Vol. 35 (2004) ACTA PHYSICA POLONICA B No 2

INTERACTION BETWEEN KINK AND RADIATION IN ϕ^4 MODEL*

Tomasz Romańczukiewicz

M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland
e-mail: trom@th.if.uj.edu.pl

(Received March 7, 2003; revised version received September 24, 2003)

The radiation from oscillating kink in (1+1) dimensional relativistic ϕ^4 model is considered. Both analytical and numerical approaches are presented and the comparison between these methods is discussed. Acceleration of the kink in external radiation is calculated and numerical results are also presented.

PACS numbers: 02.30.Jr, 02.60.Cb, 03.65.Pm

1. Introduction

Kink is a simple example of topological defect. Topological defects *i.e.* monopoles, vortices, domain walls play very important role in modern physics starting from flux tubes in QCD [1], vortices in liquid helium and superconductors, domain walls in magnetics, various defects in liquid crystals to strings in cosmology [2] (although recently it has been found cosmological strings are not as important as they had appeared to be [3]). Although these objects were studied for many years there are still many unanswered questions, mostly concerning their dynamics. There are some open problems about their interactions with other topological defects and external fields [4]. Even dynamics of their internal degrees of freedom is not fully understood and needs a lot of examinations [5, 6]. In most cases we built our knowledge from experiments and numerical calculations. In theory, topological defects are described by nonlinear partial differential equations. Most of them one can solve exactly only in special cases, for the rest of them one can apply only approximations. In this paper we focus on perturbation

* Work performed under auspices of ESF Programme "Coslab".

method around a well known analytical static solution namely the kink. The results will be compared with numerical calculations. Very similar problems were discussed in [7, 8] and [9].

Manton in [7] considered problem of kink and antikink creation and annihilation process and its relationship with the vibrational mode. He presented theory concerning the behavior of that mode and its decay. In this paper we develop methods presented in [7] and apply to different initial conditions, more suitable for computer simulations. We verify Manton's and our predictions via numerical calculations. In [8] there are another methods for calculating the radiation presented in a context of domain wall. The radiation from squeezed kink is investigated in [9].

Usually when one wants to find out whether the system is stable, one adds small perturbation and study its evolution. If the perturbation does not grow in time it means the solution is stable. Otherwise it is unstable. The problems of stability and relaxation process is well examined for dissipative systems (for example systems described by diffusion equations or wave equations with damping). The model considered in the present paper is energy conserving and therefore we can use stability only in a local sense. The relaxation process must be based upon radiation of the redundant energy into infinity. The static kink solution has the lowest possible energy in its topological sector (Bogomolny bound). This is a reason for us to treat kink as an attractor. The system with localized perturbation will tend to our kink solution in finite regions of the one-dimensional space.

In the Sections 4 and 5 of our paper we apply methods presented in the Sections 2 and 3 to kink interacting with external radiation. Because the kink is transparent to radiation in linear order of approximation our usual intuition fails. For small amplitudes, the kink instead of being pushed away by radiation pressure is accelerating toward the source of radiation.

In literature there are considered mostly kinks interacting with a constant external force [4] or a force oscillating in time [10] or [11].

2. The model

Let us consider one dimension real scalar field theory described by the equation:

$$\ddot{\phi} - \phi'' + 2\phi(\phi^2 - 1) = 0\,. \tag{1}$$

We use natural unit system ($c = 1$). There exist well known static solutions:

$$\phi_s(x) = \pm \tanh(x - x_0) \tag{2}$$

which will be referred to as kink (with $+$ sign) and antikink (with $-$). Without loosing generality one can choose $x_0 = 0$ due to translational invariance.

Let us add a small perturbation to the kink solution:

$$\phi(t,x) = \phi_s(x) + \xi(t,x)\,. \tag{3}$$

Eq. (1) can be rewritten in the form:

$$\ddot{\xi} + \boldsymbol{L}\xi + N(\xi) = 0\,, \tag{4}$$

where linear operator $\boldsymbol{L}$ has the form:

$$\boldsymbol{L} = -\frac{\partial^2}{\partial x^2} + \left[4 - 6\left(1 - \phi_s^2\right)\right] \tag{5}$$

and N denotes the part nonlinear in ξ:

$$N(\xi) = 6\phi_s\xi^2 + 2\xi^3\,. \tag{6}$$

In the first approximation we assume that ξ is small enough to neglect the term (6). We seek solutions of the linearized Eq. (4) in the form $\xi(t,x) = \mathrm{e}^{i\omega t}\eta(x)$. We obtain the eigenvalue problem (very similar to the Schrödinger equation):

$$\boldsymbol{L}\eta = \omega^2\eta\,. \tag{7}$$

The above equation has two solutions vanishing in infinity. One of them for $\omega = 0$

$$\eta_t(x) = \frac{1}{\cosh^2 x} \tag{8}$$

is called translational zero-mode because it is responsible for small translations of the kink: $\phi_s(x+\delta x) = \phi_s(x) + \delta x\eta_t(x) + \mathcal{O}(\delta x^2)$. This mode plays a very important role when one considers the evolution of a system with an external force [4] or evolution of a system itself in more dimensions (domain wall) [5] or interaction between two or more kinks. The other normalizable solution is a vibrational mode with $\omega_d = \sqrt{3}$:

$$\eta_d(x) = \frac{\sinh x}{\cosh^2 x}\,. \tag{9}$$

It is very important that in linear approximation this mode oscillates without loosing energy. It is a quasistationary solution because only couplings in higher orders are responsible for decaying of this solution due to radiation. The energy is carried away by scattering modes in continuous spectrum:

$$\eta_k(x) = \mathrm{e}^{ikx}\left(3\tanh^2 x - 3ik\tanh x - 1 - k^2\right)\,, \tag{10}$$

where k is a wave vector:

$$k^2 = \omega^2 - 4\,. \tag{11}$$

Of course this is only an approximation. In fact the radiation has a form of a finite wavelet and therefore its energy is finite.

We want to find the evolution of the system when initially only the discrete oscillating mode is excited (*i.e.* in time $t = 0$ the field has a form $\phi(0,x) = \phi_s(x) + a\eta_d(x)$ and $\dot{\phi}(0,x) = 0$, where a is assumed to be small). Let us construct a solution of the equation (4) substituting $\xi(t,x) = A_d(t)\eta_d(x) + \eta(t,x)$ and assuming that A_d and η are small we obtain the equation for η in $\mathcal{O}(A_d^2)$ order:

$$\ddot{\eta} + \boldsymbol{L}\eta + 6\phi_s A_d^2 \eta_d^2 = 0\,. \tag{12}$$

This is an inhomogeneous linear equation for η with the source term $T(t)g(x)$ where $g(x) = \eta_d^2(x)\phi_s(x)$ and $T(t) = 6A_d^2(t)$. Notice that it is proportional to the square of vibrational mode. We can make a time Fourier transform of equation (12) and obtain:

$$-\omega^2\tilde{\eta}(\omega,x) + \boldsymbol{L}\tilde{\eta}(\omega,x) + \tilde{T}(\omega)g(x) = 0\,. \tag{13}$$

It is an inhomogeneous linear equation for the spatial part but since we know the solutions (10) of the homogeneous equation we can easily construct appropriate Green's function in a standard manner:

$$G_k(x,y) = \begin{cases} -\frac{1}{W}\eta_k(x)\eta_{-k}(y) & x < y \\ -\frac{1}{W}\eta_{-k}(x)\eta_k(y) & x > y \end{cases}\,, \tag{14}$$

where $W = -2ik(k^2+1)(k^2+4)$ is Wronskian of Eq. (13). Therefore the solution has a form:

$$\tilde{\eta}(\omega,x) = -\frac{\tilde{T}(\omega)}{W}\left(\eta_{-k}(x)\int\limits_{-\infty}^{x} dy\,\eta_k(y)g(y) + \eta_k(x)\int\limits_{x}^{\infty} dy\,\eta_{-k}(y)g(y)\right)\,. \tag{15}$$

We can use an asymptotic form of $\eta_{\pm k}(x) \approx (2 - k^2 \pm 3ik)\mathrm{e}^{\pm ikx}$ for large x and obtain (the source is well localized around 0):

$$\tilde{\eta}(\omega,x) = -\frac{\tilde{T}(\omega)}{2ik(2-k^2-3ik)}\mathrm{e}^{(-ikx)}\int\limits_{-\infty}^{\infty} dy\,\eta_k(y)g(y)\,. \tag{16}$$

After calculating this integral analytically we get:

$$\tilde{\eta}(\omega,x) = -\frac{\pi k(k^2+4)(k^2-2)\tilde{T}(\omega)}{16\sinh\left(\frac{\pi k}{2}\right)(2-k^2-3ik)}\exp(-ikx)\,. \tag{17}$$

Following Manton [7] we consider an oscillating mode $A_d(t) = a\cos(\omega_d t)$. Square of the amplitude is given by $A_d^2 = \frac{1}{2}a^2\left(\cos(2\omega_d t) + 1\right)$. Since constant solution carries no energy we take into account only the time dependent part: $T(t) = 3a^2 \mathrm{e}^{2\omega_d t}$. Fourier transform has a form: $\tilde{T}(\omega) = -6\pi a^2 \delta(\omega - \omega_0)$, where $\omega_0 = 2\omega_d$. Substituting the source term into (17) and calculating inverted Fourier transform we obtain:

$$\eta(t,x) = \frac{\pi k_0(k_0^2+4)(k_0^2-2)}{32\sinh\left(\frac{\pi k_0}{2}\right)(2-k_0^2-3ik_0)} a^2 \exp\left[i(\omega_0 t - k_0 x)\right] . \tag{18}$$

Since we are only interested in real part of the above equation the radiation has a form:

$$\eta(t,x) = Qa^2\cos\left(\omega_0 t - k_0 x + \delta\right) , \tag{19}$$

where

$$\begin{aligned} k_0^2 &= \omega_0^2 - 4\,, \\ Q &= \frac{\pi k_0(k_0^2-2)}{32\sinh\left(\frac{\pi k_0}{2}\right)}\sqrt{\frac{k_0^2+4}{k_0^2+1}} = 0.0453 \end{aligned} \tag{20}$$

and δ is a phase which we are not interested in. The same radiation goes also to $-\infty$. We have made numerical simulations for various a_0, where a_0 is initial amplitude of the oscillating mode (in the following section we show that the amplitude decreases due to the radiation). In large distance from the kink we measure the outgoing field. The difference from the kink solution is sketched in figures 1 and 2 for $a_0 = 0.05$ and 0.4, respectively (for the clarity we have plotted only the local extrema of the field instead of whole oscillations). The amplitude predicted by Eq. (19) seems to fit very well in the first figure. On the second figure one can also see the damping which will be explained in the next section.

As we can see from Eq. (19) the outgoing radiation has only one single frequency which is twice the frequency of oscillating mode. This is true only in approximation, because the coupling in cubic term in (6) will result in frequency $3\omega_d$. If one wants to test this procedure numerically one has to be very careful. The first idea is to pose initial conditions in the form $\phi(0,x) = \phi_s(x) + a\eta_d(x)$ and $\dot{\phi}(0,x) = 0$. That means there is no radiation in $t = 0$, so there can be no source for $t < 0$. Source must have a form $T(t) = 3a^2\Theta(t)\mathrm{e}^{i\omega_0 t}$, where $\Theta(x)$ is a Heaviside function. Fourier transform:

$$\tilde{T}(\omega) = 3a^2\left(-\pi\delta(\omega-\omega_0) + P\frac{i}{\omega-\omega_0}\right) . \tag{21}$$

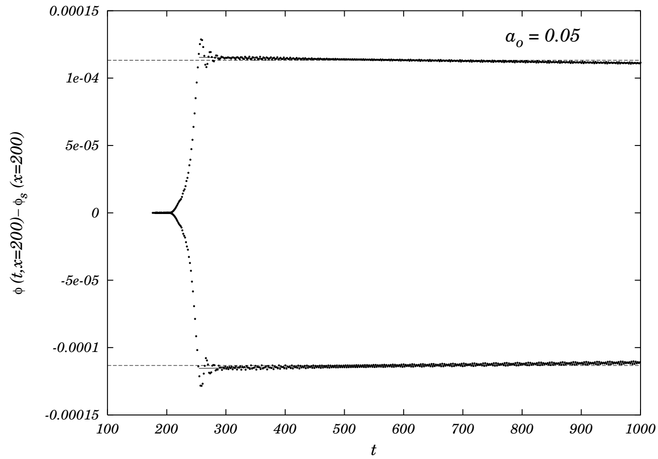

Fig. 1. Extrema of the outgoing radiation for $a_0 = 0.05$.

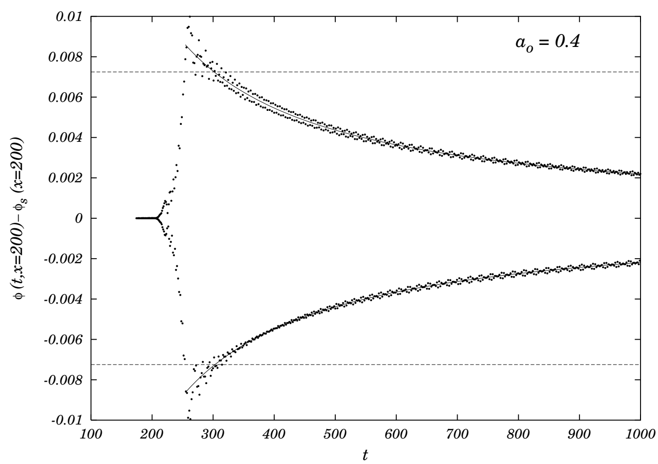

Fig. 2. Extrema of the decaying radiation for $a_0 = 0.4$ with fitted functions (34).

The field we calculate is equal to

$$\eta(t,x) \;=\; \frac{a^2}{2\pi}\mathrm{P.V.}\int\limits_{-\infty}^{\infty} d\omega\, \frac{\pi k(k^2+4)(k^2-2)\exp(i(\omega t-kx))}{32\sinh\left(\frac{\pi k}{2}\right)(2-k^2-3ik)} \times\left(-\pi\delta(\omega-\omega_0)+\frac{i}{\omega-\omega_0}\right). \tag{22}$$

Let us consider the second part of the integral in (22) containing factor $\frac{i}{\omega-\omega_0}$:

$$I(t,x) = i\int\limits_{-\infty}^{\infty} d\omega\, F(\omega)\frac{\mathrm{e}^{i\omega t}}{\omega-\omega_0}\,. \tag{23}$$

If k is a real number we have oscillations of the integrand along x axis (because of the $\exp(-ikx)$ term), but when k is imaginary we have exponential decay, therefore for large x we need to take into account only ω for which k is real, that is $\omega > \Omega$, where $\Omega = 2$. We can rewrite the above integral in a form:

$$I(t,x) = i\mathrm{e}^{i\omega_0 t}\int\limits_{\Omega}^{\infty} d\omega\, F(\omega)\frac{\mathrm{e}^{i(\omega-\omega_0)t}}{\omega-\omega_0}\,. \tag{24}$$

After introducing $\nu = (\omega-\omega_0)t$ we obtain:

$$I(t,x) = i\mathrm{e}^{i\omega_0 t}\int\limits_{(\Omega-\omega_0)t}^{\infty} d\nu\, F\left(\frac{\nu}{t}+\omega_0\right)\frac{\mathrm{e}^{i\nu}}{\nu}\,. \tag{25}$$

For large times we can expand F in $\frac{\nu}{t}$:

$$I(t,x) = i\mathrm{e}^{i\omega_0 t}\int\limits_{(\Omega-\omega_0)t}^{\infty} d\nu\, F\left(\omega_0\right)\frac{\mathrm{e}^{i\nu}}{\nu} + F'(\omega_0)\frac{\mathrm{e}^{i\nu}}{t} + \cdots\,. \tag{26}$$

Because $\Omega-\omega_0 < 0$ and t tends to infinity the first term is equal to:

$$\lim_{t\to\infty}\int\limits_{(\Omega-\omega_0)t}^{\infty} d\nu\, F\left(\omega_0\right)\frac{\mathrm{e}^{i\nu}}{\nu} = F(\omega_0)\int\limits_{-\infty}^{\infty} d\nu\frac{\mathrm{e}^{i\nu}}{\nu} = i\pi F(\omega_0)\,. \tag{27}$$

The second term we can integrate immediately after adding a term $-\epsilon\nu$ to the exponent and obtain:

$$I(t,x) = -\mathrm{e}^{i\omega_0 t}\left[\pi F(\omega_0) + \frac{\mathrm{e}^{i(\Omega-\omega_0)t}}{t}F'(\omega_0)\right]. \tag{28}$$

But $F(\omega) = \mathrm{e}^{-ikx}f(\omega)$, therefore

$$F'(\omega_0) = \left(-ixf(\omega_0)\left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} + f'(\omega_0)\right)\mathrm{e}^{-ik_0 x}\,.$$

For large x we can neglect the second term:

$$I(t,x) = -\mathrm{e}^{i\omega_0 t}\left[\pi - ix\frac{\omega_0}{k_0}\frac{\mathrm{e}^{i(\Omega-\omega_0)t}}{t}\right]F(\omega_0)\,. \tag{29}$$

Finally the radiating part in our approximation has a form

$$\eta(t,x) = Qa^2\left[1 - i\frac{\omega_0}{k_0}\frac{\mathrm{e}^{i(\Omega-\omega_0)t}}{2\pi t/x}\right]\exp[i(\omega_0 t - k_0 x + \delta)]\,. \tag{30}$$

Apart from oscillation with frequency ω_0 we have some decaying modulations with frequency $\omega_0 - \Omega \approx 1.46$. Speed of the decay is determined by the point of observation x. The larger x the slower decay. Figure 3 shows the Fourier's transform of the numerically computed field in $x = 30$. As one can see there are three easily visible peaks. The largest one is at $\omega = \omega_0 = 2\omega_d$. The next one is at another harmonic frequency $3\omega_d$. There is also one peak for $\omega = 1.15\omega_d = 2 = \Omega$ which is responsible for decaying modulations of the field. On the next figure we have plotted envelope of oscillations evaluated from Eq. (22). In figure 5 one can see the behavior of the envelope for large times which seems to be well approximated by Eq. (30). The measured frequency is really $\omega_0 - \Omega \approx 1.42$. We also presented the growth of the field for small times in figures 6 and 7. On the next pictures we have sketched numerically evaluated field (from Eq. (22), figure 9) and, for comparison, numerically calculated solution of the full partial equation (1) (figure 10). Because of the numerical errors there are only the largest modulations visible.

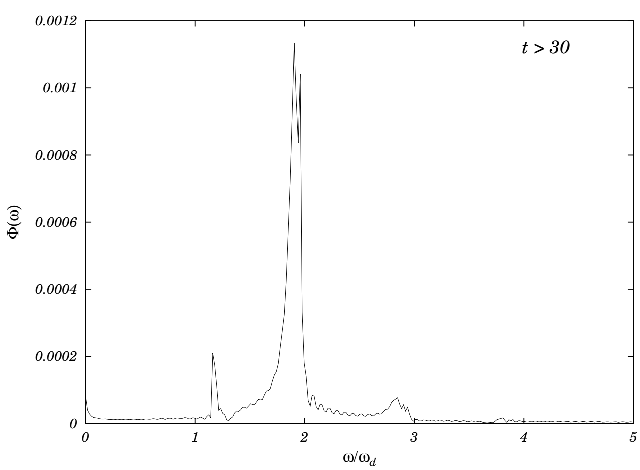

Fig. 3. Fourier transform of the radiation ($a_0 = 0.8$).

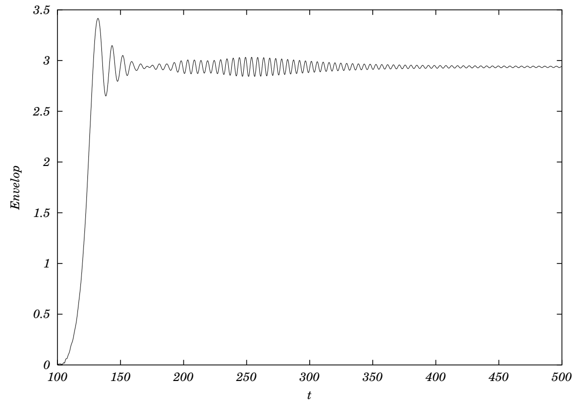

Fig. 4. Envelope of the radiation.

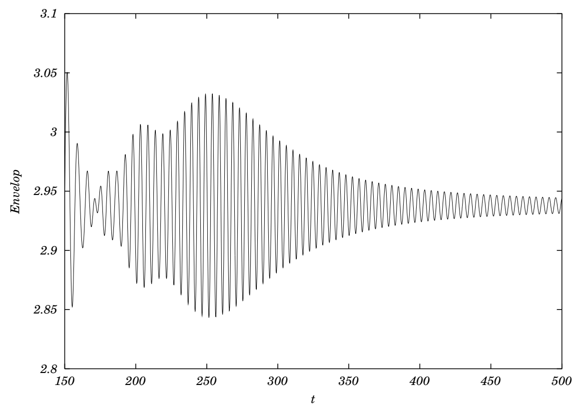

Fig. 5. Envelope for large times.

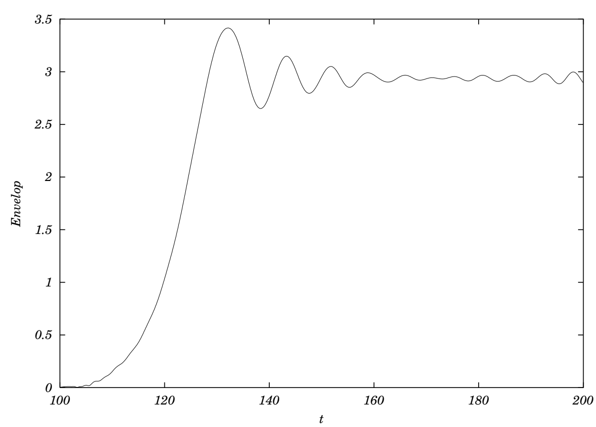

Fig. 6. Envelope for small times.

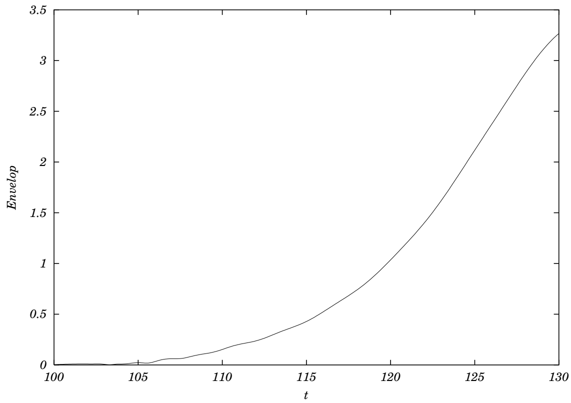

Fig. 7. Envelope for very small times.

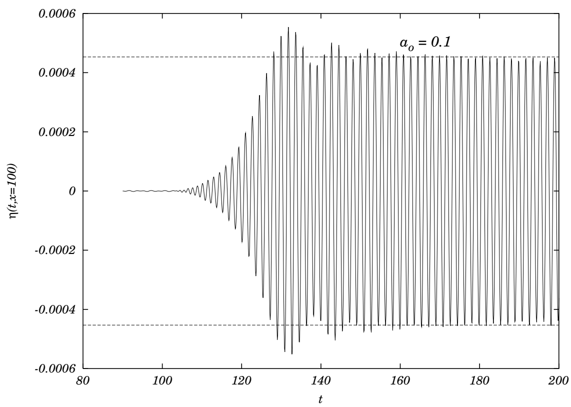

Fig. 8. Radiation calculated from integral (7).

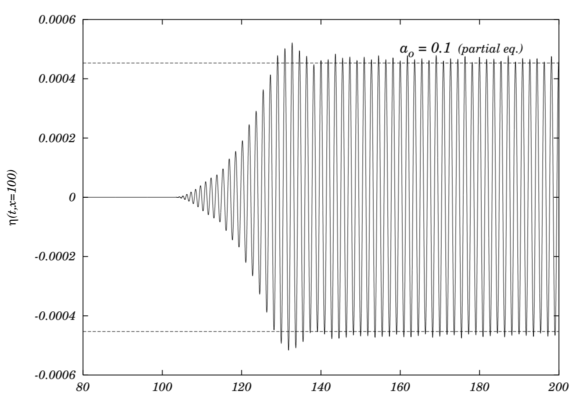

Fig. 9. Radiation computed from the full partial equation (1).

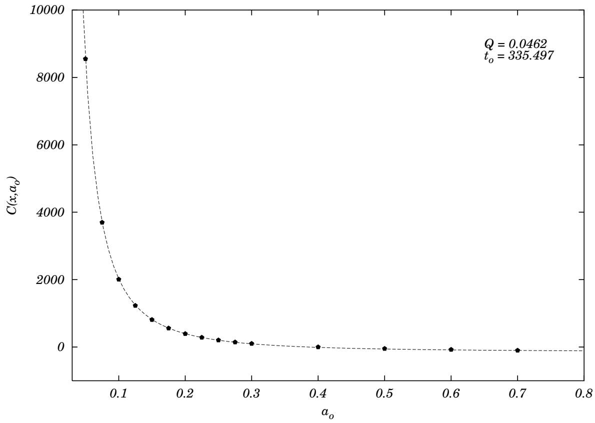

Fig. 10. $C(x, a_0)$ defined by Eq. (35) *vs* initial amplitude of the discrete mode; numerical data and fitted function.

3. Backreaction

In the above approximation we have found the field radiated to the infinity. We assumed that amplitude of the oscillating mode does not change in time. This is not true if the system is isolated. Radiation carries away energy and that leads to a decrease of the amplitude since the system is energy conserving. Let us calculate the rate at which the energy is being lost during radiation. For scalar field the energy escaping from a segment (x_1, x_2) is equal to $\frac{dE}{dt} = \dot{\phi}\phi'|_{x_1} - \dot{\phi}\phi'|_{x_2}$. After taking $\phi(t, x) = \phi_s(x) + a\eta_d(x)\cos(\omega_d t) + Qa^2\cos(\omega_0 t - k_0 x + \delta)$ and averaging over a period we obtain the energy change inside large segment:

$$\frac{dE}{dt} = -Q^2 k_0 \omega_0 a^4 = -0.020a^4 \tag{31}$$

and since energy of the vibrational mode is $E = a^2$ we find a differential equation for the amplitude of that mode:

$$\frac{da^2}{dt} = -Q^2 k_0 \omega_0 a^4\,. \tag{32}$$

We can easily find a solution with an initial condition $a(t = 0) = a_0$:

$$a(t) = \frac{a_0}{\sqrt{1 + Q^2 k_0 \omega_0 a_0^2 t}}. \tag{33}$$

From the above we obtain time dependence of radiation amplitude:

$$A(t) = \frac{1}{C(x, a_0) + Qk_0\omega_0 t}\,, \tag{34}$$

where

$$C(x,a_0)=\frac{1}{Qa_0^2}-Qk_0\omega_0t_0(x) \tag{35}$$

is constant in time. $t_0(x)$ is the time it takes for the field to travel from origin to point x where it is observed.

This very precise prediction we can compare with purely numerical results. Maxima and minima of the field with initial $a_0=0.4$ measured for $x=200$ are shown in figure 2. There are also two hyperbolic function fitted to these data. The same procedure was applied for other amplitudes. The factor that stands besides t in Eq. (34) in the theory is equal to 0.444. Numerical results show that its average value is 0.449 which is not significantly different. The value does not change much (within 2%) for different a_0 (0.05–0.7).

In figure 10 we have sketched fitted C *versus* a_0. As one can see the fitted function in the form (35) is a very good approximation of the results. We find $Q=0.04607$. In the theory above we have 0.0453 but it is still within numerical error. The last parameter in numerics is equal to $t_0=336$. The group velocity of the field is equal to $v=\frac{\partial\omega}{\partial k}\big|_{k=k_0}=\frac{k_0}{\omega_0}\approx 0.816$ and hence the distance $x=200$ than theoretical value of $t_{0,\mathrm{th}}=245$. In fact we actually can observe that the radiation comes to $x=200$ in this time (figure 1 and 2). If one plots the fitted function and the theoretical one on one graph one can find they are almost indistinguishable, especially for small a_0 where our theory is the most accurate.

4. Anharmonic corrections

As we saw in previous section, the field radiated from oscillating kink is of order $\mathcal{O}(A_d^2)$ (Eq. (19), $A_d=a\cos(\omega_0t)$). The oscillating mode reaction to scattering modes will be of order $\mathcal{O}(A_d^3)$ (in Eq. (33) $a(t)\approx a_0-\frac{1}{2}a_0^3Q^2k_0\omega_0t$). But there is still nontrivial behavior of amplitude in the order $\mathcal{O}(A_d^2)$. Let us consider only the discrete mode. Substituting $\xi(t,x)=A_d(t)\eta_d(x)$ in Eq. (4) in order $\mathcal{O}(A_d^2)$ we obtain the following equation:

$$\left(\ddot{A}_d+3A_d\right)\eta_d+6A_d^2\phi_s\eta_d^2=0\,. \tag{36}$$

We are interested only in evolution of $A_d(t)$. We can rewrite the source term in a form $\phi_s\eta_d^2=\alpha\eta_d+\eta_\perp$. To get rid of the orthogonal term $\eta_\perp$ we project this equation onto η_d. In order to do that we integrate both sides of the above equation with η_d and we obtain an ordinary differential second order equation (neglecting the perpendicular part):

$$\ddot{A}_d+3A_d+\frac{9\pi}{16}A_d^2=0\,. \tag{37}$$

We solve this using standard series method: $A_d = A^{(1)} + A^{(2)} + A^{(3)} + \cdots$. We take initial conditions: $\dot{A}(0) = 0,\ A(0) = a$. In order $\mathcal{O}(A^{(1)})$ we obtain

$$\ddot{A}^{(1)} + 3A^{(1)} = 0 \tag{38}$$

and the solution is $A^{(1)} = a\cos\sqrt{3}t$. In second order we have:

$$\ddot{A}^{(2)} + 3A^{(2)} + \frac{9\pi}{16}a^2\cos^2\sqrt{3}t = 0\,. \tag{39}$$

The whole solution has a form:

$$A_d(t) = -\frac{3\pi}{32}a^2 + \left(1 + \frac{a\pi}{16}\right)a\cos(\sqrt{3}t) + \frac{\pi}{32}a^2\cos(2\sqrt{3}t)\,. \tag{40}$$

This function's maxima are equal to $A_{\rm max} = a$ and minima $A_{\rm min} = -a - \frac{3\pi}{16}a^2$. Figure 11 shows the oscillating mode evolution calculated numerically for $a_0 = 0.2$. The field was measured in $x = x_{\rm max} = \text{arcosh}\sqrt{2} \approx 0.88$, where the function η_d has maximum equal $\frac{1}{2}$. The dashed lines represent $\frac{1}{2}A_{\rm max}$ and $\frac{1}{2}A_{\rm min}$ calculated for that amplitude. As one can see our prediction works very well.

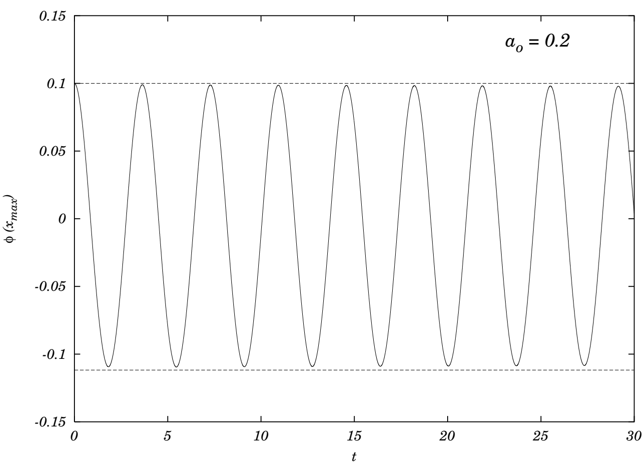

Fig. 11. Anharmonical oscillation of the discrete mode. Dashed lines are calculated theoretically.

Because the radiation depends on the square of the oscillating mode one can expect that minima are on more or less the same level while the maxima are changing (one is higher and the next one is lower and the following one is higher again). This effect is actually seen in figure 2.

5. Kink in external radiation

Let us now consider quite opposite process when kink is exposed to external radiation coming from $+\infty$: $ae^{i\omega_q t}\eta_q(x)$. Let us consider small amplitude limit. In the linear approximation there is no reflection in the potential of the soliton and therefore one cannot expect the system will behave as other similar systems in other branches of physics (there is no analogy to a particle exposed to an electromagnetic wave). In the first order the solution has a form (15), where $\tilde{T}(\omega) = -3\pi a^2\delta(\omega - 2\omega_q)$ and $g(x) = \eta_q^2(x)\phi_s(x)$. The correction oscillates with frequency $2\omega_q$. For $x \longrightarrow \pm\infty$ we can approximate the solution to form:

$$\begin{aligned}\tilde{\eta}(\omega, x \to \infty) = & -\frac{\tilde{T}(\omega)}{\mathrm{W}}\left(\eta_{-k}(x)\int\limits_{-\infty}^{\infty} dy\, \eta_k(y)\eta_q^2(y)\phi_s(y)\right.\\ & - \eta_{-k}(x)\int\limits_{x}^{\infty} dy\, \eta_k(y)\eta_q^2(y)\phi_s(y)\\ & \left. + \eta_k(x)\int\limits_{x}^{\infty} dy\, \eta_{-k}(y)\eta_q^2(y)\phi_s(y)\right)\end{aligned} \tag{41}$$

and

$$\begin{aligned}\tilde{\eta}(\omega, x \to -\infty) = & -\frac{\tilde{T}(\omega)}{\mathrm{W}}\left(\eta_k(x)\int\limits_{-\infty}^{\infty} dy\, \eta_{-k}(y)\eta_q^2(y)\phi_s(y)\right.\\ & - \eta_k(x)\int\limits_{-\infty}^{x} dy\, \eta_{-k}(y)\eta_q^2(y)\phi_s(y)\\ & \left. + \eta_{-k}(x)\int\limits_{-\infty}^{x} dy\, \eta_k(y)\eta_q^2(y)\phi_s(y)\right),\end{aligned} \tag{42}$$

where $k = k(q) = \sqrt{(2\omega_q)^2 - 4}$. In order to calculate the integrands with one finite but large limit we can use once again the asymptotic form of $\eta_k(y)$. Integrands with both infinite limits can be calculated analytically via residua and finally we can write a simply expression for asymptotic behavior of the first correction:

$$\eta_{+\infty}(t, x) = b_+(\omega_q)\cos(\omega_k t - kx + \delta_1) + c(\omega_q)\cos(\omega_k t + 2qt + \delta_2) \tag{43}$$

and

$$\eta_{-\infty}(t,x) = b_-(\omega_q)\cos(\omega_k t + kx - \delta_1) - c(\omega_q)\cos(\omega_k t + 2qt - \delta_2)\,, \quad (44)$$

where

$$b_\pm(\omega_q) = 3\pi a^2 \frac{-60480+82224\omega_q^2-31620\omega_q^4+3707\omega_q^6 \pm qk(15120-11115\omega_q^2+1851\omega_q^4)}{10\sinh\left(\frac{k\pm 2q}{2}\pi\right)(4\omega_q^2-3)\omega_q^2 k}\,, \quad (45)$$

$$c(\omega_q) = \frac{3}{2k}(q^2+1)\omega_q^2 \quad (46)$$

and $\delta_{1,2}$ are phases. The reflected part b_+ for small frequencies is more or less the same order as b_-, but for large frequencies decays exponentially $b_+(\omega_q) \sim \frac{22227}{80}\pi\omega_q a^2 \mathrm{e}^{-2\pi\omega_q}$. The transition part $b_- \sim \frac{1}{8}\omega_q^2 a^2$. c is an amplitude of a wave coming from ∞ and going to $-\infty$. This wave has exactly the same form at both sides and hence passes no energy nor momentum to the kink as one can see in details in the following calculations.

For small x we do not know an analytical solution. The solution (which for small x can differ from $\eta_{\pm k}$) oscillates around the kink solution with a frequency of the source wave $2\omega_q$. Because of the energy gathered in these oscillations kink, as seen from a distance, gains some extra mass. In following calculations we will be referring to that effect mass as M^*. The bare mass of the soliton equals $M = \frac{4}{3}$. As mentioned in Section 3 the change of energy inside a segment (x_1, x_2) equals $\frac{dE}{dt} = \dot{\phi}\phi'|_{x_1}^{x_2}$. In our case we have from the right-hand side of the kink $\phi(x) = \phi_s(x) + A\cos(\omega_q t + qx) + b_+\cos(\omega_k t - kx) + c\cos(\omega_k t + 2qt)$. The energy flowing into the segment averaged over a period is

$$\frac{dE_r}{dt} = \frac{1}{2}q\omega_q A^2 - \frac{1}{2}k\omega_k b_+^2 + q\omega_k c^2\,.$$

From the left-hand side the field is $\phi(x) = \phi_s(x) + B\cos(\omega_q t + qx) + b_-\cos(2\omega_q t + kx) - c\cos(2\omega_k t + 2qt)$. Then the energy equals:

$$\frac{dE_l}{dt} = -\frac{1}{2}q\omega_q B^2 - \frac{1}{2}k\omega_k b_-^2 - q\omega_k c^2\,.$$

The energy conservation takes the form:

$$\frac{1}{2}q\omega_q A^2 - \frac{1}{2}q\omega_q B^2 - \frac{1}{2}k\omega_k\left(b_-^2 + b_+^2\right) = M^*\frac{d\gamma}{dt}\,, \quad (47)$$

where $M^*\gamma = \frac{M^*}{\sqrt{1-v^2}}$ is an energy of the soliton. Apart from M^* which is probably different from M we have two unknown kinetic variables B and $\dot{\gamma}$. The second one is responsible for the acceleration of the kink due to interaction with the radiation. We need one more equation. We can use the momentum conservation law:

$$\frac{dP_r}{dt} = \frac{1}{2}q^2A^2 + \frac{1}{2}k^2b_+^2 + 2q^2c^2$$

and from the left-hand side of the kink:

$$\frac{dP_l}{dt} = -\frac{1}{2}q^2B^2 - \frac{1}{2}k^2b_-^2 - 2q^2c^2$$

and hence the second equation

$$\frac{1}{2}q^2A^2 - \frac{1}{2}q^2B^2 - \frac{1}{2}k^2\left(b_-^2 - b_+^2\right) = -M^*\frac{d}{dt}(v\gamma) = -F^*\,. \tag{48}$$

These equations are correct only for $v = 0$ (when we use the comoving coordinate system) because the moving kink experiences the Doppler's effect and there should be corrections in $\omega_{q,k}$, k and q. But substituting $v = 0$ to above equations we obtain the solutions very easily:

$$\left\{\begin{array}{ll} B^2 & = A^2 - 2\frac{k}{q}\left(b_-^2 + b_+^2\right) \\ F^* & = \dfrac{1}{2}k\left((k-2q)b_-^2 - (k+2q)b_+^2\right) \end{array}\right. . \tag{49}$$

We used the relation $\omega_k = 2\omega_q$. The factor before b_-

$$k - 2q = 2\left(\sqrt{\omega_q^2 - 1} - \sqrt{\omega_q^2 - 4}\right)$$

is always positive and because $b_+(\omega_q)$ is much smaller than b_- therefore the soliton accelerates in the direction from which radiation comes. The radiation pressure is negative! The force described by above equation is shown in figure 12. Only when b_- is equal to 0 the soliton is pushed away by the radiation.

The conclusion is that the soliton in most cases will stem the current. If the soliton reaches the speed when the frequency of the source wave due to the Doppler's effect $\omega' = \gamma\left(\omega_q + v\sqrt{\omega_q^2 - 4}\right)$ is the frequency for which acceleration vanishes the soliton moves with more or less constant velocity. Figure 13 shows the paths of zeros of the field (kinks) for frequencies around the zero of the pulling force. If the source wave has a frequency very close to zero of the force the kink's acceleration is small and the soliton moves with the current.

If the source wave amplitude is large the above approximation fails. There exists certain critical amplitude, depending on a frequency, for which the soliton is pushed away by the incoming wave (figure 14).

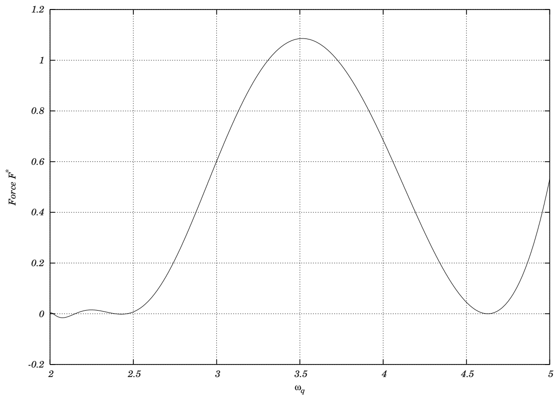

Fig. 12. Pulling force of the radiation *vs* frequency.

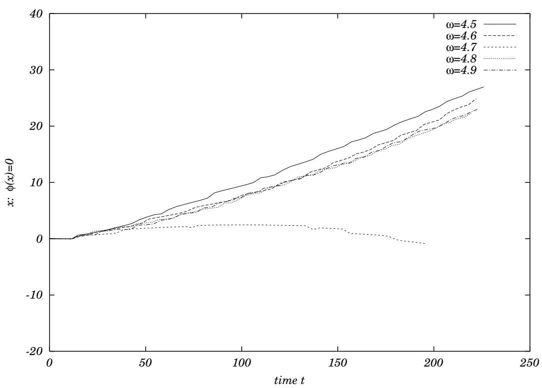

Fig. 13. Zeros of the field for different frequencies of the source wave and for the amplitude $a = 0.2$.

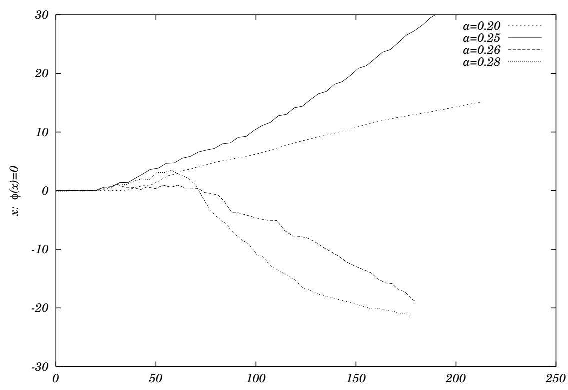

Fig. 14. Zeros of the field for $\omega = 3.5$ and for different amplitudes.

6. Conclusions

In this paper we have investigated the behavior of excited oscillating mode and its decay using both analytical and numerical methods. We have proved conformity between these two methods. We have studied the interaction of the kink with external radiation field. We have found that in most cases the kink will be pulled by the radiation. We have explained the peculiar behavior of this system. Finally we presented numerical results for large amplitudes of the source wave when the kink is pushed away by the wave.

REFERENCES

[1] See *e.g.* M. Baker, J.S. Ball, F. Zachariasen, *Phys. Rep.* **209**, 73 (1991).

[2] T.W.B. Kibble, *J. Phys. A* **9**, 1387 (1979).

[3] F.R. Bouchet, P. Peter, A. Riazuelo, M. Sakellariadou, *Phys. Rev.* **D65**, 021301 (2002).

[4] V.G. Kiselev, Ya.M. Shnir, *Phys. Rev.* **D57**, 5174 (1998).

[5] H. Arodź, *Nucl. Phys.* **B450**, 189 (1995).

[6] H. Arodź, *Acta Phys. Pol. B* **29**, 3725 (1998).

[7] N.S. Manton, H. Merabet, *Nonlinearity* **12**, 851 (1997).

[8] R. Pełka, *Acta Phys. Pol. B* **28**, 1981 (1997).

[9] M. Ślusarczyk, *Acta Phys. Pol. B* **31**, 617 (2000).

[10] R. Rajarshi, S. Sengupta, *Phys. Rev.* **D65**, 063521 (2002).

[11] F.J. Alexander, S. Habib, A. Kovner, *Phys. Rev.* **E48**, 4284 (1993).