ON THE QUANTIZATION OF DAMPED HARMONIC OSCILLATOR

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We derive results for two constants of the motion of a one-dimensional damped harmonic oscillator with position-dependent frictional coefficient and use them to obtain two alternative Lagrangian representations, which are not connected by a gauge term. The Hamiltonians corresponding to these Lagrangians lead to canonically inequivalent phase-space descriptions. We could, however, make use of a perturbation theoretic approach to quantize the classical motion using both Hamiltonians and thus demonstrate that the corresponding quantum systems are entirely different.

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1. Introduction

Studies in the quantization of dissipative systems play a role in several areas of physics ranging from electrodynamics to chromodynamics. In spite of this, the problem of quantizing the simplest dissipative system, namely, the damped harmonic oscillator (DHO), has yet remained largely unsolved [1]. The primary reason for this may be attributed to an early observation made by Lanczos [2] who noted that the forces of frictions are outside the realm of variational principle although Newtonian mechanics has no difficulty to accommodate them. In the recent past, Riewe [3] formulated the Lagrangian and Hamiltonian mechanics of dissipative systems within the framework of fractional calculus. It may be an interesting curiosity to adapt the formalism of Riewe to deal with the problem of quantizing the DHO. However, we are interested to treat a simple variant of the DHO using usual traditional machineries of classical and quantum mechanics.

The usual equation of motion for the one dimensional DHO is given by

$$m\ddot{x} + \alpha\dot{x} + kx = 0\,, \tag{1}$$

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where m is the mass of the oscillator, α the coefficient of friction of the medium in which the oscillator moves. Clearly, k is the spring constant. Equation (1) is non self-adjoint. It can easily be made self-adjoint and see that (1) follows from the action principle

$$\delta \int_{t_1}^{t_2} L dt = 0 \,, \tag{2}$$

with the Lagrangian

$$L = \frac{1}{2} e^{\frac{\alpha}{m} t} \left(\dot{x}^2 - \frac{k}{m} x^2 \right) . \tag{3}$$

Equation (3) gives the so-called Caldirola–Kanai Lagrangian [4]. From (3) the canonical momentum

$$p_x = e^{\frac{\alpha}{m} t} \, \dot{x} \,. \tag{4}$$

Equation (4) exhibits that the canonical momentum is different from kinetic momentum, also the former is explicitly time dependent. These two points do not permit one to unambiguously quantize the DHO of (1) by canonical procedure. Since the Lagrangian is explicitly time dependent one also cannot quantize the system by the path integral method. Although it appears that there is no direct route to quantize the motion of the DHO, one may consider the one dimensional DHO in the presence of its time-reversed image and write an explicitly time-independent Lagrangian for the system of equations often designated as the Bateman dual system [5]. The presence of the mirror-image equation implies that energy drained from the DHO is absorbed by the image oscillator. Thus here one tries to handle the dissipative systems as if they were conserved. Since the Lagrangian or equivalently Hamiltonian of the Bateman dual system is explicitly time independent, it could be quantized by the usual canonical procedure [6]. Blasone and Jizba [7], however, studied the quantization of the damped-antidamped harmonic oscillator system by the use of Feynman–Hibbs kernel formula.

In the above context we introduce a model for the DHO with position dependent dissipative coefficient and quantize the system by the canonical procedure. The equation of our interest is given by

$$m\ddot{x} - \alpha k x \dot{x} + kx = 0 \,, \qquad \alpha < 0 \,. \tag{5}$$

In Sec. 2 we show that (5) follows from two alternative Lagrangians both of which are explicitly time independent. Therefore, one can use the Hamiltonian corresponding to any of the Lagrangians to quantize the system. The alternative Lagrangians are not connected by gauge terms. Despite that, they are harmless at the classical level in the sense that alternative

Lagrangians via the action principle give the same equation of motion [8]. Morandi *et al.* [9] have established that alternative Lagrangian descriptions of a given physical system lead to entirely different quantum mechanical systems. Keeping this in view we exhibit in Sec. 3 the quantum dynamical systems corresponding to the alternative Lagrangians representing (5) and calculate their eigen-energies using a perturbation theoretic approach. In Sec. 4 we make some concluding remarks.

2. Alternative Lagrangians and Hamiltonians

The differential equation in (5) represents an autonomous mechanical system in one spatial dimension. We know from Darboux [10] that the Lagrangian for a one-dimensional autonomous system always exists. We shall construct here two alternative Lagrangian representations of (5) and thus obtain the corresponding Hamiltonians. Our construction procedure will be based on a simple variant of the inverse variational method [11] in which the Lagrangian is constructed using a first integral of the dynamical equation. We shall make use of the method of characteristics for solving the first-order partial differential equations to derive two different expressions for the constant of the motion or first integral of the dynamical equation. This will allow us to arrive at alternative Lagrangian descriptions of (5). Equation (5) can be written in the autonomous form

$$v\frac{dv}{dx} = -\frac{kx}{m}(1-\alpha v)\,, \tag{6}$$

with $v = dx/dt$, the instantaneous velocity of the oscillating mass. A constant of the motion for this system is a function $K = K(x, v)$ such that $dK/dt = 0$. The function $K(x, v)$ satisfies the first-order partial differential equation

$$v\frac{\partial K}{\partial x} - \frac{kx}{m}(1-\alpha v)\frac{\partial K}{\partial v} = 0\,. \tag{7}$$

The general solution for this equation is given by [12]

$$K(x, v) = G(C)\,, \tag{8}$$

where G is an arbitrary function of the characteristic curve C of (7). The characteristic curve can be obtained as

$$C = -\frac{m}{\alpha^2}\left[\ln(1-\alpha v) + \alpha v\right] + \frac{1}{2}kx^2\,. \tag{9}$$

The function G can be chosen such that in the limit of no dissipation one can get the usual energy expression $E = \lim_{\alpha\to 0} K = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ for the

undamped harmonic oscillator. One way of doing this is to select G as the identity function so that $G(C) = C$. Thus the first constant of the motion for (5) is given by

$$K^{(1)}(x, v) = C\,. \tag{10}$$

The Lagrangian $L(x, v)$ of an autonomous system is related to the constant of the motion $K(x, v)$ according to [11]

$$L(x, v) = v \int^{v} \frac{K(x, \xi)}{\xi^2} d\xi\,. \tag{11}$$

From (10) and (11) we get

$$L^{(1)}(x, v) = \frac{m}{\alpha^2} \ln(1 - \alpha v) + \frac{mv}{\alpha} \left[1 - \ln(1 - \alpha v)\right] - \frac{1}{2} k x^2 \tag{12}$$

as the Lagrangian for our first choice regarding the function G. Rather than taking G as an identity function we could also take

$$G(C) = -\frac{m}{\alpha^2} e^{-\frac{\alpha^2}{m} C} + \frac{m}{\alpha^2}\,. \tag{13}$$

The choice in (13) gives a second constant of the motion

$$K^{(2)}(x, v) = \frac{m}{\alpha} v e^{\alpha v} e^{-\frac{k}{2m}\alpha^2 x^2} + \frac{m}{\alpha^2} \left(1 - e^{\alpha v} e^{-\frac{k}{2m}\alpha^2 x^2}\right) \tag{14}$$

for which we can write a second Lagrangian

$$L^{(2)}(x,\, v) = \frac{m}{\alpha^2} e^{\alpha v} e^{-\frac{k}{2m}\alpha^2 x^2} - \frac{m}{\alpha^2}\,. \tag{15}$$

Clearly the Lagrangians $L^{(1)}(x,\, v)$ and $L^{(2)}(x,\, v)$ are gauge inequivalent. It is easy to check that both of them give the Lagrangian for the undamped harmonic oscillator in the limit $\alpha \to 0$. The Hamiltonians corresponding to $L^{(1)}(x,\, v)$ and $L^{(2)}(x,\, v)$ are given by

$$H^{(1)} = \frac{p_1}{\alpha} - \frac{m}{\alpha^2} \left(1 - e^{-\frac{\alpha p_1}{m}}\right) + \frac{1}{2} k x^2 \tag{16}$$

and

$$\begin{aligned} H^{(2)} \;=\; & \frac{1}{\alpha}\, p_2 \ln\left(1 + \frac{\alpha}{m} p_2\right) + \frac{\alpha k}{2m}\, p_2 x^2 \\ & - \frac{1}{\alpha} \left(p_2 + \frac{m}{\alpha}\right) + \frac{m}{\alpha^2} + \frac{m}{\alpha^2} \ln\left(1 + \frac{\alpha}{m}\, p_2\right) + \frac{1}{2} k x^2\,, \end{aligned} \tag{17}$$

with the canonical momenta

$$p_1 = \frac{m}{\alpha} \ln \left(\frac{1}{1 - \alpha v} \right) \tag{18}$$

and

$$p_2 = \frac{m}{\alpha} e^{\alpha v} e^{-\frac{k}{2m}\alpha^2 x^2} - \frac{m}{\alpha} \,. \tag{19}$$

One can verify that both $H^{(1)}$ and $H^{(2)}$ via the Hamilton's canonical equations provide us with the same equation of motion as given in (5). These Newton-equivalent Hamiltonians, however, lead to canonically inequivalent phase-space descriptions [13, 14]. The Hamilton's equations $\dot{x} = \{x,\, H_1\}$ and $\dot{p}_1 = \{p_1,\, H_1\}$ correspond to the equation of motion (5) endowed with the Poisson structure

$$\{x,\, p_1\}_{x,p_1} = 1\,, \qquad \{v,\, p_1\}_{x,p_1} = 0\,, \quad \text{and} \quad \{x,\, v\}_{x,p_1} = \frac{1}{m} e^{-\frac{\alpha p_1}{m}} \,. \tag{20}$$

From (18) and (19) we can write

$$p_1(x,\, p_2) \;=\; \frac{m}{\alpha} \ln \left(1 - \frac{k}{2m}\alpha^2 x^2 - \ln \left(1 + \frac{\alpha p_2}{m} \right) \right)^{-1} \,, \tag{21}$$

$$v(x,\, p_2) \;=\; \frac{1}{\alpha} \ln \left(1 + \frac{\alpha p_2}{m} \right) + \frac{k}{2m}\alpha^2 x^2 \,. \tag{22}$$

Equations (21) and (22) enable us to find the noncanonical Poisson brackets

$$\{x,\, p_1(x,\, p_2)\}_{x,p_2} = \left(1 - \frac{\alpha^2}{m} H_2 \right)^{-1} \,, \tag{23a}$$

$$\{x,\, v(x,\, p_2)\}_{x,p_2} = \frac{1}{m} \left(1 + \frac{\alpha p_2}{m} \right)^{-1} \,, \tag{23b}$$

and

$$\{v(x,\, p_2),\, p_1(x,\, p_2)\}_{x,p_2} = 0\,. \tag{23c}$$

Thus, with Horzela [15], we have constructed two alternative/inequivalent Hamiltonian descriptions of the same Newtonian system, namely, $(x,\, p_1,\, H_1)$ or $(x,\, p_2,\, H_2)$. The associated Lagrangians are not connected by a gauge term and the Hamiltonians are not related by a canonical transformation. Quantization of systems characterized by inequivalent Hamiltonians is an interesting curiosity.

3. Quantum energy levels using $H^{(1)}$ and $H^{(2)}$

Both $H^{(1)}$ and $H^{(2)}$ are explicitly time independent. One, therefore, may attempt to quantize these Hamiltonian descriptions by means of the canonical procedure. But alternative Hamiltonians call for the use of a more general quantization scheme than the canonical one [14, 15]. About sixty years ago Wigner[16] provided us with such a general scheme which assumes that Heisenberg's equations of motion have a "more immediate physical significance" than the Heisenberg–Born–Jordan commutation relation. The validity of Heisenberg's equations of motion allows for rules of quantization (Wigner quantization) more general than the canonical one. In this context we display in Fig. 1 the phase diagrams of the Hamilton's equation $\dot{x} = \frac{\partial H_1}{\partial p_1}$, $\dot{p}_1 = -\frac{\partial H_1}{\partial x}$ and $\dot{x} = \frac{\partial H_2}{\partial p_2}$, $\dot{p}_2 = -\frac{\partial H_2}{\partial x}$ along with the phase diagram of the undamped ($\alpha = 0$) harmonic oscillator. We have chosen to work with $k/m = 1$. The dotted curve ($\alpha = -0.4$) represents the phase trajectory ($x(t)$, $p_1(t)$) while the dashed one is a similar trajectory for ($x(t)$, $p_2(t)$). The solid curve denotes the phase trajectory of the harmonic oscillator. The

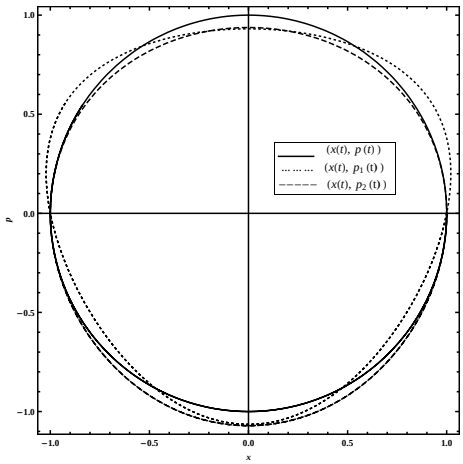

Fig. 1. Phase-space of the damped ($H^{(1)}$, $H^{(2)}$) and undamped (H) harmonic oscillators. The dotted- and dashed curves show the trajectories ($x(t)$, $p_1(t)$) and ($x(t)$, $p_2(t)$) while the solid curve ($x(t)$, $p(t)$) represents the harmonic oscillator's phase diagram.

dotted and dashed curves do not deviate appreciably from the phase diagram of the harmonic oscillator which can be quantized by the canonical procedure. We have verified that this is true for any physically acceptably value of α. In view of this we can expand both $H^{(1)}$ and $H^{(2)}$ in powers of α such that

$$H^{(1)} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - \frac{\alpha}{6m}p^3 + \frac{\alpha^2}{24m^3}p^4\,, \tag{24a}$$

and

$$H^{(2)} = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \frac{\alpha}{2m}kpx^2 - \frac{\alpha}{6m^2}p^3 + \frac{\alpha^2}{3m^3}p^4\,. \tag{24b}$$

In writing (24) we have neglected all terms involving powers of α higher than 2. Here

$$p = \lim_{\alpha\to 0} p_i = mv\,, \qquad i = 1,\, 2\,. \tag{25}$$

The Hamiltonians in (24) can be written in the form

$$H^{(i)} = H_0 + H^{(i)\prime}\,, \qquad i = 1\,, 2 \tag{26}$$

with $H_0 = \frac{p^2}{2m} + \frac{k^2}{2}x^2$, the harmonic oscillator Hamiltonian and $H^{(i)\prime}$, a small perturbation. This decomposition allows us to calculate energy eigenvalues using non degenerate perturbation theory within the framework of canonical quantization procedure. To achieve this we write the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}^{(i)}\psi\,, \qquad i = 1 \text{ and } 2\,, \tag{27}$$

where $\hat{H}^{(i)}$ is the Hamiltonian operator corresponding to $H^{(i)}$. The time-dependent wave function $\psi(x,\, t)$ can be written as a product, $\psi(x,\, t) = \phi(x)f(t)$, so as to introduce the time independent Schrödinger equation

$$\hat{H}^{(i)}\phi_n(x) = E_n^{(i)}\phi_n(x) \tag{28}$$

with $E_n^{(i)}$, the n-th eigenvalue of the Schrödinger operator $\hat{H}^{(i)}$. The calculation of $E_n^{(i)}$ will enable us to demonstrate whether the quantum systems associated with the alternative Hamiltonian descriptions are same or different.

The operator corresponding to H_0 is

$$\hat{H}_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\,, \qquad \omega = \sqrt{\frac{k}{m}}\,, \tag{29}$$

where the creation and annihilation operators $a^\dagger$ and a are defined by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\,, \tag{30a}$$

and

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\,. \tag{30b}$$

These operators satisfy relations

$$[a^\dagger, a] = 1\,, \qquad \begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle\,, \\ a|n\rangle &= \sqrt{n}|n-1\rangle\,, \qquad a|0\rangle = 0\,. \end{aligned} \tag{31}$$

The n-th eigenvalue of $\hat{H}_0$ is given by

$$E_n^{(0)} = \hbar\omega\left(n + \frac{1}{2}\right)\,. \tag{32}$$

The operator corresponding to $H^{(1)'}$ can be obtained simply by replacing p and x with $\hat{p}$ and $\hat{x}$ while the first term in $H^{(2)'}$ needs an operator ordering ($px^2 \to \frac{1}{3}(\hat{p}\hat{x}\hat{x} + \hat{x}\hat{p}\hat{x} + \hat{x}\hat{x}\hat{p})$) for use in quantum mechanics. However, it is evident that the first-order terms in α of $H^{(i)'}$ involve only the odd powers of a and $a^\dagger$ and as such these terms do not contribute anything to the energy eigenvalues. Thus we need to calculate the energy correction using the second-order perturbation theory according to which

$$E_n^{(2)} = \sum_m \frac{|\langle m|H^{(i)'}|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}\,, \qquad n \neq m\,. \tag{33}$$

From (24), (26) and (33) we get

$$E_n^{(2)}|_{(1)} = -\frac{\alpha^2}{96m}\hbar^2\omega^2\left(68n^3 + 102n^2 + 118n + 42\right)\,, \tag{34a}$$

$$E_n^{(2)}|_{(2)} = 8E_n^{(2)}|_{(1)}\,. \tag{34b}$$

Here $E_n^{(2)}|_{(1)}$ gives the energy correction to the oscillator level described by $H^{(1)}$ and $E_n^{(2)}|_{(2)}$, that to the oscillator represented by $H^{(2)}$. In both cases the perturbation causes lowering of the harmonic oscillator energy level. In the second system lowering is eight times as that of the first. The two quantum systems corresponding to the classical system in (5) are thus entirely different.

4. Conclusion

The Lagrangian for the one dimensional DHO in (1) does never follow from an explicitly time-independent Lagrangian. This provides an awkward analytical constraint for quantizing the motion. We have found that a one dimensional DHO with the position-dependent frictional coefficient as given by (5) admits alternative analytic representations [8] characterized by two explicitly time-independent Lagrangians.

From the viewpoint of time inversion $t \to -t$, Newtonian equations can be classified into symmetry preserving and symmetry violating. For symmetry violating systems the Lagrangian will be explicitly time dependent. On the other hand symmetry preserving systems can be represented by time-independent Lagrangians. For example, symmetry violating equation $\ddot{x} + \gamma\dot{x} = 0$ demands for its consistent description a specified direction of time or 'time's arrow' with the Lagrangian given by $L = \frac{1}{2}\dot{x}^2 e^{\gamma t}$, while the symmetry preserving equation $\ddot{x} + \gamma\dot{x}^2 = 0$ can be analytically represented by $L = \frac{1}{2}\dot{x}^2 e^{2\gamma x}$. Equation (5) is time reversal symmetry violating. Despite that we could find time-independent Lagrangians for it. The reason for this is not immediately clear. However, we note that our equation is invariant under the joint operations — space reflection and time reversal. The Hamiltonians corresponding to the alternative Lagrangians obtained by us, although give the same classical equation of motion, have different phase-space structure. As a result their quantization leads to completely different quantum systems.

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