CONTINUUM REDUCTION IN LARGE N GAUGE THEORIES*

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These are notes associated with three lectures given at the XLIX Cracow School of Theoretical physics where a pedagogical explanation of the Gross–Witten transition, Eguchi–Kawai reduction and continuum reduction were given, followed by a description of the numerical computation of fermionic observables in the 't Hooft limit of large N gauge theory.

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1. Introduction

It has been a long held hope [1, 2] that the large N limit of QCD is simpler than QCD with $N = 3$ but an analytical solution in $d = 4$ is still to come. A simplification specific to $d = 2$ resulted in an analytical solution to the large N limit of QCD in $d = 2$ [3].

Relatively recently, one has been able to show that continuum reduction [4] holds in $d = 4$ and this has been used to make progress in a numerical solution of large N QCD. In order to understand continuum reduction, it will be useful to understand generalized Eguchi–Kawai reduction [5]. This is best motivated by studying the large N limit of two-dimensional U(N) lattice gauge theory following Gross and Witten [6].

The lectures start off with a pedagogical explanation of the large N gauge theories on a two dimensional lattice. We will not include fermions anticipating that they do not play a dynamical role in $N \to \infty$ limit. After this, the generalized Eguchi–Kawai reduction will be explained and it is a simple extension of the original argument by Eguchi and Kawai [5]. We will

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then show that reduction to a single site works only in $d = 2$. Although reduction to a single site does not work in $d > 2$, reduction to a finite physical volume will work and this is explained in Section 5.

The infinite volume limit at finite N is replaced by a infinite N limit at finite volume. Spontaneous chiral symmetry breaking can be realized at finite volume. Details pertaining to fermions in the large N limit form the last part of the lectures.

2. Large N gauge theories in two dimensions

Consider $\mathrm{U}(N)$ gauge theory on an infinite d dimensional lattice defined by the partition function

$$Z = \int \prod_x \prod_\mu dU_{x,\mu} e^S \,, \qquad S = bN \sum_p \mathrm{Tr}\left(U_p + U_p^\dagger\right) , \tag{1}$$

where S is the Wilson action, x labels a site, μ labels a direction and p denotes a plaquette. U_p is the parallel transporter around the plaquette. The inverse 't Hooft coupling is denoted by $b = \frac{1}{g^2 N}$. Under a local gauge transformation,

$$U_{x,\mu} \to g_x U_{x,\mu} g^\dagger_{x+\hat{\mu}} \tag{2}$$

and the action is invariant.

We follow Gross and Witten [6] and gauge fix the two dimensional theory by going to the $A_1 = 0$ gauge[1]. This corresponds to setting $U_{x,1} = 1$ for all x. In this gauge,

$$\sum_p \mathrm{Tr}\, U_p = \sum_x \mathrm{Tr}\left(U_{x,2} U^\dagger_{x+\hat{1},2}\right) . \tag{3}$$

We still have a remnant gauge symmetry which corresponds to gauge transformations that are independent of x_1, namely g_{x_2}, which can be fixed by setting

$$U_{x,2}|_{x_1=-\infty} = 1 \forall x_2 \,. \tag{4}$$

If we make the change of variables,

$$U_{x+\hat{1},2} = U_x U_{x,2} \,, \tag{5}$$

then the partition function becomes

$$Z = \int \prod_x dU_x e^{bN \sum_x \mathrm{Tr}\left(U_x + U_x^\dagger\right)} = \prod_x \int dU_x e^{bN\, \mathrm{Tr}\left(U_x + U_x^\dagger\right)} , \tag{6}$$

[1] This can be done on a infinite lattice but not on a finite lattice. We will address this point in Section 4.

and it factorizes with plaquettes being independently and identically distributed. The only remaining symmetry is a global $\mathrm{U}(N)$ symmetry, yet (6) is invariant under

$$U_x \to V_x U_x V_x^\dagger \tag{7}$$

for any V_x. This is special to Yang–Mills action in two dimensions and is not the original gauge symmetry defined in (2). Note that (7) along with (5) implies (2) only if $V_x = g$ for all x and this is the remaining global symmetry. If we had included fermions, they would have coupled to $U_{x,2}$ in our gauge and we will not have (7) as a symmetry.

The factorization of the partition function enables us to compute expectation values over individual factors and use that result to get any general expectation value. In particular,

$$\begin{aligned}\langle U_{ij}\rangle &= \frac{\int dU U_{ij} e^{bN\,\mathrm{Tr}\left(U+U^\dagger\right)}}{\int dU e^{bN\,\mathrm{Tr}\left(U+U^\dagger\right)}} = \frac{\int dU dV (VUV^\dagger)_{ij} e^{bN\,\mathrm{Tr}\left(U+U^\dagger\right)}}{\int dU e^{bN\,\mathrm{Tr}\left(U+U^\dagger\right)}} \\ &= \frac{1}{N}\langle \mathrm{Tr}\, U\rangle \delta_{ij} \equiv w(b,N)\delta_{ij}.\end{aligned} \tag{8}$$

We have used (7) in the second equality;

$$\int dV V_{ij} V_{kl}^\dagger = \frac{1}{N}\delta_{il}\delta_{jk} \tag{9}$$

in the third equality and $w(b,N)$ is the expectation value of a single plaquette.

If we define

$$z(b,N) = \int dU e^{bN\,\mathrm{Tr}\left(U+U^\dagger\right)} \tag{10}$$

as the single plaquette partition function, then

$$w(b,N) = \frac{1}{2N^2}\frac{d}{db}\ln z(b,N)\,. \tag{11}$$

The integral in (10) can be performed [7] and the result is

$$z(b,N) = \det M\,, \qquad M_{i,j} = I_{i-j}(2Nb)\,, \qquad i,j = 1,\cdots,N\,. \tag{12}$$

Consider a rectangular $L\times T$ Wilson loop with corners at x, $x+T\hat{1}$, $x+L\hat{2}$ and $x+T\hat{1}+L\hat{2}$. The parallel transporter around this loop in our gauge is

$$\begin{aligned}W_x(L,T) = \quad & U_{x+T\hat{1},2}U_{x+T\hat{1}+1\hat{2},2}\cdots U_{x+T\hat{1}+(L-1)\hat{2},2} \\ & U^\dagger_{x+(L-1)\hat{2},2}U^\dagger_{x+(L-2)\hat{2},2}\cdots U^\dagger_{x,2}\,.\end{aligned} \tag{13}$$

It follows from (5) that

$$U_{x+T\hat{1}+(L-1)\hat{2},2}U^{\dagger}_{x+(L-1)\hat{2},2} = U_{x+(T-1)\hat{1}+(L-1)\hat{2}}U_{x+(T-2)\hat{1}+(L-1)\hat{2}} \cdots U_{x+\hat{1}+(L-1)\hat{2}}U_{x+(L-1)\hat{2}} \,. \tag{14}$$

Using (8) and averaging over all the U_x variables appearing in the above equation, we find that

$$\frac{1}{N}\langle \mathrm{Tr}\, W_x(L,T)\rangle = [e(b,N]^T \frac{1}{N}\langle \mathrm{Tr}\, W_x(L,T-1)\rangle \,. \tag{15}$$

Repeating the above steps L times we arrive at

$$\frac{1}{N}\,\mathrm{Tr}\, W_x(L,T) = [e(b,N)]^{LT} \,, \tag{16}$$

since $W_x(L,0)=1$. The above equation says that the area law is exact in two dimensional Yang–Mills theory for all values of N.

The continuum limit at a fixed N is obtained by taking $b \to \infty$. If we first take $N \to \infty$ at a fixed b and then take $b \to \infty$, we obtain the continuum limit of the large N gauge theory à la 't Hooft. In the large N limit, it is instructive to solve for $z(b,N)$ using the method of steepest descent resulting in the stationary condition

$$2b\sin\alpha_i = \frac{1}{N}\sum_{j\neq i}\cot\left|\frac{\alpha_i-\alpha_j}{2}\right| \tag{17}$$

for the eigenvalues α_i of the U(N) matrix, U, appearing in (10). Due to the symmetry (7) of the single site partition function, all expectation values will be only functions of the eigenvalues of U and can be evaluated by substituting for α_i the values that solve (17). Since the single site partition function is dominated by the stationary point, it follows that expectation values factorize in the large N limit:

$$\langle F(U)\rangle = F(\langle U\rangle) \,. \tag{18}$$

The partition function on the infinite lattice can be further reduced from a product of independently and identically distributed plaquettes in (6) to a single site partition function as in (10). Expectation of the Wilson loop operator defined in (13) reduces to the *folded* operator

$$W_x(L,T) = \left\langle \mathrm{Tr}\, U^{LT}\right\rangle , \tag{19}$$

and it is clear from (18) that we will obtain (16). This is the motivation behind the idea of Eguchi–Kawai reduction [5].

Before we proceed with a discussion of Eguchi–Kawai reduction, it will be useful to finish this section with a property of (17). We can replace, α_i, by a continuum function $\alpha(x)$, $x \in [0,1]$ in the large N limit. Furthermore, we can define the density of eigenvalues,

$$\rho(\alpha) = \frac{dx}{d\alpha}\,. \tag{20}$$

Then (17) reduces to

$$2b \sin\alpha = P \int_{-\pi}^{\pi} d\beta \rho(\beta) \cot\frac{\alpha-\beta}{2}, \tag{21}$$

an equation for ρ where P refers to the principal part of the integral. This equation is solved in [6] and the result is

$$\rho(\alpha) = \begin{cases} \frac{2b}{\pi}\cos\frac{\alpha}{2}\sqrt{\frac{1}{2b} - \sin^2\frac{\alpha}{2}} & \text{if } b \geq \frac{1}{2} \text{ and } |\alpha| < 2\sin^{-1}\sqrt{\frac{1}{2b}}\,, \\ \frac{1}{2\pi}\left(1 + 2b\cos\alpha\right) & \text{if } b \leq \frac{1}{2} \text{ and } |\alpha| \leq \pi\,. \end{cases} \tag{22}$$

The lattice theory undergoes a phase transition at $b = 1/2$. The continuum theory does not exhibit this phase transition. But the lattice strong coupling limit and the weak coupling limit are separated by this phase transition. In order to obtain the correct continuum limit of the large N theory, we need to keep $b > 1/2$ and $\rho(\alpha)$ has a finite region of support around $\alpha = 0$ that does not extend up to $\alpha = \pm\pi$.

3. Generalized Eguchi–Kawai reduction

The discussion of the large N limit of Yang–Mills theories in Section 2 suggests:

1. Factorization of observables;
2. Domination of the path integral by a single *classical* configuration.

Witten argues for the above two points in [8]. Consider, for example, an observable that we encountered in Section 2, namely, $\langle \mathrm{Tr}\, U_x \,\mathrm{Tr}\, U_y \rangle$. This quantity, in perturbation theory, has two pieces, connected and disconnected. $\langle \mathrm{Tr}\, U_x \rangle$ is of order N^2 since there are N^2 gluon species that can run around the loop. Therefore, the disconnected piece is of order N^4. The connected piece has only one loop with two insertions, one for $\mathrm{Tr}\, U_x$ and another for $\mathrm{Tr}\, U_y$, and therefore it is of order N^2. Therefore,

$$\langle \mathrm{Tr}\, U_x \,\mathrm{Tr}\, U_y \rangle = \langle \mathrm{Tr}\, U_x \rangle \langle \mathrm{Tr}\, U_y \rangle \tag{23}$$

in the large N limit. The same argument would also imply that

$$\langle e^2 \rangle - \langle e \rangle^2 = 0\,, \qquad e = \frac{1}{V}\sum_x \mathrm{Tr}\, U_x\,. \tag{24}$$

If fluctuations go to zero in the large N limit, it is not necessary to do a path integral since one classical field configuration must dominate as was seen using steepest descent in Section 2. The above argument of Witten was made rigorous in [5] and the following statement is a generalization of the Eguchi–Kawai reduction:

Consider U(N) Yang–Mills gauge theory with Wilson action given by (1) on a finite lattice of size $L_1 \times L_2 \cdots L_d$ with a fixed lattice coupling $b = \frac{1}{g^2 N}$ and periodic boundary conditions in all directions. Also consider another theory with only one difference from the previous one: $L_{\mu_i} = \infty$ for $i = 1, \cdots, k \leq d$. Now consider an arbitrary closed Wilson loop operator. The operators associated with the same Wilson loop on the finite lattice and the lattice with k infinite directions could be different due to possible folding. The folding comes from the use of periodic boundary conditions on the finite lattice. The large N limit will be the same in both cases provided the Z_N symmetries associated with the Polyakov loops in the μ_i; $i = 1, \cdots, k$ directions are not broken on the finite lattice.

We provide relevant steps for a proof of the above statement by following the steps in [5]. Consider a closed Wilson loop that contains the link $U_{x,\mu}$ once and let us write it is as

$$W = \mathrm{Tr}\, U_{x,\mu} C^\dagger_{x,\mu}\,, \tag{25}$$

where $C_{x,\mu}$ is an open path with more than one link that connects x and $x+\hat{\mu}$ and does not contain $U_{x,\mu}$. The terms in the Wilson action that contain $U_{x,\mu}$ can be written as $\mathrm{Tr}\left(U_{x,\mu}S^\dagger_{x,\mu} + S_{x,\mu}U^\dagger_{x,\mu}\right)$ where $S_{x,\mu}$ is the sum of the parallel transporters over all the three link paths that connect x and $x+\hat{\mu}$. $S_{x,\mu}$ does not contain $U_{x,\mu}$ if none of the finite directions are of unit length[2]. The group measure is invariant under a small change of the form

$$U_{x,\mu} \to e^{i\epsilon T^j} U_{x,\mu}\,, \tag{26}$$

where T^j is a group generator and ϵ is a small parameter. Therefore,

[2] Equation (30) remains unaltered if $S_{x,\mu}$ contains $U_{x,\mu}$.

$$\left\langle \mathrm{Tr}\left(T^j U_{x,\mu} C^\dagger_{x,\mu}\right)\right\rangle = \Bigg\langle \mathrm{Tr}\left(T^j e^{i\epsilon T^j} U_{x,\mu} C^\dagger_{x,\mu}\right) \times e^{bN\,\mathrm{Tr}\left(\left[e^{i\epsilon T^j}-1\right]U_{x,\mu}S^\dagger_{x,\mu}+S_{x,\mu}U^\dagger_{x,\mu}\right]\left[e^{-i\epsilon T^j}-1\right]\right)}\Bigg\rangle, \quad (27)$$

and to the lowest order in ϵ,

$$\left\langle \mathrm{Tr}\left(T^j T^j U_{x,\mu} C^\dagger_{x,\mu}\right)\right\rangle + bN\left\langle \mathrm{Tr}\left(T^j U_{x,\mu} C^\dagger_{x,\mu}\right)\mathrm{Tr}\left(T^j U_{x,\mu} S^\dagger_{x,\mu} - S_{x,\mu} U^\dagger_{x,\mu} T^j\right)\right\rangle = 0\,. \quad (28)$$

Summing the above equation over all values of j and using the identity,

$$\sum_{j=1}^{N^2} T^j_{ab} T^j_{cd} = \delta_{ad}\delta_{bc}\,, \quad (29)$$

we get

$$\left\langle \mathrm{Tr}\left(U_{x,\mu} C^\dagger_{x,\mu}\right)\right\rangle + b\left\langle \mathrm{Tr}\left(U_{x,\mu} C^\dagger_{x,\mu} U_{x,\mu} S^\dagger_{x,\mu}\right)\right\rangle - b\left\langle \mathrm{Tr}\left(C^\dagger_{x,\mu} S_{x,\mu}\right)\right\rangle = 0\,. \quad (30)$$

The above equation, referred to as the Schwinger–Dyson equation for Wilson loops, relates the expectation value of the original Wilson loop to other Wilson loops that correspond to the modification of the original Wilson loop by attaching the various plaquette parallel transporters that contain $U_{x,\mu}$. Even if we start with a Wilson loop where each link occurs only once, the above equations will generate Wilson loops where certain links appear more that once. In fact, this can already been seen in the second term of (30). If we repeat the above procedure starting with a Wilson loop where $U_{x,\mu}$ appears twice, namely,

$$W = \mathrm{Tr}\, U_{x,\mu} C^\dagger_{x,\mu} U_{x,\mu} D^\dagger_{x,\mu}\,, \quad (31)$$

where $C_{x,\mu}$ and $D_{x,\mu}$ are two open paths that connect x to $x+\hat{\mu}$. In this case, the Schwinger–Dyson equation will have an additional term of the form

$$\frac{1}{N}\left\langle \mathrm{Tr}\left(U_{x,\mu} C^\dagger_{x,\mu}\right)\mathrm{Tr}\left(U_{x,\mu} D^\dagger_{x,\mu}\right)\right\rangle\,. \quad (32)$$

This is an expectation value of products of Wilson loops. Noting that expectation value of an closed loop is of the order of N, we see that the new term is the same order in N as the ones in (30) and we also note that the new term factorizes in the large N limit. The coupled set of infinite number of Schwinger–Dyson equations obtained in this process will also involve Polyakov loops in the finite directions for the following reason: Since it will involve Wilson loops of arbitrary size, it will contain loops of the form

$$W = \mathrm{Tr}\, U_{x,\mu} C^\dagger_{x,\mu} U_{x+L_\mu\hat{\mu},\mu} D^\dagger_{x,\mu}\,, \tag{33}$$

where $C_{x,\mu}$ is an open path that connects $(x + L_\mu\hat{\mu})$ to $x + \hat{\mu}$ and $D_{x,\mu}$ is an open path that connects x to $(x + (L_\mu + 1)\hat{\mu})$. Since $U_{x+L_\mu\hat{\mu},\mu} = U_{x\mu}$ by periodic boundary conditions, this will result in a term of the form

$$\frac{1}{N}\left\langle \mathrm{Tr}\left(U_{x,\mu} C^\dagger_{x,\mu}\right) \mathrm{Tr}\left(U_{x+L_\mu\hat{\mu},\mu} D^\dagger_{x,\mu}\right)\right\rangle. \tag{34}$$

The difference between the infinite set of coupled equations in the two cases, one with k infinite directions and the other being finite in all directions, is the presence of additional Polyakov loops in the k finite directions. Polyakov loops in the μ direction are not invariant under a global Z_N symmetry in that direction where we replace all $U_{x,\mu}$ in a fixed hyperplane perpendicular to μ by $e^{i\frac{2\pi k}{N}} U_{x,\mu}$ with $0 < k < N$. Since this is a symmetry of the gauge action, the Polyakov loops appearing in (34) will have zero expectation value if the Z_N symmetry in the μ direction is not spontaneously broken. This completes our discussion of the statement concerning generalized Eguchi–Kawai reduction.

4. Reduction to a single site

The arguments presented in Section 3 show that one can reduce the large N theory from an infinite lattice down to a single site lattice if the Z_N symmetries on a single site lattice are not broken. Since we independently showed that this reduction was possible in two dimensions in Section 2, it follows that the two Z_N symmetries are not broken on a single site lattice in two dimensions for all values of b and therefore also in the continuum limit. On the other hand, the Z_N symmetries are broken in the weak coupling limit in three or more dimensions and we will present the argument following [9].

Consider the U(N) Wilson gauge action on a single site d dimensional lattice, namely,

$$S_{\mathrm{EK}} = bN \sum_{\mu\neq\nu=1}^{d} \mathrm{Tr}\left[U_\mu U_\nu U^\dagger_\mu U^\dagger_\nu\right]. \tag{35}$$

The action depends on d U(N) matrices and the gauge transformation is

$$U_\mu \to g U_\mu g^\dagger. \tag{36}$$

Note that the eigenvalues of U_μ are gauge invariant. We cannot fix a gauge such that one of the $U_\mu = 1$ since we are on a finite lattice. The action has an additional $U^d(1)$ symmetry given by

$$U_\mu \to e^{i\phi_\mu} U_\mu \,, \tag{37}$$

with $-\pi < \phi_\mu < \pi$. The four Polyakov loop operators given by

$$P_\mu = \mathrm{Tr}\, U_\mu \tag{38}$$

are gauge invariant but not invariant under (37). If the $U^d(1)$ symmetry is not broken, then the eigenvalues of all U_μ are uniformly distributed on the unit circle and $P_\mu = 0$. In order to see if this symmetry is spontaneously broken in the weak coupling limit, we set

$$U_\mu = e^{ia_\mu} D_\mu e^{-ia_\mu} \,, \qquad D_\mu^{jk} = e^{i\theta_\mu^j} \delta^{jk} \,, \tag{39}$$

and expand to the quadratic term in the hermitian matrix a_μ. We fix the gauge by setting $a_1 = 0$. The group measure is given by

$$\prod_\mu dU_\mu = \left[\prod_\mu \prod_i d\theta_\mu^i\right] \left[\prod_\mu \prod_{i>j} p_\mu^{ij}\right] \left[\prod_{\mu=2}^{d} \prod_{i>j} da_\mu^{ij} da_\mu^{ij*}\right] , \tag{40}$$

where

$$p_\mu^{ij} = \sin^2 \tfrac{1}{2} \left(\theta_\mu^i - \theta_\mu^j\right) . \tag{41}$$

The quadratic piece of the action is

$$S = -32bN \sum_{i<j} \sum_{\mu,\nu=2}^{d} a_\mu^{ij*} \left(p_\mu^{ij} p^{ij} \delta_{\mu\nu} - p_\mu^{ij} p_\nu^{ij}\right) a_\nu^{ij} , \tag{42}$$

where $p^{ij} = \sum_\mu p_\mu^{ij}$. The result of the integration over a_μ, ignoring normalization factors, is

$$\left[\prod_\mu \prod_{i>j} \frac{1}{p_\mu^{ij}}\right] \left[\prod_{i>j} p^{ij}\right]^{2-d} . \tag{43}$$

Therefore, up to second order in a_μ, the partition function is

$$Z = \left[\prod_\mu \prod_i d\theta_\mu^i\right] e^{(2-d)\sum_{i<j} \ln p^{ij}} . \tag{44}$$

All $\theta_\mu^i = 0$ has the maximum probability for $d > 2$ since $p^{ij} = 0$ for all i and j implying that the Z_N symmetries are broken in the weak coupling limit if $d > 2$.

5. Continuum reduction

Consider the continuum theory in a finite torus of size $l_1 \times l_2 \cdots l_d$ obtained from the theory on a $L_1 \times L_2 \cdots L_d$ periodic lattice at a fixed coupling b and taking the limit $L_1, L_2, \cdots, L_d, b \to \infty$ such that

$$\begin{aligned} l_i &= \frac{L_i}{\sqrt{b}}, \qquad i = 1,2, \qquad \text{in } \ d = 2\,, \\ l_i &= \frac{L_i}{b}, \qquad i = 1,2,3 \qquad \text{in } \ d = 3\,, \\ l_i &= L_i a(b)\,, \quad i = 1,2,3,4 \quad \text{in } \ d = 4 \end{aligned} \tag{45}$$

are kept fixed. The lattice spacing, $a(b)$, in $d = 4$ is given by

$$a(b) = \frac{1}{\Lambda} \left(\frac{48\pi^2 b}{11} \right)^{51/121} e^{-24\pi^2 b/11}\,, \tag{46}$$

in weak coupling perturbation theory [10].

In two dimensions, the variables, θ^i_μ are uniformly distributed in the weak coupling limit as can be seen by setting $d = 2$ in (44). Therefore, the single site lattice $L_1 = L_2 = 1$ will give the same results in the large N limit as any $L_1 \times L_2$ lattice for all values of the coupling b implying that the continuum theory will be independent of l_1 and l_2. Since the Z_N (U(1) in the limit $N \to \infty$) symmetries are unbroken for all l_1 and l_2, it follows that two dimensional large N QCD is in the confined phase for all temperatures.

The lack of a finite temperature phase transition separating the confined phase from the deconfined phase is special to $d = 2$. Numerical analysis has strongly established the existence of the deconfining phase transition for finite N in $d = 3$ [11, 12] and $d = 4$ [13–15]. This transition is expected to have a large N limit both in $d = 3$ [16] and $d = 4$ [17]. Therefore, we could not have expected single site reduction to work in $d > 2$.

Consider a symmetric lattice $L_1 = L_2 = \cdots L_d = L$ ($d > 2$) and a lattice coupling $b_1(L)$ such that no Z_N symmetry is broken for $b < b_1(L)$ but not all Z_N symmetries are unbroken for $b > b_1(L)$. We know that such a coupling exists since all Z_N symmetries will be unbroken for $b = 0$ and all will be broken for $b = \infty$. The theory is in the confined phase for $b < b_1(L)$ and since we expect the continuum theory to have a confined phase $b_1(L)$ should approach ∞ as $L \to \infty$. Fixing, $b < b_1(L)$ we can consider a $L_1 \times L_2 \cdots L_d$ lattice such that $L_i > L$ for all i and the arguments provided in Section 3 shows that there will be no dependence on L_i and the theory is in the confined phase. Now consider a $L \times \infty^{d-1}$ lattice with a coupling of $b = b_1(L)$. As L is varied we will remain at the phase transition point separating the confined phase from the deconfined phase where the Z_N symmetry is broken in the direction with finite extent L. Therefore, $l_1 = L/b_1(L)$ in $d = 3$ and

$l_1 = La(b_1(L))$ in $d = 4$ should have a finite limit as $L \to \infty$ and it should be the inverse of the deconfining temperature. This has been numerically verified in $d = 3$ [4] and $d = 4$ [18]. The order of the deconfining phase transition at infinite N can be obtained by a numerical computation of the latent heat associated with the transition. The presence of a latent heat will result in a jump in the average value of the action density at the transition. A non-zero latent heat has been computed in $d = 4$ [19] by numerical methods and numerical studies are currently under way in $d = 3$ [20].

We will refer to the confined phase of the continuum theory as the 0c phase. There is no dependence on the physical size of the box in this phase. There is a transition from the 0c to the 1c phase when one of the directions has a length less than l_1. This is the conventional deconfined phase and there is no dependence on the physical size of the box in the $(d-1)$ directions. Let $l < l_1$ be the length of the direction along which the U(1) symmetry is broken. Let $(d-2)$ of the other directions be finite and let one direction be finite. Fixing l, we can vary the length of the second finite direction and go from a U(1) symmetric phase in that direction to a U(1) broken phase. Let $l_2(l) \geq l$ be the length of the second finite direction such the U(1) symmetry in that direction is unbroken for lengths larger than $l_2(l)$ and broken for lengths smaller than $l_2(l)$. The existence of $l_2(l)$ in the continuum theory has been verified by numerical means using the lattice theory for $d = 3$ [21]. The temperature in the deconfined phase is $1/l$ and one can remain in the deconfined phase for all temperatures above $1/l_1$ as long as one keeps the extent of the other $(d-1)$ directions larger than $l_2(l)$. If we pick one of the other $(d-1)$ directions to be less than $l_2(l)$ two of the U(1) symmetries are broken and we refer to this as the 2c phase. In this manner we can have the continuum theory in a kc phase with $0 \leq k \leq d$ where k of the d U(1) symmetries are broken. One can view the $(d-1)$c and the dc phase as the low temperature and the high temperature phase of large N QCD in a Bjorken universe [22]. The complete phase diagram has not been mapped out in $d = 3$ or $d = 4$.

6. Fermions

As long as the fermions are in the fundamental representation and we only have a finite number of flavors, N_f, fermion loops are suppressed in the large N limit compared to gluon loops since $NN_f \ll N^2$ as N gets large [1,2]. Physical quantities associated with the fermionic sector of large N QCD can be computed using fermionic observables in a gauge background generated using the pure gluonic action. Continuum reduction continues to hold while computing physical quantities in the fermionic sector.

6.1. Chiral condensate

Chiral symmetry is expected to be broken in $d = 2$ and $d = 4$ in the confined phase of large N limit of QCD. The theory in $d = 2$ is in the confined phase for any finite torus. The large N degrees of freedom must therefore be responsible for spontaneous chiral symmetry breaking in finite volume. Consider the lattice model on a single site. We have two $\mathrm{U}(N)$ matrices, namely, U_1 and U_2. The Wilson action in (1) reduces to $S = 2bN\mathrm{Re}\ \mathrm{Tr}\left[U_1 U_2 U_1^\dagger U_2^\dagger\right]$. The fermionic operator on the infinite lattice splits into momentum blocks with each block being of the form $D_f\left(U_1 e^{ip_1}, U_2 e^{ip_2}; m_q\right)$ where $-\pi < p_1, p_2 \leq \pi$ is the momentum of the block and m_q is the quark mass. The chiral condensate is given by

$$\chi(b, N, m_q) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dp_1 \int_{-\pi}^{\pi} dp_2 \frac{1}{N} \left\langle \mathrm{Tr}\, D_f \left(U_1 e^{ip_1}, U_2 e^{ip_2}; m_q\right)\right\rangle , \tag{47}$$

where the expectation value is obtained using the single site Wilson gauge action. In order to obtain the chiral condensate, we will have to take $N \to \infty$ limit before we take the $m_q \to 0$ limit. In the limit of $N \to \infty$, the two $\mathrm{U}(1)$ symmetries are not broken and $\langle \mathrm{Tr}\, D_f\left(U_1 e^{ip_1}, U_2 e^{ip_2}; m_q\right)\rangle$ is independent of p_1 and p_2. Therefore, the chiral condensate in the massless limit is given by

$$\Sigma = \lim_{m_q \to 0} \lim_{N \to \infty} \frac{1}{N} \langle \mathrm{Tr}\, D_f\left(U_1, U_2; m_q\right)\rangle. \tag{48}$$

The fermionic operator $D_f(U_1, u_2; 0)$ on the single site lattice will have $2N$ eigenvalues and it will have N paired eigenvalues, $\pm\lambda_i$; $i = 1, \cdots, N$ if the global topology is zero and the fermionic operator obeys chiral symmetry on the lattice. Let us assume that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ and let $p(\lambda_1, \lambda_2, \cdots, \lambda_N)$ be the joint probability of the N eigenvalues after averaging over U_1 and U_2 using the single site gauge action as the measure. Chiral random matrix theory [23] predicts the joint probability distribution, $p_{\mathrm{chRMT}}(z_1, z_2, \cdots, z_N)$, when N is large and where

$$z_i = \Sigma(b) N \lambda_i \tag{49}$$

with $\Sigma(b)$ being the only adjustable parameter and is the chiral condensate at the lattice coupling b. This prediction has been numerically verified [24] and the resulting chiral condensate agrees with the known analytical result [25, 26].

Turning now to $d = 4$, we first note following the discussion in Section 5 that we need to consider a theory on a L^4 lattice at a fixed lattice coupling b such that $b < b_1(L)$ and one is in the Z_N symmetric phase. The fermionic operators on the L^4 lattice will be such that

$$\psi(x + L\hat{\mu}) = e^{ip_\mu}\psi(x) \tag{50}$$

since the operator in the infinite lattice will split into momentum blocks with the momentum translating into the above boundary conditions on fermions. We can replace the boundary conditions on fermions by periodic boundary conditions and replace the gauge fields by

$$U_\mu(x) \to U_\mu(x)e^{ip_\mu/L}, \tag{51}$$

where we have used a gauge transformation to uniformly distribute e^{ip_μ} over L links. Since we are in the U(1) symmetric phase of the large N gauge theory, the chiral condensate will not depend on p_μ and we can set them to zero and compute the chiral condensate. One can use matching with chiral random matrix theory predictions to numerically estimate the chiral condensate in the large N limit of QCD in $d = 4$. The eigenvalues, $\pm\lambda_i$, $i = 1, \cdots, 4NL^4$, will match with the variables, z_i, according to

$$z_i = \Sigma(b)NL^4\lambda_i\,. \tag{52}$$

The chiral condensate, $\Sigma(b)$, has been numerically extracted and its scaling behavior has been studied as a function of b [24].

The fermion propagator in the deconfined (1c) phase will depend on the boundary conditions, namely (50), in the direction where the Z_N symmetry is broken. Fermions, in this sense, do play a dynamical role in the 1c phase and anti-periodic boundary conditions with respect to the value of the Polyakov loop in that direction will be favored. This has been numerically verified in [27]. Since the chiral condensate does not depend on the temperature in the confined phase, the chiral phase transition in going from 0c to 1c will be first order. Restoration of chiral symmetry in the 1c phase has been numerically verified by the presence of a non-zero gap in the fermion spectrum for all values of temperature in the deconfined phase [27].

6.2. Meson propagator

Let

$$M(x) = \bar{u}(x)\Gamma\sum_z S_{xz}(U_\mu)d(z) \tag{53}$$

denote a meson at x made out of two different flavors in some spin representation given by Γ. The meson is defined using some gauge field dependent smearing operator, $S_{xy}(U_\mu)$, that commutes with Γ and transforms as

$$S_{xz}(U_\mu^g) = g_x S_{xz}(U_\mu) g_z^\dagger \tag{54}$$

under a gauge transformation, $d(z) \to g_z d(z)$ and $\bar{u}(x) \to \bar{u}(x) g_x^\dagger$. The meson propagator in momentum space is

$$\begin{aligned}
G(p) &= \sum_{x,y} e^{ip(x-y)} \langle M(x) M^\dagger(y) \rangle \\
&= \int \frac{d^4 q}{16\pi^4} \sum_{x,y,z,w} \left\langle \mathrm{tr} \left[\Gamma e^{i\left(\frac{p}{2}+q\right)x} S_{xz}(U_\mu) G^f_{zw}(U_\mu, m_q) S^\dagger_{wy}(U_\mu) e^{-i\left(\frac{p}{2}+q\right)y} \right.\right. \\
&\quad \left.\left. \times \Gamma^\dagger e^{-i\left(\frac{p}{2}-q\right)y} G^f_{yw}(U_\mu, m_q) e^{i\left(\frac{p}{2}-q\right)x} \right] \right\rangle \\
&= \int \frac{d^4 q}{16\pi^4} \left\langle \mathrm{Tr} \left[\Gamma S\left(U_\mu e^{i\left(\frac{p}{2}+q\right)_\mu}\right) G^f\left(U_\mu e^{i\left(\frac{p}{2}+q\right)_\mu}, m_q\right) S^\dagger\left(U_\mu e^{i\left(\frac{p}{2}+q\right)_\mu}\right) \right.\right. \\
&\quad \left.\left. \times \Gamma^\dagger G^f\left(U_\mu e^{i\left(-\frac{p}{2}+q\right)_\mu}, m_q\right) \right] \right\rangle \\
&= \left\langle \mathrm{Tr} \left[\Gamma S\left(U_\mu e^{i\frac{p_\mu}{2}}\right) G^f\left(U_\mu e^{i\frac{p_\mu}{2}}, m_q\right) S^\dagger\left(U_\mu e^{i\frac{p_\mu}{2}}\right) \right.\right. \\
&\quad \left.\left. \times \Gamma^\dagger G^f\left(U_\mu e^{-i\frac{p_\mu}{2}}, m_q\right) \right] \right\rangle .
\end{aligned} \tag{55}$$

The first equality above assumes that we have translational invariance upon averaging over the gauge fields. We have introduced an integral over q in the second equality and the integrand does not depend upon q. The tr in the second equality indicates a sum over spin and color indices only. We extend this to Tr in the third equality where the sum is now over space, spin and color indices. The exponential factors on either side of the smeared d quark propagator and on either side of the u quark propagator in the second equality are viewed as a gauge transformations and results in the gauge transformed fields in the third equality. These factors are thought of as momenta carried by the quarks: d quark has a momentum equal to $q + \frac{p}{2}$ and the u quark has a momentum equal to $q - \frac{p}{2}$. This corresponds to a meson momentum equal to p and q is the momentum around the quark loop in the meson propagator. Due to the U(1) gauge invariance in the confined phase, we can replace $U_\mu e^{iq_\mu}$ by U_μ in the third equality and the integrand does not depend on q. This results in the final equality in the above equation.

The meson momentum can take any value in the range $[-\pi, \pi]$ since the above equation was derived on an infinite lattice. Momenta that are integer multiples of $\frac{2\pi}{L}$ are the ones allowed by periodic boundary conditions on the L^4 lattice. Momenta that fill in the gaps between the integer multiples correspond to boundary conditions of the form given by (50).

Pion mass and the vector meson mass as a function of the quark mass has been numerically studied using the above procedure. This has resulted in an estimate of the pion decay constant [28] and the vector meson mass in the chiral limit [29].

6.3. Numerical details

We will focus on $d = 4$ and present the necessary numerical details to perform the computations described in the previous subsections. Gauge fields are generated using a combination of Cabibbo–Marinari SU(2) heat-bath [30] and SU(N) over-relaxation [18]. A description of overlap fermionscan be found in [33–35]. The ϵ function appearing in the overlap-Dirac operator is best approximated using the 20$^{\text{th}}$ order Zolotarev approximation [36, 37]. The action of ϵ on a vector can be performed using the multiple mass conjugate gradient algorithm [38, 39]. The low lying eigenvalues of the massless overlap-Dirac can be computed using the Ritz algorithm [40]. The trace involved in the computation of the meson propagator can be stochastically estimated using a single random vector.

7. Other topics and future directions

Some progress in the understanding of the physical transition from strong coupling (hadron resonances) to weak coupling (perturbative QCD) has been achieved by the study of Wilson loops [41]. Much more work needs to be done in this topic and there is recent progress in connecting the transition in Wilson loops to a chiral transition in two-dimensional fermions coupled to the four dimensional gauge field [42].

A computation of the string tension using folded Wilson loops has been performed in $d = 3$ [43] and $d = 4$ [44]. Typically, one obtains the string tension from Polyakov loop correlators [46] since large Wilson loops have a small expectation value due to two reasons: large area and perimeter divergence. The results at large N obtained using folded Wilson loops are in good agreement with the results obtained at smaller N using Polyakov loop correlations and then performing a large N extrapolation [46,47]. Based on our discussion in Section 3, we note that folded Wilson loops can also be computed in the 1c phase as long as the Wilson loop is in the plane where the Z_N symmetries are not broken and such loops are referred to as spatial loops. Numerical computation of the spatial string tension in the 1c phase of $d = 3$ show that the string tension grows linearly with the temperature [45].

Fermions play a dynamical role in the large N limit when they are in the adjoint representation. Since the gauge action induced by the fermion determinant will tend to cancel the original gauge action, one expects single site reduction to hold in $d = 4$ if the fermions are in the adjoint representation. Strong arguments in this direction have been put forward recently in the continuum [48, 49]. Taking the zero volume limit in the continuum is not necessarily the same as working on a single site lattice. Therefore, it is interesting to numerically study large N gauge theory with fermions in the adjoint representation on a single site lattice. Recent progress on this topic can be found in [50, 51].

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