle follows a radial geodesic and falls towards the centre and then recedes farther in the opposite direction until reaches the point $r = r_0$ at the distance

$$L = 2a \ln \left(Ea + \sqrt{(Ea)^2 - 1} \right),$$

where E is the integral of energy for the geodesic line subject to $\cosh r_0 = Ea > 1$. Then the particle falls down back and returns to the starting point. In other terms the particle oscillates between the opposite points at $r = r_0$ like a pendulum. The period of these oscillations is universal¹¹ (*i.e.* is independent of r_0) and is $2\pi a$ in the proper time s and 2π in the coordinate time t . The three-velocity of the particle has modulus equal to (in units $c = 1$)

$$[1 - (Ea)^{-2} \cosh^2 r]^{1/2}$$

and not far from the centre the velocity becomes relativistic. The fundamental reason for these bizarre features of test particle motion in the covering AdS space is that no analogue of the Hopf–Rinow theorem for Riemannian manifolds exists for Lorentzian spacetimes and that AdS space is not globally hyperbolic [34]. Hence for two test particles, each performing this kind of motion, it is very hard to define the Newtonian interaction.

This is, however, not the end of the story. AdS space is globally linearization stable [35] and nonlinearly stable for finite time [36] (at present it is only believed that it is globally nonlinearly stable). Yet if a spacetime is weakly asymptotic to the exact AdS space to the infinite past and future, then it is globally isometric to the exact AdS spacetime [37]. A regular (*i.e.* no singularities) perturbation of AdS remains close to it for long time (or possibly globally) but cannot tend to this spacetime at the infinity. Perturbations in AdS neither disappear at the infinity nor tend to a stationary perturbation in a far future, they are for ever traveling through the background. This is in marked contrast to the $\Lambda = 0$ case where small global perturbations of Minkowski space disperse in time and asymptotically tend to this spacetime. In flat spacetime the Newtonian interaction of a system of massive bodies can be unambiguously defined because if in a distant region of space a gravitational perturbation arises, it passes through the system in a finite time interval and then fades away at the infinity. Yet in AdS space the perturbation will be present for ever and inextricably disturb interactions between the bodies.

The corollary is that dS space due to its geometrical structure and AdS space due to both its geometrical structure, test particle motion and the behaviour of the gravitational perturbations (in particular the non-existence of

¹¹ This periodicity in the covering AdS is a residual effect of the time periodicity of the original AdS space containing closed time-like curves.

stationary excitations vanishing at the infinity), do not admit the Newtonian limit in the strict sense of the notion in the framework of general relativity with $\Lambda \neq 0$.

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