ZZ BRANES FROM A WORLDSHEET PERSPECTIVE*

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We show how non-compact space-time (ZZ branes) emerges as a limit of compact space-time (FZZT branes) for specific ratios between the square of the boundary cosmological constant and the bulk cosmological constant in the $(2, 2m - 1)$ minimal model coupled to two-dimensional Euclidean quantum gravity. Furthermore, we show that the principal (r, s) ZZ brane can be viewed as the basic (1,1) ZZ boundary state tensored with a (r, s) Cardy boundary state for a general (p, q) minimal model coupled to two-dimensional quantum gravity. In this sense there exists only one ZZ boundary state, the basic (1,1) boundary state.

1. Introduction

Two-dimensional Euclidean quantum gravity serves as a good laboratory for the study of potential theories of quantum gravity in higher dimensions. Although it contains no dynamical gravitons and does not face the problem of being non-renormalizable, it can address many of the other conceptional questions which confronts a *quantum field theory* of gravity. How does one define the concept of distance in a theory where one is instructed to integrate over all geometries, how does one define the concept of correlation functions when one couples matter to gravity and the resulting theory is supposed to be diffeomorphism invariant? These are just two of many questions which can be addressed successfully and which are as difficult to answer in two dimensions as in higher dimensions. In addition two-dimensional quantum

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gravity coupled to a minimal conformal field theory is nothing but a so-called non-critical string theory and serves as a good laboratory for the study of non-perturbative effects in string theory.

The quantization of 2d gravity was first carried out for compact two-dimensional geometries using matrix models, combinatorial methods and methods from conformal field theory. Later on Zamolodchikov, Zamolodchikov and Fateev and also Teschner (FZZT) used Liouville quantum field theory to quantize the disk geometry [2], thereby reproducing results already obtained from matrix models. Then the Zamolodchikovs (ZZ) turned their attention to a previously unadressed question crucial to quantum gravity, namely how to quantize non-compact 2d Euclidean geometries [1]. They asked if the quantization of the disk geometry could be generalized to the Lobachevskiy plane, also known as Euclidean AdS_2 or the pseudosphere. The pseudosphere is a non-compact space with no genuine boundary. However, one has to impose suitable boundary conditions at infinity in order to obtain a conformal field theory. The crucial difference compared to the quantization of the compact disk is that one can invoke the assumption of factorization when discussing the correlator of two operators. One assumes that

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle \rightarrow \langle \mathcal{O}_1(x)\rangle\langle \mathcal{O}_2(y)\rangle \tag{1}$$

when the *geodesic distance on the pseudosphere* between x and y goes to infinity. This additional requirement results in a number of self-consistent boundary conditions at infinity compatible with the conformal invariance of quantum Liouville theory.

It is the purpose of this article to address the question of quantizing non-compact 2D Euclidean geometries using a different approach than the Zamolodchikovs. However, our results will relate to the random geometries obtained by the Zamolodchikovs.

In modern string terminology boundary conditions are almost synonymous to "branes" and in this spirit the conventional partition function for the disk and the partition function for the pseudosphere were reinterpretated in non-critical string theory as FZZT and ZZ branes, respectively. In this context it was first noticed that there is an intriguing relationship between the FZZT and the ZZ branes [3–6], as well as between the analytical continuation of the disk amplitude to the complex plane and the space of conformal invariant boundary conditions one can impose.

It is the objective of this article to analyze these relations from a worldsheet perspective.

2. From compact to non-compact geometry

The disk and cylinder amplitudes for generic values of the coupling constants in minimal string theory were first calculated using matrix model techniques. In order to compare with continuum calculations performed in the context of Liouville theory, it is necessary to work in the so-called conformal background [7]. In the following we will, for simplicity, concentrate on the disk and the cylinder amplitudes in the $(2, 2m-1)$ minimal conformal field theories coupled to 2d quantum gravity. In the conformal background the disk amplitude is given by:

$$w_\mu(x) = (-1)^m \hat{P}_m(x, \sqrt{\mu})\sqrt{x + \sqrt{\mu}} = (-1)^m \left(\sqrt{\mu}\right)^{(2m-1)/2} P_m(t)\sqrt{t+1}\,, \tag{2}$$

where $t = x/\sqrt{\mu}$ and where [4, 7]

$$P_m^2(t)\,(t+1) = 2^{2-2m}(T_{2m-1}(t) + 1)\,. \tag{3}$$

$T_p(t)$ being the first kind of Chebyshev polynomial of degree p. In Eq. (2) x denotes the boundary cosmological coupling constant and μ the bulk cosmological coupling constant, the theory viewed as 2d quantum gravity coupled to the $(2, 2m-1)$ minimal CFT. The zeros of the polynomial $P_m(t)$ are all located on the real axis between -1 and 1 and more explicitly we can write:

$$
\begin{aligned}
P_m(t) &= \prod_{n=1}^{m-1} (t - t_n)\,, \\
t_n &= -\cos\left(\frac{2n\pi}{2m-1}\right)\,, \quad 1 \le n \le m-1\,.
\end{aligned}
\tag{4}
$$

The zeros of $P_m(t)$ can be associated with the $m-1$ principal ZZ branes in the notation of [4]. In order to *understand this*, *i.e.* in order to understand why the special values t_n (and only these values) of the boundary cosmological constant are related to non-compact worldsheet geometries, it is useful to invoke the so-called loop–loop propagator $G_\mu(x, y; d)$ [12–16]. It describes the amplitude of an "exit" loop with boundary cosmological constant y to be separated a distance d from an "entrance" loop with boundary cosmological constant x (the entrance loop conventionally assumed to have one marked point). $G_\mu(x, y; d)$ satisfies the following equation:

$$\frac{\partial}{\partial d}\, G_\mu(x, y; d) = -\frac{\partial}{\partial x}\, w_\mu(x) G_\mu(x, y; d)\,, \tag{5}$$

with the following solution:

$$G_\mu(x, y; d) = \frac{w_\mu(\bar{x}(d))}{w_\mu(x)} \frac{1}{\bar{x}(d) + y}, \qquad d = \int_{\bar{x}(d)}^{x} \frac{dx'}{w_\mu(x')}, \tag{6}$$

where $\bar{x}(d)$ is called the running boundary coupling constant.

For the $(2, 2m-1)$ minimal model coupled to 2d gravity (6) reads:

$$G_\mu(t, t'; d) \propto \frac{1}{\sqrt{\mu}} \frac{1}{\bar{t}(d) + t'} \frac{\sqrt{1 + \bar{t}(d)}}{\sqrt{1 + t}} \frac{\prod_{n=1}^{m-1} (\bar{t}(d) - t_n)}{\prod_{n=1}^{m-1} (t - t_n)}, \tag{7}$$

where we use the notation of (2), *i.e.* $t = x/\sqrt{\mu}$, $t' = y/\sqrt{\mu}$ and $\bar{t}(d) = \bar{x}(d)/\sqrt{\mu}$. For $m = 2$, *i.e.* pure gravity d measures the geodesic distance. For $m > 2$ this is not true. Rather, it is a distance measured in terms of matter excitations. This is explicit by construction in some models of quantum gravity with matter, for instance the Ising model and the $c = -2$ model formulated as an $O(-2)$ model [18, 19]. However, we can still use d as a measure of distance and we will do so in the following. When $d \to \infty$ it follows from (6) that the running boundary coupling constant $\bar{t}(d)$ converges to one of the zeros of the polynomial $P_m(t)$, *i.e.*

$$\bar{t}(d) \xrightarrow[d \to \infty]{} t_k, \qquad t_k = -\cos\left(\frac{2k\pi}{2m-1}\right). \tag{8}$$

The cylinder amplitude (7) vanishes for generic values of t' in the limit $d \to \infty$. However, as shown in [8] we have a unique situation when we choose $t' = -t_k$ since in this case the term $1/(\bar{t}(d) + t')$ in (7) becomes singular for $d \to \infty$. After some algebra we obtain the following expression:

$$G_\mu(t, t' = -t_k, d \to \infty) \propto \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{1+t}} \sum_{n=1}^{m-1} (-1)^n \sin\left(\frac{2n\pi}{2m-1}\right)$$

$$\times \left[\frac{1}{\sqrt{1+t} + \sqrt{1+t_n}} - \frac{1}{\sqrt{1+t} - \sqrt{1+t_n}}\right]. \tag{9}$$

Notice, $G_\mu(t, t' = -t_k, d \to \infty)$ is independent of which zero t_k the running boundary coupling constant approaches in the limit $d \to \infty$, apart from an overall constant of proportionality.

Formula (9) describes an AdS-like non-compact space with cosmological constant μ and with one compact boundary with boundary cosmological constant x as explained in [8] in the case of pure gravity. In the last section we will comment on the fact that we have to set $t' = -t_k$ in order to generate

an AdS-like non-compact space in the limit $d \to \infty$ and that t_k serves as an attractive fixed point for the running boundary coupling constant. Now, we will explain how the cylinder amplitude (9) is related to the conventional FZZT–ZZ cylinder amplitude in the Liouville approach to quantum gravity.

3. The cylinder amplitudes

Like the disk amplitude (2), the cylinder amplitude in the $(2, 2m-1)$ minimal CFT coupled to 2d quantum gravity was first calculated using the one-matrix model. Quite remarkable it was found to be universal, *i.e.* the same in all the $(2, 2m-1)$ minimal models coupled to quantum gravity [7,17]:

$$Z_\mu(t_1, t_2) = -\log\left[\left(\sqrt{t_1 + 1} + \sqrt{t_2 + 1}\right)^2 \sqrt{\mu}\, a\right], \qquad (10)$$

where a is a (lattice) cut-off.

The amplitude $Z_\mu(t_1, t_2)$ is only one of many cylinder amplitudes which in principle exist when we consider a $(2, 2m-1)$ minimal conformal field theory coupled to 2d gravity. If we consider the cylinder amplitude of the $(2, 2m-1)$ minimal conformal field theory before coupling to gravity we have available $m-1$ Cardy boundary states $|r\rangle_{\text{Cardy}}$, $r = 1, \ldots, m-1$, on each of the boundaries, and a corresponding cylinder amplitude for each pair of Cardy boundary states [10]:

$$Z_{\text{matter}}(r, s; q) = \sqrt{2}\, b \sum_{l=1}^{m-1} (-1)^{r+s+m+l+1} \frac{\sin(\pi r l b^2) \sin(\pi s l b^2)}{\sin(\pi l b^2)}\, \chi_l(q), \quad (11)$$

where

$$b = \sqrt{\frac{2}{2m-1}} \qquad (12)$$

and where we consider a cylinder with a circumference of 2π and length $\pi\tau$ in the closed string channel. The generic non-degenerate Virasoro character $\chi_p(q)$ is

$$\chi_p(q) = \frac{q^{p^2}}{\eta(q)}, \qquad q = e^{-2\pi\tau}, \qquad (13)$$

where $\eta(q)$ is the Dedekind function. However, the degenerate Virasoro character $\chi_l(q)$ in Eq. (11) is given by [11]:

$$\chi_l(q) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} \left(q^{(2n/b + 1/2(1/b - l\, b))^2} - q^{(2n/b + 1/2(1/b + l\, b))^2}\right). \qquad (14)$$

In order to couple the cylinder amplitude in Eq. (11) to 2d quantum gravity one has, in the conformal gauge, to multiply $Z_{\mathrm{mat}}(r, s; q)$ by a contribution $Z_{\mathrm{ghost}}(q)$ obtained by integrating over the ghost field, as well as by a contribution $Z_{\mathrm{Liouv}}(t_1, t_2; q)$ obtained by integrating over the Liouville field. Explicitly we have

$$Z_{\mathrm{ghost}}(q) = \eta^2(q), \quad Z_{\mathrm{Liouv}}(t_1, t_2; q) = \int_0^\infty dP \; \bar{\Psi}_{\sigma_1}(P) \, \Psi_{\sigma_2}(P) \chi_P(q), \quad (15)$$

where $\Psi_\sigma(P)$ is the FZZT boundary wave function [2], such that

$$\bar{\Psi}_{\sigma_1}(P) \, \Psi_{\sigma_2}(P) = \frac{4\pi^2 \cos(2\pi P \sigma_1) \cos(2\pi P \sigma_2)}{\sinh(2\pi P/b) \sinh(2\pi P b)}, \quad (16)$$

and where σ is related to the boundary cosmological constant by

$$\frac{x}{\sqrt{\mu}} \equiv t = \cosh(\pi b \, \sigma). \quad (17)$$

One finally obtains the full cylinder amplitude by integrating over the single real moduli τ of the cylinder:

$$Z_\mu(r, t_1; s, t_2) = \int_0^\infty Z_{\mathrm{ghost}}(q) Z_{\mathrm{Liouv}}(t_1, t_2; q) Z_{\mathrm{mat}}(r, s; q). \quad (18)$$

This cylinder amplitude depends not only on the Cardy states r, s, but also on the values of the boundary cosmological constants t_1, t_2 as well as the bulk cosmological constant μ.

From the discussion above it is natural that the matrix model (for a specific value of m) only leads to a single cylinder amplitude since it corresponds to an explicit (lattice) realization of the conformal field theory, and thus only to one realization of boundary conditions. In the language of Cardy states we want to identify *which* boundary condition is realized in the scaling limits of the one-matrix model. We do that by calculating the cylinder amplitude (18) and then comparing the result with the matrix model amplitude.

The calculation, using (11), (15) and (18), is in principle straight forward, but quite tedious, see [9] for some details. The result is for $r + s \leq m$ (for $r + s > m$ we have a slightly more complicated formula, which we will not

present here, but all conclusions are valid also in this case)[1]

$$Z_\mu(r, t_1; s, t_2) = - \sum_{k=1-r}^{r-1}{}' \sum_{l=1-s}^{s-1}{}' \log\left(\left[(\sqrt{t_1+1} + \sqrt{t_2+1})^2 - f_{k,l}(t_1, t_2)\right]\sqrt{\mu}a\right),$$

$$(19)$$

where a is the cut-off (as in (10)) and the summations are in steps of two, indicated by the primes in the summation symbols.

$$f_{k,l}(t_1, t_2) = 4\left[\sqrt{(t_1+1)(t_2+1)} + 2\cos^2\left(\frac{(k+l)\pi b^2}{4}\right)\right]\sin^2\left(\frac{(k+l)\pi b^2}{4}\right).$$

$$(20)$$

From Eqs. (19) and (20) it follows that *we have agreement with the matrix model amplitude (10) if and only if $r = s = 1$.* The $r = 1$ boundary condition is in the concrete realizations of conformal field theories related to the so-called fixed boundary conditions and for the matter part of the cylinder amplitude it corresponds to the fact, that only the conformal family of states associated with the identity operator propagates in the open string channel.

Following Martinec [3] it is now possible to calculate the FZZT–ZZ amplitude by replacing one of the FZZT wave functions in (15) with

$$\Psi_{\hat{n}}(P) \propto \Psi_{\sigma(\hat{n})}(P) - \Psi_{\sigma(-\hat{n})}(P),$$

$$(21)$$

where (in the $(2, 2m-1)$ models)

$$\sigma(\hat{n}) = i\left(\frac{1}{b} + \hat{n}\,b\right),$$

$$(22)$$

and where $\hat{n} = 1, \ldots, m-1$ is an integer labeling the different principal ZZ-branes.

Notice, the boundary cosmological constants $t_{\hat{n}}$ and $t_{-\hat{n}}$ corresponding to the complex valued $\sigma(\hat{n})$ and $\sigma(-\hat{n})$ are real and are actually the same for a given value of $\hat{n}$:

$$t_{\hat{n}} = t_{-\hat{n}} = -\cos\left(\frac{2\hat{n}\pi}{2m+1}\right),$$

$$(23)$$

i.e. they are the zeros of the polynomial $P_m(t)$ in formula (2). We now obtain the following FZZT–ZZ cylinder amplitude[2] for $r + s \le m$, differentiated

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[1] The prime in the summation symbol $\sum'$ means that the summation runs in steps of two.

[2] The upper sign in (24) is for $0 \le k + l + \hat{n}$, while the lower sign is for $k + l + \hat{n} \le 0$.

after the boundary cosmological constant on the FZZT brane:

$$Z'_\mu(r, \hat{n}; s, t) \;\propto\; \sum_{k=-(r-1)}^{r-1}{}' \sum_{l=-(s-1)}^{s-1}{}' \; \frac{(\pm)}{\sqrt{\mu}\sqrt{1+t}}$$

$$\times \left[\frac{1}{\sqrt{t+1}+\sqrt{1+t_{k+l+\hat{n}}}} - \frac{1}{\sqrt{t+1}-\sqrt{1+t_{k+l+\hat{n}}}} \right]. \quad (24)$$

The differentiation after the boundary cosmological constant is performed in order to compare with the corresponding amplitude $G_\mu(t, t' = -t_{\hat{n}}, d \to \infty)$ given by (9), which is the amplitude of a cylinder with one marked point on the compact boundary.

Let us now consider the FZZT-ZZ cylinder amplitude with an $r = 1$ Cardy matter boundary condition imposed on the FZZT boundary. This is the natural choice if we want to compare with the matrix model results since the Cardy matter boundary condition captured by the matrix model is precisely $r = 1$. In this case the summation over s is not present in Eq. (24) and comparing formula (24) with the expression (9) for $G_\mu(t, -t_{\hat{n}}, d \to \infty)$ one can show that

$$G_\mu(t, -t_{\hat{n}}, d \to \infty) \propto \sum_{r=1}^{m-1} S_{1,r}\, Z'_\mu(r, \hat{n}; 1, t)\,, \quad (25)$$

where $S_{k,l}$ is the modular S-matrix in the $(2, 2m-1)$ minimal CFT, *i.e.* [11]

$$S_{k,l} = \sqrt{2}\, b\, (-1)^{m+k+l} \sin(\pi k l\, b^2)\,. \quad (26)$$

This result is valid for any $(2, 2m-1)$ minimal CFT coupled to quantum gravity and is valid independent of which zero t_k the running boundary coupling constant approaches in the limit $d \to \infty$. The proof of (25) is straight forward but tedious and will not be given here (see [9] for some details).

The natural interpretation of Eq. (25) is that the matter boundary state of the exit loop in the loop–loop amplitude $G_\mu(t, -t_{\hat{n}}, d)$ is projected on the following linear combination of Cardy boundary states in the limit $d \to \infty$:

$$|a\rangle = \sum_{r=1}^{m-1} S_{1,r}\, |r\rangle_{\mathrm{Cardy}} \propto |1\rangle\rangle\,, \quad (27)$$

where the last state is the Ishibashi state corresponding to the identity operator and where we have used the orthogonality properties of the modular S-matrix and the relation between Cardy states and Ishibashi states:

$$|r\rangle_{\mathrm{Cardy}} = \sum_{k=1}^{m-1} \frac{S_{r,k}}{\sqrt{S_{1,k}}}\, |k\rangle\rangle\,. \quad (28)$$

The Ishibashi state corresponding to the identity operator is in a certain way the simplest boundary state available, and it is remarkable that it is precisely this state which is captured by the explicit transition from compact to non-compact geometry enforced by taking the distance $d \to \infty$.

4. The nature of ZZ branes

A ZZ-brane is defined as the tensorproduct of a ZZ boundary state $|r, s\rangle_{\text{zz}}$ and a Cardy matter state $|k, l\rangle_{\text{Cardy}}$. Hence, in addition to specifying a ZZ boundary condition, we have to impose a Cardy matter state at infinity. In the original article by the Zamolodchikovs only Liouville field theory was considered [1]. However, their line of reasoning relied crucially on the interpretation of quantum Liouville theory as describing 2d quantum gravity. Invarians under diffeomorphisms demands that the total central charge is zero. Hence, for a given value of the Liouville central charge we should think of the corresponding matter and ghost fields (which have central charges such that the total central charge is zero) as having been integrated out. In the article of the Zamolodchikovs the nature of the various boundary states at infinity was unclear. The successive work in the context of non-critical string theory [4–6] showed how to reduce the possible ZZ branes to a number of principal ZZ branes. However, the origin of precisely these principal ZZ branes remained somewhat of a mystery.

In (p, q) minimal non-critical string theory the principal ZZ-branes are defined as

$$|1, 1\rangle_{\text{Cardy}} \otimes |r, s\rangle_{\text{zz}} , \tag{29}$$

where $1 \leq r \leq p-1$, $1 \leq s \leq q-1$ and $rq - sp > 0$. It turns out that we may interpret the $(p-1)(q-1)/2$ different principal ZZ branes in (p, q) minimal string theory as matter dressed basic $(1, 1)$ ZZ boundary states [23]:

$$|1, 1\rangle_{\text{Cardy}} \otimes |r, s\rangle_{\text{zz}} = |r, s\rangle_{\text{Cardy}} \otimes |1, 1\rangle_{\text{zz}} . \tag{30}$$

Eq. (30) should be understood in the following way: With regard to expectation values of physical observables it does not matter whether we use the right hand side or the left hand side of Eq. (30). Thus, in this sense there exists only one ZZ boundary condition, the basic $(1, 1)$ boundary condition. Furthermore, we have the following generalization of (30):

$$|k, l\rangle_{\text{Cardy}} \otimes |r, s\rangle_{\text{zz}} = \left(\sum_{i=|r-k|+1}^{\text{top}(r,k;p)}{}' \sum_{j=|s-l|+1}^{\text{top}(s,l;q)}{}' |i, j\rangle_{\text{Cardy}} \right) \otimes |1, 1\rangle_{\text{zz}} , \tag{31}$$

where

$$\text{top}(a, b; c) \equiv \min(a + b - 1, 2c - 1 - a - b) . \tag{32}$$

Notice, this summation is precisely the same which appears in the fusion of two primary operators in the (p, q) minimal conformal field theory:

$$O_{k,l} \times O_{r,s} = \sum_{i=|r-k|+1}^{\text{top}(r,k;p)}{}' \sum_{j=|s-l|+1}^{\text{top}(s,l;q)}{}' [O_{i,j}]. \tag{33}$$

Why are Eqs. (30) and (31) true? (we refer to [23] for the full details of the proof.)

Recall the definition of the Cardy matter boundary states in the (p, q) minimal conformal field theory:

$$|k, l\rangle_{\text{Cardy}} \equiv \sum_{i,j} \frac{S(k, l; i, j)}{\sqrt{S(1, 1; i, j)}} |i, j\rangle\rangle, \tag{34}$$

where the summation runs over all the different Ishibashi states $|i, j\rangle\rangle$ in the (p, q) minimal model and

$$S(k, l; i, j) = 2\sqrt{\frac{2}{pq}}(-1)^{1+kj+li} \sin(\pi b^2 lj) \sin(\pi ki/b^2), \tag{35}$$

is the modular S-matrix in the (p, q) minimal model. The Cardy matter boundary states are labeled by two integers (k, l), which satisfy that $1 \leq k \leq p-1$, $1 \leq l \leq q-1$ and $kq - lp > 0$.

On the other hand the principal ZZ boundary states are defined as

$$|r, s\rangle_{\text{zz}} = \int_0^\infty dP \, \frac{\sinh(2\pi r P/b) \, \sinh(2\pi s Pb)}{\sinh(2\pi P/b) \, \sinh(2\pi Pb)} \, \Psi_{1,1}(P) |P\rangle\rangle, \tag{36}$$

where $b = \sqrt{p/q}$. $\Psi_{1,1}(P)$ is the basic ZZ wave function [1]:

$$\Psi_{1,1}(P) = \beta \frac{iP\mu^{-iP/b}}{\Gamma(1 - 2iPb)\Gamma(1 - 2iP/b)}, \tag{37}$$

where the constant β is independent of the cosmological constant μ and P. Finally, $|P\rangle\rangle$ denotes the Ishibashi state corresponding to the non-local primary operator $\exp(2(Q/2 + iP)\phi)$ in Liouville theory, where $Q = b + 1/b$.

Notice, the ranges of the indices k, l labeling the different Cardy matter boundary states and the indices r, s labeling the principal ZZ branes are the same. As noted already by the Zamolodchikovs in [1], the modular bootstrap equations for the ZZ boundary states are surprisingly similar to the bootstrap equations for the Cardy matter boundary states in the minimal

models. The key point is now that the physical operators in minimal string theory carry both a matter "momentum" and a Liouville "momentum" and these are not independent, but related by the requirement that the operators scale in a specific way. In particular, the Liouville momenta P of the physical observables are imaginary and the imaginary i explains the shift from sin to sinh going from (34) to (36). The coupling between the matter and Liouville momenta implies, that physical expectation values will be the same irrespectively of whether we use the left or the right side of Eq. (30).

Our interpretation of the term

$$\frac{\sinh(2\pi rP/b) \ \sinh(2\pi sP\, b)}{\sinh(2\pi P/b) \ \sinh(2\pi P\, b)} \tag{38}$$

in the definition of the principal (r, s) ZZ boundary state (36) as a dressing factor arising from the integration over the matter and the ghost fields becomes evident when considering the cylinder amplitude. For simplicity we only consider this amplitude in $(2, 2m - 1)$ minimal string theory. The cylinder amplitude does not factorize into a matter part and a Liouville part. The integration over the single real moduli τ correlates matter with geometry. If one imposes the $(1, s)$ Cardy matter state on a ZZ boundary and performs the integrations over both τ, the matter and the ghost fields, the ZZ boundary wave function get dressed exactly with the term (38) with $r = 1$ [23].

5. Discussion

We have shown how it is possible to construct an explicit transition from compact to non-compact geometry in the framework of 2d quantum gravity coupled to conformal field theories. The non-compact geometry is AdS-like in the sense that the average area and the average length of the exit loop diverge exponentially with d when $d \to \infty$ as shown in [8] (for pure gravity), and the corresponding amplitude can be related to the FZZT–ZZ cylinder amplitude with the simplest Ishibashi state living on the ZZ brane. The $d \to \infty$ limit plays an instrumental role and we would like to address two important aspects of this.

Firstly, our construction also adds to the understanding of the relation (21) discovered by Martinec. In Liouville theory there is a one-to-one correspondence between the ZZ boundary states labeled by (m, n) and the degenerate primary operators $V_{m,n}$ [1]. This correspondance completely determines the Liouville cylinder amplitude with two ZZ boundary conditions: The spectrum of states flowing in the open string channel between two ZZ boundary states is obtained from the fusion algebra of the corresponding degenerate operators. Similarly, there is a one-to-one correspondence between

the FZZT boundary states labeled by $\sigma > 0$ and the non-local "normalizable" primary operators $V_\sigma = \exp((Q + i\sigma)\phi)$, where ϕ is the Liouville field. The conformal dimension of the spin-less degenerate primary operator $V_{m,n}$ is given by

$$\Delta_{m,n} = \frac{Q^2 - (m/b + nb)^2}{4}, \qquad (39)$$

while the conformal dimension of the spin-less non-local primary operator V_σ is given by

$$\Delta_\sigma = \frac{Q^2 + \sigma^2}{4}. \qquad (40)$$

Since $\Delta_{m,n} = \Delta_\sigma$ for $\sigma = i(m/b + nb)$, one is naively led to the wrong conclusion, that a FZZT boundary state turns into a ZZ boundary state, if one tunes $\sigma = i(m/b + nb)$. However, the operator $V_{m,n}$ is degenerate and in addition to setting $\sigma = i(m/b + nb)$ we therefore have to truncate the spectrum of open string states, that couple to the FZZT boundary state, in order to obtain a ZZ boundary state. This is precisely captured in the relation (21) concerning the principal ZZ boundary states. The world-sheet geometry characterizing the FZZT brane is compact, while the world-sheet geometry of the ZZ-brane is non-compact. Hence, truncating the spectrum of open string states induces a transition from compact to non-compact geometry.

In order to clarify how this truncation is obtained in our concrete realization of a transition from compact to non-compact geometry, we have to discuss the boundary cosmological constant of the exit loop. The cylinder amplitude (6) may be expressed as

$$G_\mu(x, y; d) = \int_0^\infty dl\, e^{-yl} G_\mu(x, l; d), \qquad (41)$$

where the cylinder amplitude $G_\mu(x, l; d)$ with fixed length l of the exit loop is given by

$$G_\mu(x, l; d) = e^{-\bar{x}(d)l} \frac{w_\mu(\bar{x}(d))}{w_\mu(x)}. \qquad (42)$$

Hence, an interpretation of the running boundary coupling constant $\bar{t} = \bar{x}(d)/\sqrt{\mu}$ (measured in units of $\sqrt{\mu}$) as a boundary cosmological constant *induced* on the exit loop seems obvious. Notice, this induced boundary cosmological constant approaches one of the values t_k associated with the ZZ-branes in the limit $d \to \infty$. However, an AdS geometry emerges in the limit $d \to \infty$ if and only if we set the boundary cosmological constant of the exit loop $y/\sqrt{\mu} = -t_k$.

The induced boundary cosmological constant approaches one of the zeros t_k in the limit $d \to \infty$ regardless of whether we set $y/\sqrt{\mu} = -t_k$ *i.e.* regardless of whether we generate an AdS geometry or not. Hence, these discrete values of the boundary cosmological constant induced at infinity seems to be generic to non-compact geometries. This suggests, that we should regard the boundary cosmological constants associated with the ZZ-branes as induced.

Secondly, in [4] it was advocated that the algebraic surface

$$T_p \left(\frac{w}{C_{p,q}(\mu)} \right) = T_q(t) \,, \tag{43}$$

where $C_{p,q}(\mu)$ is a constant, is the natural "target space" of (p, q) non-critical string theory. For $(p, q) = (2, 2m-1)$ Eq. (43) reads

$$w^2 = \mu^{(2m-1)/2} P_m^2(t)(t + 1) \,, \tag{44}$$

and in this case the extended target space is a double sheeted cover of the complex t-plane except at the singular points, which are precisely the points $(t_k, w=0)$ associated with the zeros of the polynomial $P_m(t)$. One is also led to this extended target space from the world-sheet considerations made here. We want the running boundary coupling constant to be able to approach any of the fixed points t_k in the limit $d \to \infty$, *i.e.* we want all the fixed points to be attractive. This is only possible if we consider the running boundary coupling constant $\bar{t}(d) = \bar{x}(d)/\sqrt{\mu}$ as a function taking values on the algebraic surface defined by (44). The reason is that t_k is either an attractive or a repulsive fixed point depending on which sheet we consider and some of the fixed points are attractive on one sheet, while the other fixed points are attractive on the other sheet. Hence, we are forced to view $\bar{t}(d)$ as a map to the double sheeted Riemann surface defined by Eq. (44) in the $(2, 2m-1)$ minimal model coupled to quantum gravity.

The picture becomes particularly transparent if we use the uniformization variable z introduced for the (p, q) non-critical string in [4] by

$$t = T_p(z), \quad w/C_{p,q}(\mu) = T_q(z) \,, \tag{45}$$

i.e. in the case of $(p, q) = (2, 2m-1)$:

$$z = \frac{1}{\sqrt{2}} \sqrt{t + 1} \,. \tag{46}$$

The map (45) is one-to-one from the complex plane to the algebraic surface (43), except at the singular points of the surface where it is two-to-one. The singular points are precisely the points corresponding to ZZ branes. If we

change variables from x to z in Eq. (5) (choosing $\mu = 1$ for simplicity) we obtain

$$\frac{\partial}{\partial d}\,\tilde{G}_\mu(z, z'; d) = -\frac{\partial}{\partial z}\,\tilde{P}_m(z)\tilde{G}_\mu(z, z'; d)\,, \tag{47}$$

where $\tilde{G}_\mu(z, z'; d) = zG_\mu(x, y; d)$ and where the polynomial $\tilde{P}_m(z)$ is

$$\tilde{P}_m(z) \propto \prod_{k=1}^{m-1}(z^2 - z_k^2)\,, \quad z_k = \sin\left(\frac{\pi}{2}b^2\,k\right). \tag{48}$$

Each zero t_k of $P_m(t)$ gives rise to two zeros $\pm z_k$ of $\tilde{P}_m(z)$. The zeros $\pm z_k$ are the fixed points of the running "uniformized" boundary cosmological constant $\bar{z}$ associated with the characteristic equation corresponding to Eq. (47). For a given value of k one of the two zeros $\pm z_k$ is an attractive fixed point, while the other is repulsive. Moving from one sheet to the other sheet on the algebraic surface (44) corresponds to crossing the imaginary axis in the z-plane. Hence, for a given value of k the two fixed points $\pm z_k$ are each associated with a separate sheet and $\bar{z}$ will only approach the attractive of the two fixed points $\pm z_k$, if $\bar{t}(d)$ belongs to the correct sheet.

Quite remarkable Eq. (47) was derived in the case of pure 2d gravity (the $(2, 3)$ model corresponding to $c = 0$) using a completely different approach to quantum gravity called CDT[3] (causal dynamical triangulations) [20] and the uniformization transformation relating the CDT boundary cosmological constant z to the boundary cosmological constant t was derived and given a world-sheet interpretation in [21], but again from a different perspective. From the CDT loop–loop amplitude determined by (47) one can define a CDT "ZZ brane" with non-compact geometry [22].

REFERENCES

[1] A.B. Zamolodchikov, A.B. Zamolodchikov, `hep-th/0101152`.

[2] V. Fateev, A.B. Zamolodchikov, A.B. Zamolodchikov, `hep-th/0001012`.

[3] E.J. Martinec, `hep-th/0305148`.

[4] N. Seiberg, D. Shih, *J. High Energy Phys.* **0402**, 021 (2004) [`hep-th/0312170`].

[5] D. Kutasov, K. Okuyama, J. Park, N. Seiberg, D. Shih, `hep-th/0406030`.

[6] A. Basu, E.J. Martinec, *Phys. Rev.* **D72**, 106007 (2005) [`hep-th/0509142`].

[7] G.W. Moore, N. Seiberg, M. Staudacher, *Nucl. Phys.* **B362**, 665 (1991).

[8] J. Ambjorn, S. Arianos, J.A. Gesser, S. Kawamoto, *Phys. Lett.* **B599**, 306 (2004) [`hep-th/0406108`].

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[3] It should be noted that CDT seemingly has an interesting generalization to higher dimensional quantum gravity theories [24].

[9] J. Ambjorn, J.A. Gesser, *Phys. Lett.* **B653**, 439 (2007) [`hep-th/0706.3231`].

[10] R.E. Behrend, P.A. Pearce, V.B. Petkova, J. Zuber, *Nucl. Phys.* **B579**, 707 (2000) [`hep-th/9908036`].

[11] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Springer, New York 1997, p. 890.

[12] H. Kawai, N. Kawamoto, T. Mogami, Y. Watabiki, *Phys. Lett.* **B306**, 19 (1993) [`hep-th/9302133`].

[13] N. Ishibashi, H. Kawai, *Phys. Lett.* **B314**, 190 (1993) [`hep-th/9307045`].

[14] S.S. Gubser, I.R. Klebanov, *Nucl. Phys.* **B416**, 827 (1994) [`hep-th/9310098`].

[15] Y. Watabiki, *Nucl. Phys.* **B441**, 119 (1995) [`hep-th/9401096`].

[16] J. Ambjorn, Y. Watabiki, *Int. J. Mod. Phys.* **A12**, 4257 (1997) [`hep-th/9604067`].

[17] J. Ambjorn, J. Jurkiewicz, Yu.M. Makeenko, *Phys. Lett.* **B251**, 517 (1990).

[18] J. Ambjorn, C. Kristjansen, Y. Watabiki, *Nucl. Phys.* **B504**, 555 (1997) [`hep-th/9705202`].

[19] J. Ambjorn, K.N. Anagnostopoulos, J. Jurkiewicz, C.F. Kristjansen, *J. High Energy Phys.* **9804**, 016 (1998) [`hep-th/9802020`].

[20] J. Ambjorn, R. Loll, *Nucl. Phys.* **B536**, 407 (1998) [`hep-th/9805108`].

[21] J. Ambjorn, J. Correia, C. Kristjansen, R. Loll, *Phys. Lett.* **B475**, 24 (2000) [`hep-th/9912267`].

[22] J. Ambjorn, R. Janik, W. Westra, S. Zohren, *Phys. Lett.* **B641**, 94 (2006) [`gr-qc/0607013`].

[23] J. Ambjorn, J.A. Gesser, [`hep-th/0707.3431`].

[24] J. Ambjorn, J. Jurkiewicz, R. Loll, *Phys. Rev.* **D72**, 064014 (2005), [`hep-th/0505154`]; *Phys. Rev. Lett.* **93**, 131301 (2004), [`hep-th/0404156`]; *Phys. Rev. Lett.* **95**, 171301 (2005), [`hep-th/0505113`]; *Phys. Lett.* **B607**, 205 (2005), [`hep-th/0411152`]; *Phys. Rev.* **D64**, 044011 (2001), [`hep-th/0011276`].