FERMIONS IN 2d LORENTZIAN QUANTUM GRAVITY

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We implement Wilson fermions on 2d Lorentzian triangulation and determine the spectrum of the Dirac–Wilson operator. We compare it to the spectrum of the corresponding operator in the Euclidean background. We use fermionic particle to probe the fractal properties of Lorentzian gravity coupled to $c = 1/2$ and $c = 4$ matter. We numerically determine the scaling exponent of the mass gap $M \sim N^{-1/d_{\rm H}}$ to be $d_{\rm H} = 2.11(5)$, and $d_{\rm H} = 1.77(3)$ for $c = 1/2$ and $c = 4$, respectively.

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1. Introduction

The formulation of a theory of quantum gravity is one of the most challenging problems in theoretical physics. Simplicial gravity is a non-perturbative approach to this problem. It is a natural extension of Feynman's idea of defining quantum amplitudes via functional path integrals.

Simplicial gravity is a lattice regularization of Feynman integrals over a set of geometries [1–3]. The idea is to look for non-perturbative fixed points of the renormalization group at which a continuum limit can be taken. It is crucial in a lattice regularization to preserve the gauge invariance of the underlying continuum theory. Simplicial gravity, similarly as lattice regularization of QCD, properly treats the problem of gauge invariance. The underlying continuum theory is invariant with respect to the change of coordinates. Lattice formulation is coordinate free by construction. A remnant of coordinates are labels on lattice simplices and vertices. Lattice theory is defined in a way which is invariant with respect to relabeling. This invariance is a left over of the diffeomorphism invariance of the continuous formulation. Statistical weights of simplicial manifolds take into account the volume of the discrete symmetry group. For example, in two dimensions, were the sum over simplicial diagrams (dynamical triangulations) can be explicitly generated by a perturbative expansion of a matrix model [4, 5],

the statistical weights are automatically generated as combinatorial factors of the corresponding Feynman diagrams. These factors play the role of the Faddeev–Popov determinants. The 2d case is analytically solvable. The continuum limit of 2d lattice gravity [6] is equivalent to Liouville theory being a quantum version of Euclidean gravity regularized by completely different means [7, 8]. This equivalence is treated as a strong indication that the sum over simplicial manifolds provides a proper definition of the integration measure over Riemannian manifolds.

Real gravity has the Lorentzian signature. One can obtain this signature by Wick rotation. One way of doing this is to calculate quantities in the Euclidean sector and then perform analytic continuation to the Lorentzian one. This strategy is used in quantum field theory but it is not clear whether it can be straightforwardly applied to quantum gravity. An alternative approach is to impose the causal structure on simplicial manifolds which enter the Feynman integrals [9,10]. This leads to a formulation called Lorentzian simplicial gravity for which the causality is achieved by introducing a time-slicing into the lattice structure. This formulation is very close in spirit to the Hamiltonian formulation in the temporal gauge. In two dimensional case one can determine an explicit form of the Hamiltonian of the underlying continuum theory [11]. Similarly as for the Euclidean case the model is analytically solvable in two dimensions [11–13]. The resulting continuum theory differs from Liouville gravity. One can determine mathematical relations between Euclidean and Lorentzian gravity in terms of a singular renormalization of coupling constants [13].

Both the Euclidean [1–3] and Lorentzian models [14–16] have natural extensions to higher dimensional cases. It is a matter of debate which of the two versions may serve as a theory of quantum gravity in higher dimensional case. Both have been a subject of intensive studies. In the ultimate theory of gravity an important role is played by the interaction of gravity with matter fields. Matter fields are known to modify fractal properties of gravity and scaling properties of the underlying theory. Results of explicit calculations of 2d Euclidean case, are summarized in the KPZ formula [8]. They show that the scaling properties of matter field are indeed modified by Euclidean gravity. On the contrary, numerical simulations of Lorentzian case indicate that the scaling properties of matter in Lorentzian background stay intact even if one crosses the $c = 1$ barrier [17, 18].

Numerical studies of higher-dimensional Euclidean gravity have shown the importance of matter fields for of the critical properties of the underlying continuum theory. For example, matter fields remove the conformal instability of Euclidean sector and modify the phase structure of the model [19–21]. So far numerical simulations have been performed only for bosonic matter. An introduction of fermionic matter may be crucial for defining a fixed

point of gravity at which a continuum limit can be taken. First step towards defining fermions on simplicial quantum gravity was done in 2d Euclidean gravity [22–24]. In this paper we extend these studies to the 2d Lorentzian case. We shall use fermions to probe fractal properties of geometry.

2. The model

Let us briefly recall the model of 2d Lorentzian gravity [9, 10]. The integration measure of the 2d Lorentzian gravity is defined as a sum over triangulations which have a time sliced structure. Additionally, for technical reasons this structure is periodic in temporal direction.

Each time slice consists of a random number of vertices on a circle. The number of vertices V_t on a slice t and V_{t+1} on the consecutive slice give the number of triangles $N_{t+1/2} = V_t + V_{t+1}$ lying in between. The strip between slices consists of a random combination of triangles built of edges which join vertices of the two time slices (see Fig. 1). The temporal index runs

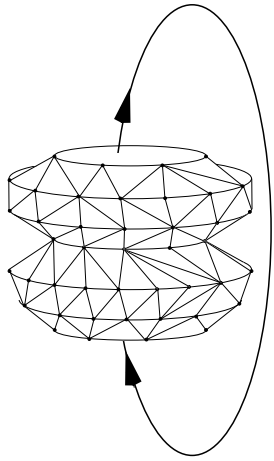


Fig. 1. Schematic drawing of the 2d Lorentzian lattice.

periodically over $t = 1, \ldots, T$. Consequently, the total numbers of triangles and vertices are related to each other as $N = 2V$. Topology of each time slice is that of circle in contrast to Euclidean gravity where it can be a set of disconnected circles. The effect of branching, which plays a dominant role in Euclidean case, is thus suppressed here. In consequence, fractal properties of Lorentzian gravity are completely different from those of Euclidean gravity, as reflected by the Hausdorff dimension which changes from $D_{\mathrm{H}} = 2$ [17, 18] for the former to $D_{\mathrm{H}} = 4$ [25] for the latter case. In a sense, Lorentzian gravity has not enough freedom to produce structures which would significantly

deviate from flat geometry of the canonical dimension $d = 2$. It is generally very difficult to change fractal properties of the Lorentzian gravity. This can be achieved by strengthening the influence of the matter sector on geometry by increasing its conformal charge c. It was shown in [17,18] that a multiple Ising field with q-families, and the conformal charge $c = q/2 > 1$, modifies fractal properties of the underlying geometry leading to a space–time with the Hausdorff dimension $D_{\rm H} \approx 3$. The MC simulations [17,18] were done for $c = 4$ which is for technical reasons an optimal choice: $c = 4$ is large enough to allow for observing for relatively small lattices the effects of crossing the $c = 1$ barrier and on the other hand it is still not very large from the point of view of MC simulations, in particular of the computer time needed to update the matter sector, which grows linearly with c. We stick here to $c = 4$.

As mentioned we shall use fermionic particle to probe geometrical properties of the Lorentzian background. More precisely, we shall do this by studying the scaling properties of the lowest part of the spectrum of the Dirac–Wilson operator.

We consider a system of q species of Ising fields on dynamical Lorentzian triangulations with N triangles and T time slices. The canonical partition function of this system reads:

$$Z_{(q)}(\beta) = \sum_{l \in L_{N,T}} \Big(Z_l(\beta) \Big)^q , \tag{1}$$

where the sum runs over all triangulations l from the set of Lorentzian triangulations $L_{N,T}$ with N triangles and T time slices. Each triangulation l is dressed with q species of independent Ising spins and thus the weight of each triangulation in the ensemble is given by the q-th power of the partition function of a single Ising field on this triangulation:

$$Z_l(\beta) = \sum_{\{\sigma\}_l} \exp\left(\beta \sum_{(ab)} \sigma_a \sigma_b \right). \tag{2}$$

Here a, b and (ab) denote vertices and links of the Lorentzian triangulation l, respectively. Spins live on vertices. Each spin σ_a assumes two values $\sigma_a = \pm 1$. The sum $\{\sigma\}_l$ runs over all $2^{N/2}$ spin configurations of one spin family on the lattice l. Although spin families are independent on a given triangulation, they are not independent in the ensemble of triangulatons since they interact through dynamical lattices, which are summed over in the partition function (1).

The partition function for an individual spin family can be rewritten as a partition function for Ising spins living on vertices of the dual lattice l or

equivalently on triangles of the original lattice. The dual temperature β_* is related to β as $\tanh\beta_* = \exp -2\beta$. The equivalence between the original and dual model holds up to finite size effects [23]. The Ising model is also equivalent to a model of Wilson fermions for Majorana fields located on triangles:

$$\mathcal{Z}(K) = \sum_{l\in L_{N,T}} \Big(\mathcal{Z}_l(K)\Big)^q , \tag{3}$$

where each $\mathcal{Z}_l$ stands for for the partition function for Majorana fermions on a lattice l:

$$\begin{aligned} \mathcal{Z}_l(K) &= \int \prod_i d\bar\Psi_i d\Psi_i \exp\left(-\frac{1}{2}\sum_i \bar\Psi_i \Psi_i + K \sum_{\langle ij\rangle} \bar\Psi_i H_{ij} \Psi_j \right) \\ &= \int \prod_i d\bar\Psi_i d\Psi_i \exp\left(-\sum_{ij} \bar\Psi_i D_{ij} \Psi_i \right) \end{aligned} \tag{4}$$

with fermions located at the centers of triangles $i, j, \ldots$. The Dirac–Wilson operator $D = \mathbb{1}/2 + KH$ consists of a mass part and a hopping term controlled by the hopping parameter K. The sum in the hopping term runs over all oriented pairs $\langle ij\rangle$ of nearest triangles on the triangulation l. The hopping operator H_{ij} can be expressed in local frames as:

$$H_{ij} = \frac{1}{2}\,(1 + \vec{n}_{ij}\vec{\gamma})\,\mathcal{U}_{ij}\,, \tag{5}$$

where $\vec{n}_{ij}$ is a vector of the local derivative which goes between the neighbours i and j and $\mathcal{U}_{ij}$ is a spin connection in the spinorial representation. The components of the spinors Ψ_i are given in the local frames. The spin connection matrices allow for parallel transport of spinors between neighbouring frames and for recalculating spinor components. The hopping parameter K is related to the Ising temperature as:

$$K = \frac{\mathrm{e}^{-2\beta}}{\sqrt{3}} = \frac{\tanh(\beta_*)}{\sqrt{3}}\,. \tag{6}$$

The critical temperature of the Ising model corresponds to the critical value of the hopping parameter for which fermions become massless. The critical value for the Euclidean gravity can be analytically determined $\beta_{\rm cr} = \frac{1}{2}\ln\frac{131}{85} = 0.21627\ldots$ [26]. It corresponds to the critical value of the hopping parameter $K_{\rm cr} = 85\sqrt{3}/393 = 0.3746\ldots$ which should be compared with the critical value on the regular triangulated lattice: $K_{\rm cr} = 1/3 = 0.3333\ldots$ [26]. As one can see, the interaction with a random lattice dresses the critical value

of the hopping parameter similarly as interactions with gauge fields in lattice QCD. As we shall see the dressing of the hopping parameter is different for Lorentzian gravity.

The matrix of the Dirac–Wilson operator can be easily read off from the equation (4). The spectrum of the Dirac operator is related to the propagation of a fermionic particle through the lattice. The smallest eigenvalues are related to the effective mass of this particle. For an infinite lattice and at the critical value of the hopping parameter the theory describes a massless Majorana fermion. For a finite lattice there exists a non-vanishing mass gap which separates the lowest part of the spectrum from zero. This mass gap is minimal for certain value of the hopping parameter which we will refer to as pseudo-critical. We will denote this value as K_* and the corresponding mass gap as M_*. The two values change with the lattice size N and are expected to approach their limiting values $K_* \to K_{\rm cr}$ and $M_* \to 0$ for $N \to \infty$. In particular one expects the scaling:

$$M_* \sim N^{-1/d_{\rm H}} \tag{7}$$

with an exponent $d_{\rm H}$ which is related to the fractal properties of the underlying geometry. For an isotropic system, like for instance Euclidean gravity on a regular lattice, this exponent corresponds to the Hausdorff dimension $D_{\rm H}$. Lorentzian lattice is anisotropic. Its fractal dimensions in the temporal and spatial directions change with the matter content [17,18]. The spatial and temporal asymmetry becomes very transparent when one crosses the $c = 1$ barrier. In this case the system forms a bubble which is supplemented by a narrow long neck. Denote the temporal extension of the bubble by $T_{\rm B}$ and spatial by $L_{\rm B}$. The temporal and spatial extensions of the bubble scale differently with the size, $N_{\rm B}$, of the bubble $T_{\rm B} \sim N_{\rm B}^{1/D_{\rm H}}$ and $L_{\rm B} \sim N_{\rm B}^{1/\delta_h}$. The fractal dimensions $D_{\rm H}$ and δ_h are not independent. Using the relation $N_{\rm B} \sim T_{\rm B} L_{\rm B}$ one can see that

$$\frac{1}{D_{\rm H}} + \frac{1}{\delta_h} = 1\,. \tag{8}$$

In particular, for $c = 1/2$ the two exponents are $D_{\rm H} = 2$ and $\delta_h = 2$, merely reflecting the fact that the bubble is not developed and the temporal size of the bubble corresponds to the temporal extension of the system $T_{\rm B} \sim T$ and correspondingly $L_{\rm B} \sim N/T$. The situation changes dramatically for $c = 4$. In this case, $D_{\rm H} = 3$ and $\delta_h = 3/2$. In view of this asymmetry the following question arises. The scaling of the lowest part of the spectrum of the Dirac operator is expected to be controlled by the lowest momentum and thus in this case one can expect the mass exponent to be $d_{\rm H} = \delta_h$. On the other hand as discussed in [17,18] matter fields coupled to Lorentzian gravity even

above $c = 1$ barrier have flat space exponents which means that the fields behave effectively as in a flat 2d background. According to this hypothesis one should observe the value $d_{\mathrm{H}} = 2$ of the mass exponent. Which of the scenarios is realized in the system, is one of the questions addressed here.

3. Numerical set-up

Let us shortly describe our 'experimental' set-up. We use a MC generator to simulate a system of a given size N and a given temporal extent T. The average number of vertices per slice is $L = N/2T$ and hence the lattice asymmetry is $\tau = T/L = 2T^2/N$. The bulk thermodynamic properties of the system are expected to be independent of τ. This parameter can be thus used to minimize finite size effects.

Geometry of the lattice is updated by the standard local algorithm based on a pair of mutually reciprocal moves: split and join operations [17]. The transformations preserve the temporal length of the system T but change the lattice size $N \leftrightarrow N + 2$. In order to ensure ergodicity of this algorithm one allows the system size to fluctuate. In practice one does it by simulating a system with a partition function

$$z(\beta) = \sum_{l \in L_T} \exp\left(-\lambda n - \frac{1}{2\sigma^2}(n - N)^2\right) \left(Z_l(\beta)\right)^q \tag{9}$$

with a volume n which may fluctuate. In order to avoid too large fluctuations an external potential $U(n) = \lambda n + (n - N)^2/(2\sigma^2)$ is added to the action in (9). This potential constrains the volume fluctuations to a neighbourhood of N. The width of the distribution of n is of order σ. If the parameter λ is optimally tuned, the maximum of the distribution lies exactly at N.

The algorithm generates a smeared distribution of volumes n but measurements are performed only at $n = N$. The condition $n = N$ cuts out from the ensemble (9) a sub-ensemble with the partition function which is equal to (1) up to a constant factor irrelevant for statistical averages at N.

We concentrate the MC measurements on the spectrum of the Dirac–Wilson operator D_{ij} (4). Each triangle on the lattice is dressed with a two-component spinor, and hence for a lattice with N triangles the operator is represented by a $2N \times 2N$ matrix. The evaluation of the spectrum requires a time proportional to N^3. This is a very time consuming operation. We use the reduction to the Hessenberg form and then QR decomposition procedure [27]. Many problems of interest are related to the behaviour of the lowest part of the spectrum. We use the Lanczos algorithm to determine the position of the lowest eigenvalues. The Lanczos algorithm is most efficient for this purpose [27].

4. Results

The spectrum of the Dirac–Wilson operator is complex. As far as the lowest part of the spectrum is concerned it is more convenient to study the Majorana–Wilson operator instead of the Dirac–Wilson one

$$\mathcal{D} = CD \tag{10}$$

because it has a purely imaginary spectrum. C is the charge conjugation matrix. Indeed, if one chooses a representation in which the two-dimensional γ matrices are real: $\gamma_1 = \sigma_3$, $\gamma_2 = \sigma_1$, so is the charge conjugation matrix, $C = i\sigma_2 = \epsilon$ and the whole matrix of the Majorana–Wilson operator $\mathcal{D}$. Since the matrix $\mathcal{D}$ is also antisymmetric, it is anti-Hermitian.

From here on, when we refer to the lowest eigenvalues, we mean the closest to zero eigenvalues of the Majorana–Wilson operator. In fact, as mentioned already, it is rather this operator than the Dirac–Wilson one, which is related to the Ising spins and the conformal field with $c = 1/2$.

The construction of the Dirac–Wilson operator requires an introduction of a field of local frames on a simplicial manifold which locally defines gamma matrices and a spin connection $\mathcal{U}$. An explicit construction of the operator is given in [22, 23].

A snapshot of the spectrum of the Dirac–Wilson operator generated in a MC simulation of Lorentzian gravity is shown in Fig. 2. It should be compared with the spectrum on Euclidean lattice. As one can see there are some visual differences for eigenvalues with large absolute values. The differences obviously have the origin in the different properties of Lorentzian and Euclidean lattices on small distances. What is of physical interest is the small eigenvalue behaviour of the spectrum because it is responsible for the large distance behaviour and the universal critical properties of the system. This behaviour is governed by the scaling of the part of the spectrum closest to the origin of the complex plane. As will be shown, it is given by the mass exponent which has a different value for the Lorentzian than for the Euclidean case. The spectrum changes with the hopping parameter K. The main effect of this change on the shape of the spectrum is that it gets rescaled in the complex plane around the point $(1/2, 0)$ as follows directly from the form of the operator $\mathbb{1}/2 + KH$ which is a sum of a constant operator $\mathbb{1}/2$ and a random operator H multiplied by the factor K. In addition to this effect, the matter sector influences geometry of the lattice and hence also the randomness encoded in the operator H.

When the hopping parameter changes from $K = 0$ (which corresponds to $\beta = \infty$) to $K = 1/\sqrt{3}$ ($\beta = 0$) the spectrum broadens from a spectrum localized at the point $(1/2, 0)$ to an extended shape. In the course of the broadening the claws-shaped part of the spectrum passes close to the origin

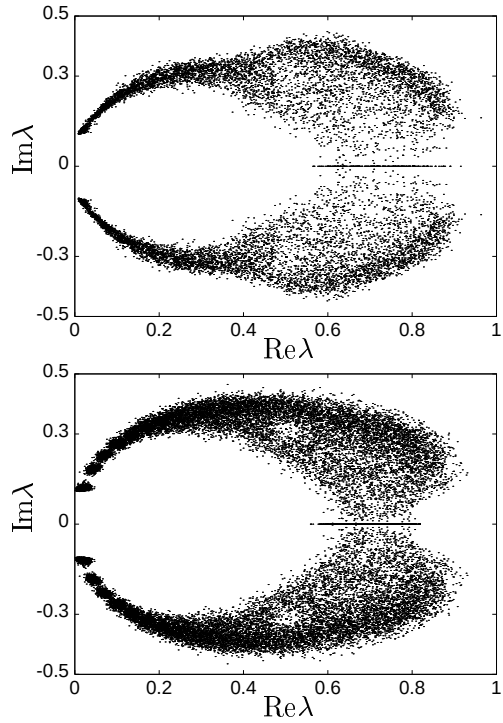


Fig. 2. Spectra of the Dirac–Wilson operator measured in MC simulations for Lorentzian lattice with $N = 64$ triangles, $\tau = 2$ and $K = 0.3486$ (top), and for Euclidean lattice with $N = 64$ triangles, $K = 0.364$ (bottom).

of the complex plane. The value of K, at which the distance of eigenvalues to the origin is smallest, corresponds to a pseudo-critical value K_*. A similar effect is seen in the movement of the lowest end of the spectrum of the Majorana–Wilson operator which first moves towards zero when K grows from zero to K_* and then moves away from zero when K_* further increases. We use this observation to determine the mass gap M_* for the system with a given volume in the following way. We determine the mean-value M of the distribution of the lowest eigenvalue of the Majorana–Wilson operator for the system with a size N and a hopping parameter K. Then we plot the dependence $M = M(N, K)$ as a function of K (see Fig. 3). Combining the quadratic interpolation with the jack-knife method we find the minimum $M_*(N)$ of the plotted function We repeat the same procedure for different volumes N to obtain the dependence of the pseudo-mass on the volume of the system $M_*(N)$. Eventually we fit the experimentally determined points $M_*(N)$ to the scaling formula (7) to determine the optimal value of the

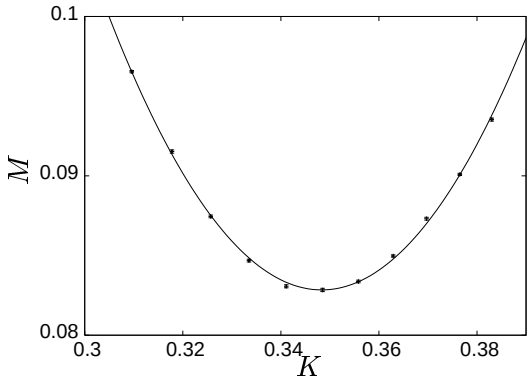


Fig. 3. Mas gap M as a function of hopping parameter K for a lattice with $N = 200$ triangles, asymmetry parameter $\tau = 4$ and the conformal charge $c = 1/2$. A parabola is fitted to the data.

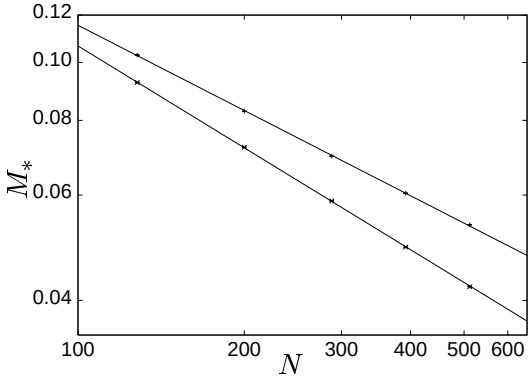


Fig. 4. Mas gap M_* of fermionic particle as a function of lattice volume N, for lattices with deformation parameter $\tau = 4$ and conformal charge $c = 1/2$ (upper line) and $c = 4$ (lower line). The fit $M_*(N) = aN^{-1/d_{\rm H}}$ gives $a = 1.02(2)$ and $d_{\rm H} = 2.11(5)$ for $c = 1/2$ and $a = 1.44(2)$ and $d_{\rm H} = 1.77(3)$.

scaling exponent $d_{\rm H}$. As an example, in Fig. 4 we show the experimental data points and the best fits to (7) for $c = 1/2$ and $c = 4$. The system size varies in the range from $N = 128$ to $N = 512$ and the asymmetry parameter is $\tau = 4$ for the presented data. A value of the pseudo-critical hopping parameter depends on the matter content. For example for the lattice size $N = 128$ it is $K = 0.3482(2)$ and $K = 0.3536(8)$ for $c = 1/2$ and $c = 4$, respectively. The corresponding value for Euclidean lattice of the same size is $K = 0.370(1)$.

The shape of the spectrum of the Dirac–Wilson operator for the pseudo-critical value of the hopping parameter K does not change visually when the system size N grows, except for the two claw ends which approach the origin of the complex plane as $N^{-1/d_{\rm H}}$. The two ends eventually close at the origin when N becomes infinite. This effect corresponds to the appearance of a massless particle on an infinite lattice. The shape of the bulk part of the spectrum does not change but merely becomes denser for larger lattices, which means that the typical distance between eigenvalues becomes smaller for larger system volumes.

The best fit to the scaling formula (7) gives a value $d_{\rm H} = 2.11(5)$ for $c = 1/2$. This value should be compared with $d_{\rm H} = 2.87(3)$ measured for Euclidean gravity interacting with the $c = 1/2$ matter [24]. Fermionic particle in the Lorentzian background, in contrast to Eucildean gravity, detects thus a flat space exponent.

For the case $c = 4$ for which geometry of the lattice changes dramatically, the value of the mass exponent is $d_{\rm H} = 1.77(3)$. Its value clearly moves towards the spatial scaling dimension $\delta_h = 3/2$ which is dictated by asymmetry of fractal properties of the lattice (8). As mentioned before, Lorentzian lattice coupled with matter field for $c = 4$ consists of two distinct parts: of the bubble whose extensions scale as $T_{\rm B} \sim N_{\rm B}^{1/D_{\rm H}}$ and $L_{\rm B} \sim N_{\rm B}^{1/\delta_h}$ and of a narrow neck whose spatial width does not scale. The presence of the narrow neck introduces a finite size effect to the measurements of the spectrum of the Dirac operator. This effect is probably the source of the deviation of the measured value $d_{\rm H} = 1.77(3)$ from $\delta_h = 3/2$. One should try to reduce the finite size effect by going to larger lattice for which the contribution of the neck to the spectrum should gradually decrease because the number of triangles on the bubble grows with lattice size much faster than the number of triangles in the neck. This is however very time consuming because the time for collecting the spectrum grows typically as the third power of the system size.

5. Summary

We implemented fermions to Lorentzian gravity and determined the spectrum of Dirac–Wilson operator. We calculated the mass gap exponent $d_{\rm H} = 2.11(5)$ for a single massless fermionic particle propagating in the Lorentzian background and interacting with it. The computations were done for lattices of size up to $N = 512$ triangles. The measured value of the mass exponent seems to be consistent with the canonical dimension $d = 2$, if one takes into account a possibility of finite size effects. For $c = 4$, above the $c = 1$ barrier, we measured $d_{\rm H} = 1.77(3)$. The value of the exponent moved towards the index $\delta_h = 3/2$ which controls the scaling of the lowest spatial momentum. In fact, if we explicitly introduce a finite size correc-

tion to the formula (7) of the form $M_* \sim N^{-1/d_H}(1 + c/N)$ the value of the exponent gets shifted to $d_H = 1.7(1)$ indicating indeed the presence of a deviation from the straight line in the measured volume range. It would be very helpful to extend the simulations to larger systems to see whether the observed tendency will indeed bring the exponent to the expected value 3/2. As we mentioned, the measurements of the spectrum for larger volume are very time consuming due to the strong dependence of the required CPU time on the volume. We plan to perform these computation in the future.

The measurements of the spectrum of the Dirac operator enable one to directly detect in the matter sector changes of the fractal structure of two-dimensional Lorentzian gravity. In other measurements of the critical indices of the matter sector one namely sees the flat space critical exponents even above the $c = 1$ barrier, for example, the Onsager exponents [17, 18] for the Ising model. The spectrum of the Dirac–Wilson operator is sensitive to changes of fractal structure and therefore it provides a practical tool for a detection of fractal properties of the geometrical background.

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