CONTRACTIONS OF EXCEPTIONAL LIE ALGEBRAS AND SEMIDIRECT PRODUCTS

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For any semisimple subalgebra $\mathfrak{s}'$ of exceptional Lie algebras $\mathfrak{s}$ satisfying the constraint $\mathrm{rank}(\mathfrak{s}') = \mathrm{rank}(\mathfrak{s}) - 1$ we analyze the branching rules for the adjoint representation, and determine the compatibility of the components with Heisenberg algebras. The analysis of these branching rules allows to classify the contractions of exceptional algebras onto semidirect products of semisimple and Heisenberg Lie algebras. Applications to the Schrödinger algebras are given.

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1. Introduction

Since its introduction in Quantum Mechanics, group theory has shown to be a powerful tool to understand and interpret physical phenomena, from the crystalline structure of solids and the interpretation of atomic spectra to the classification of particles and the establishment of nuclei models. In all these applications, the groups are related usually to the symmetries of the system, either as spectrum-generating or dynamical groups, where the Casimir operators of the corresponding Lie algebra and those of distinguished subalgebras play a central role to describe the Hamiltonian or construct mass formulae [1–4].

Semidirect products $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_N$ of semisimple and Heisenberg Lie algebras for N independent sets of boson creation and annihilation operators occupy a central position in applications of non-semisimple Lie algebras, and constitute an adequate tool to combine inner and outer symmetries of physical systems. Among the various problems where these structures appear, the most important application of Lie algebras of this type is given in the vector coherent state theory (VCS) [5]. For the symplectic algebras $\mathfrak{s} = \mathfrak{sp}(2N, \mathbb{R})$, these semidirect products have been used as dynamical algebras of the

N-dimensional harmonic oscillator [2] or the description of microscopic nuclear collective motions [7]. Other examples are given by the Schrödinger algebras $\widehat{S}(n)$ in $(n+1)$-dimensional space-time, the quantum relativistic kinematical algebra or the Hamilton algebras used in relativity theory [8–11]. Due to the deep connection of VCS theory with the boson realizations of Lie algebras, it is of interest to analyze whether such semidirect products appear as contractions of semisimple Lie algebras, since this provides an alternative procedure to expand boson realizations.

In this work we determine all semidirect products $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_N$ that arise as a contraction of a complex exceptional Lie algebra. It will be seen that this problem is related to the classification of maximal rank reductive Lie algebras of exceptional Lie algebras, although the describing representations R are subjected to additional constraints concerning the branching rules of representations.

Any Lie algebra considered in this work is finite dimensional over the fields $\mathbb{K}=\mathbb{R},\mathbb{C}$. An algebra will be called indecomposable if it is not the direct sum of ideals.

2. Casimir operators and contractions

It is well known from classical theory that any semisimple Lie algebra $\mathfrak{g}$ has exactly $\mathcal{N}(\mathfrak{g})=l$ independent Casimir operators, *i.e.*, polynomials in the generators that commute with all elements of the algebra, where l denotes the rank of the algebra[1].

Given a Lie algebra $\mathfrak{g}=\left\{X_1,\dots,X_n \mid [X_i,X_j]=C_{ij}^k X_k\right\}$ in terms of generators and commutation relations, we are primarily interested in (polynomial) operators $C_p=\alpha^{i_1\dots i_p}X_{i_1}\dots X_{i_p}$ in the generators of $\mathfrak{s}$ such that the constraint $[X_i,C_p]=0$, $(i=1\dots n)$ is satisfied. Such an operator can be shown to lie in the centre of the enveloping algebra of $\mathfrak{s}$, and is traditionally referred to as Casimir operator [12]. However, in some applications, the relevant invariant functions are not polynomials (*e.g.* the inhomogeneous Weyl group [13–16]). Thus the approach with the universal enveloping algebra has to be generalized in order to cover arbitrary Lie groups. A quite convenient method is the analytical realization. The generators of the Lie algebra $\mathfrak{s}$ are realized in the space $C^{\infty}(\mathfrak{g}^*)$ by means of the differential operators:

$$\widehat{X}_i = C_{ij}^k\, x_k\, \frac{\partial}{\partial x_j}\,, \tag{1}$$

where $\{x_1,\dots,x_n\}$ are the coordinates of a dual basis of $\{X_1,\dots,X_n\}$. The invariants of $\mathfrak{g}$ (in particular, the Casimir operators) are solutions of the following system of partial differential equations:

[1] We recall that the rank is defined as the common dimension of the Cartan subalgebras.

$$\widehat{X}_i F = 0 \,, \qquad 1 \le i \le n \,. \tag{2}$$

Whenever we have a polynomial solution of (2), the symmetrization map defined by

$$\mathrm{Sym}\,(x_{i_1} \ldots x_{i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} x_{\sigma(i_1)} \ldots x_{\sigma(i_p)} \tag{3}$$

allows to recover the Casimir operators in their usual form, *i.e.*, as elements in the centre of the enveloping algebra of $\mathfrak{g}$. A maximal set of functionally independent invariants is usually called a fundamental basis. The number $\mathcal{N}(\mathfrak{g})$ of functionally independent solutions of (2) is obtained from the classical criteria for differential equations, and is given by:

$$\mathcal{N}(\mathfrak{g}) := \dim\, \mathfrak{g} - \mathrm{rank}\left(C_{ij}^k x_k \right) \,, \tag{4}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k \right)$ is the matrix associated to the commutator table of $\mathfrak{g}$ over the given basis.

Contractions of Lie algebras have developed from a formal procedure to justify certain physical systems to a technique of considerable importance [17–19]. It does not only allow to relate different symmetry or classification schemes by means of limiting precesses, but also provides useful information on the behavior of certain observables, codified in functions or Lagrangians. Various types of contractions have been developed in the literature, and their equivalence or relations have been explored. In this work we will only focus on generalized Inönü–Wigner contractions, that are the adequate type for physical applications [17, 20].

Let $\mathfrak{g}$ be a Lie algebra and $\Phi_t \in \mathrm{End}(\mathfrak{g})$ a family of non-singular linear maps, where $t \in [1, \infty)$[2]. For any $X, Y \in \mathfrak{g}$ we define

$$[X, Y]_{\Phi_t} := \Phi_t^{-1} \left[\Phi_t(X), \Phi_t(Y) \right] \,, \tag{5}$$

which obviously represent the brackets of the Lie algebra over the transformed basis, and defines an isomorphic algebra. Suppose that the limit

$$[X, Y]_\infty := \lim_{t \to \infty} \Phi_t^{-1} \left[\Phi_t(X), \Phi_t(Y) \right] \tag{6}$$

exists for any $X, Y \in \mathfrak{g}$. Then equation (6) defines a Lie algebra $\mathfrak{g}'$ called the contraction of $\mathfrak{g}$ (by Φ_t), non-trivial if $\mathfrak{g}$ and $\mathfrak{g}'$ are non-isomorphic, and trivial otherwise [17, 20]. A contraction for which there exists some

[2] Other authors use the parameter range $t' \in (0, 1]$, which is equivalent to this by simply changing the parameter to $t' = 1/t$.

basis $\{X_1, \dots, X_n\}$ such that the contraction matrix A_Φ is diagonal, that is, adopts the form

$$(A_\Phi)_{ij} = \delta_{ij} t^{n_j}\,, \qquad n_j \in \mathbb{Z}\,, \quad t > 0\,, \tag{7}$$

is called a generalized Inönü–Wigner contraction [17]. Among the various properties of contractions, we enumerate a numerical inequality satisfied by them that will play a central role (for others see *e.g.* [18]): For an arbitrary contraction $\mathfrak{g} \rightsquigarrow \mathfrak{g}'$ the following must hold:

$$\mathcal{N}(\mathfrak{g}) \leq \mathcal{N}(\mathfrak{g}')\ . \tag{8}$$

This limiting process can also be used to construct Casimir invariants of contractions. Let $F(X_1, \dots, X_n)$ is a Casimir operator of order p. Expressing the latter over the transformed basis $\{\Phi_t(X_1), \dots, \Phi_t(X_n)\}$, the limit

$$F'(X_1, \dots, X_n) := \lim_{t\to\infty} \frac{1}{t^p} F(\Phi_t(X_1), \dots, \Phi_t(X_n)), \tag{9}$$

can be easily shown to be an invariant of the contraction [21].

3. Representations compatible with Heisenberg algebras

We briefly review in this section the structure of representations of semisimple Lie algebras that can appear as describing representations of semidirect products of the shape $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{\Gamma\oplus\Gamma_0} \mathfrak{h}_N$, where $\mathfrak{h}_N$ denotes the $(2N+1)$-dimensional Heisenberg Lie algebra.

We convene that for arbitrary representations V, W of $\mathfrak{s}$, the symbol $\mathrm{mult}_V(W)$ denotes the multiplicity of W in V, *i.e.*, the number of copies of W appearing the the decomposition of V into irreducible representations. Following [22], we say that a $2N$-dimensional representation Γ of a semisimple Lie algebra $\mathfrak{s}$ is compatible with the $(2N+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_N$ if there exists a Lie algebra $\mathfrak{g}$ with Levi decomposition

$$\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{\Gamma\oplus\Gamma_0} \mathfrak{h}_N\,, \tag{10}$$

where Γ_0 is the identity or trivial representation. In principle, the latter decomposition (10) does not exclude the case $\mathrm{mult}_\Gamma(\Gamma_0) \neq 0$.

Certainly the compatibility problem is mainly of interest for semisimple Lie algebras, since for this class we have complete reducibility of representations. Indeed, by the Weyl theorem any finite dimensional representation of a semisimple algebra is completely reducible, *i.e.*, $\Gamma = \bigoplus W_i$, where the W_i are irreducible representations that we call constituents of Γ. If W_i is isomorphic to its contragredient (or dual) representation W_i^*, then we will say that W_i is self-dual [23]. The structure and properties of these constituents where analyzed in [22], where they were applied to the case of inhomogeneous Lie algebras. We recall the main properties: Any constituent W_i of a compatible representation Γ satisfies one of the following conditions:

$$\begin{array}{ll} 1. & \text{If } W_i \not\simeq W_i^*, \text{then } \mathrm{mult}_\Gamma(W_i) = \mathrm{mult}_\Gamma(W_i^*)\ , \\ 2. & \text{if } W_i \simeq W_i^* \text{and } W_i \wedge W_i \not\supseteq \Gamma_0, \text{then } \mathrm{mult}_\Gamma(W_i) = 2p+2\,,\ p \geq 0\,, \\ 3. & \text{if } W_i \simeq W_i^* \text{and } \mathrm{mult}_\Gamma(W_i) \text{ is odd, then } W_i \wedge W_i \supseteq \Gamma_0\,. \end{array} \tag{11}$$

In particular, the only self-dual constituents W_i allowed to have multiplicity one are those satisfying condition 3. Among the exceptional algebras, only E_7 possesses fundamental representations that satisfy this constraint, given in Table I. We observe that the representation $\Gamma_6 = [0,0,0,0,0,1,0]$ gives rise to the semidirect product $E_7 \overrightarrow{\oplus}_{\Gamma_6 \oplus \Gamma_0} \mathfrak{h}_{28}$, well known in the literature as the $E_{\frac{7}{2}}$ Lie algebra.

TABLE I

Self-dual fundamental representations of E_7 satisfying condition 3.

Fundamental representation Γ	$\dim \Gamma$
$[0,0,0,0,0,1,0]$	56
$[0,0,0,1,0,0,0]$	27664
$[0,0,0,0,0,0,1]$	912

For the real forms of the semisimple algebras, the situation is quite similar. If $\widehat{\mathfrak{s}}$ is such a real form, then the (real) representation R is $\mathfrak{h}_N$-compatible if $R \otimes \mathbb{C}$ is a $\mathfrak{h}_N$-compatible representation of $\mathfrak{s}$ [22].

4. Semidirect products obtained by contraction

Lie algebras with the Levi decomposition $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_N$ are of special interest among the semidirect products, since many of its structural features can be derived from the Levi subalgebra. In particular, the number of Casimir operators depends only on the rank of $\mathfrak{s}$, and the particular structure of the invariants can be deduced from that of the Casimir operators of $\mathfrak{s}$ (see *e.g.* [7, 24] and references therein). Lie algebras of this type play a relevant role in various physical models and applications. For example, the Hamilton algebras $\mathfrak{so}(N) \overrightarrow{\oplus}_{\Gamma \oplus \Gamma_0} \mathfrak{h}_N$ are used in the context of relativity groups for noninertial states in Quantum Mechanics [9], or the Schrödinger algebras $\widehat{S}(N) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(N)) \overrightarrow{\oplus}_{\Gamma \oplus \Gamma_0} \mathfrak{h}_N$ appearing as invariance algebras of the Schrödinger equation in $(N+1)$-dimensional space-time [25, 26].

Let $\mathfrak{s}$ be a (complex) semisimple Lie algebra. For the indecomposable semidirect product[3] $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_{\Gamma\oplus\Gamma_0}\mathfrak{h}_N$ the number of Casimir operators is given by

$$\mathcal{N}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{s}) + 1\,. \tag{12}$$

In some sense, the Levi subalgebra $\mathfrak{s}$ determines these Casimir invariants, to which the central charge (the generator of the centre of the Heisenberg algebra) is added [7, 22]. The particularities of Lie algebras of this type also allow to determine realizations of these algebra in terms of creation and annihilation operators, the number of which is determined solely by N. Therefore, it is of interest to determine whether such a semidirect product can be obtained as a contraction of semisimple Lie algebras, which could provide alternative boson realizations, or whether it is, on the contrary, a stable Lie algebra [8].

Consider a semisimple Lie algebra $\mathfrak{s}$ and let $\mathfrak{s} \rightsquigarrow \mathfrak{s}'\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_N$ be a contraction. Because of the cohomology and rigidity properties of semisimple Lie algebras (see *e.g.* [8, 28–30]), there exists a subalgebra $\mathfrak{s}''$ of $\mathfrak{s}$ such that $\mathfrak{s}'' \simeq \mathfrak{s}'$. Since equivalent algebras give rise to the same branching rules, the embedding $\mathfrak{s} \supset \mathfrak{s}''$ implies that the adjoint representation $\Gamma = \mathrm{ad}(\mathfrak{s})$ of $\mathfrak{s}$ obeys the branching rule

$$\Gamma = \Gamma' \oplus R \oplus \Gamma_0\,, \tag{13}$$

where Γ' denotes the adjoint representation of $\mathfrak{s}''$[4]. We call $\Gamma^{\mathrm{ch}} = R \oplus \Gamma_0$ the characteristic representation of $\mathfrak{s}'$ in $\mathfrak{s}$. The embedding provides the first condition on the subalgebra, namely that $\mathrm{rank}(\mathfrak{s}') \leq \mathrm{rank}(\mathfrak{s})$[5].
Now, using the properties of contractions of Lie algebras [17], we further obtain the constraint:

$$\mathcal{N}(\mathfrak{s}) \leq \mathcal{N}(\mathfrak{s}'\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_N)\,. \tag{14}$$

Since $\mathfrak{s}$ is semisimple, the branching rule (13) and formula (12) imply the inequality

$$\mathrm{rank}(\mathfrak{s}') \leq \mathrm{rank}(\mathfrak{s}) \leq \mathrm{rank}(\mathfrak{s}') + 1\,. \tag{15}$$

This means that in order to classify contractions of semisimple Lie algebras $\mathfrak{s}$ that are isomorphic to semidirect products with Heisenberg algebras, only the semisimple subalgebras that have either the rank of $\mathfrak{s}$ or its rank

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[3] An analogous result holds for the real forms of the algebra [22].

[4] Without loss of generality we can further suppose that $\mathfrak{s}'' = \mathfrak{s}'$, *i.e.*, that the subalgebra remains unaltered by the contraction.

[5] Even if the rank coincides, this does not mean that $\mathfrak{s}'$ is a maximal semisimple subalgebra of $\mathfrak{s}$.

minus one have to be considered. However, by formula (4), the number of invariants of a Lie algebra has the same parity as the dimension. Since contractions preserve the latter, we easily see that maximal rank subalgebras do not give rise to contractions of the preceding type. In fact, if $\mathrm{rank}(\mathfrak{s}') = \mathrm{rank}(\mathfrak{s})$, then by formula (12) we would have $\mathcal{N}(\mathfrak{s}' \overrightarrow{\oplus}_{R\oplus\Gamma_0} \mathfrak{h}_N) = \mathrm{rank}(\mathfrak{s}) + 1$, but this in contradiction with the fact that $\mathcal{N}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{s})$ and that both Lie algebras have the same dimension. As a consequence of this analysis, we obtain the following

Proposition 1 *Let $\mathfrak{s}' \overrightarrow{\oplus}_{\Gamma^{\mathrm{ch}}} \mathfrak{h}_N$ be the contraction of a semisimple Lie algebra $\mathfrak{s}$. Then $\mathfrak{s}'$ is a semisimple subalgebra of $\mathfrak{s}$ satisfying the condition* $\mathrm{rank}(\mathfrak{s}') = \mathrm{rank}(\mathfrak{s}) - 1$.

The previous result further has a notable relation with the classification of maximal rank reductive Lie algebras[6]. In fact, if $\mathfrak{s} \supset \mathfrak{s}'$ is a semisimple subalgebra having $\mathrm{rank}(\mathfrak{s}) - 1$, one may ask which condition must be satisfied in order that the direct sum $\mathfrak{s}' \oplus \mathfrak{u}(1)$ is a subalgebra of $\mathfrak{s}$. It can be easily shown [23] that the embedding $\mathfrak{s} \supset \mathfrak{s}' \oplus \mathfrak{u}(1)$ holds if and only if the decomposition of the adjoint representation of $\mathfrak{s}$ when reduced with respect to $\mathfrak{s}'$ is of type (13), *i.e.*, if the characteristic representation Γ^{ch} contains the identity representation Γ_0.

Let $\mathfrak{u}(1)$ be the Abelian Lie algebra generated by the element Y_0 of $\mathfrak{s}$ that after contraction generates the centre of the Heisenberg algebra. Since it transforms trivially by the action of the subalgebra $\mathfrak{s}'$[7], we can consider the Lie algebra $\mathfrak{s}' \oplus \mathfrak{u}(1)$. It is clearly a subalgebra of $\mathfrak{s}$, and since $\mathfrak{s}'$ is semisimple, it is reductive. If now $\mathfrak{h}'$ denotes an arbitrary Cartan subalgebra of $\mathfrak{s}'$, then $\mathfrak{h}' \oplus \mathfrak{u}(1)$ is an Abelian algebra of dimension $\mathrm{rank}(\mathfrak{s})$. This proves that the reductive subalgebra is of maximal rank. As a consequence of the branching rule (13), it turns out that $\mathfrak{h}' \oplus \mathfrak{u}(1)$ is a Cartan subalgebra of $\mathfrak{s}$. The proof follows at once using the following well known lemma [31]:

Lemma 1 *Let $\mathfrak{s}$ be a simple Lie algebra and $\mathfrak{r} = Z(\mathfrak{r}) \oplus [\mathfrak{r}, \mathfrak{r}]$ be a reductive Lie algebra of maximal rank. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{r}$, then the direct sum $\mathfrak{h} \oplus Z(\mathfrak{r})$ is a Cartan subalgebra of $\mathfrak{s}$.*

Actually the result remains valid if the multiplicity of the identity representation in (13) is greater than one. In this case, the proof follows using root theory and combining it with the preceding argument.

[6] Recall that a Lie algebra $\mathfrak{g}$ is called reductive if it decomposes as the direct sum $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_0$ of its centre $Z(\mathfrak{g})$ and a semisimple ideal $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$. It is called of maximal rank if it contains a Cartan subalgebra of $\mathfrak{s}$.

[7] This further implies that this element does not commute with other generators that transform trivially by the action of the subalgebra.

Branching rules

In this paragraph we consider some results on branching rules that will be used later in our classification of contractions of exceptional algebras. To this extent, a result already considered in [22] is useful, since it gives a compatibility condition for representations of subalgebras.

Proposition 2 *Let $\mathfrak{s} \supset \mathfrak{s}'$ be an embedding of semisimple Lie algebras and let Γ be a representation of $\mathfrak{s}$ compatible with $\mathfrak{h}_N$ for some N. Then the induced representation $\Gamma|_{\mathfrak{s}'}$ is a compatible representation of $\mathfrak{s}'$.*

This property further implies that direct sums of compatible representations are still compatible. However, for our purpose this result is not sufficient, since the two Lie algebras $\mathfrak{s}\overrightarrow{\oplus}_{\Gamma\oplus\Gamma_0}\mathfrak{h}_N$ and $\mathfrak{s}'\overrightarrow{\oplus}_{\Gamma|_{\mathfrak{s}'}\oplus\Gamma_0}\mathfrak{h}_N$ will have different dimension. In order to analyze the contractions of simple Lie algebras onto the semidirect products of this shape, we have to combine the compatibility of representations with the decomposition of the adjoint representation of Lie algebras with respect to a subalgebra.

Let us study more closely the situations that can appear. Let $\mathfrak{s} \supset \mathfrak{s}'$ be an embedding of semisimple Lie algebras. The two possible cases for the rank of $\mathfrak{s}'$ have to be analyzed separately, since they will imply different properties for the branching rules.

1. Let $\mathrm{rank}(\mathfrak{s}) = \mathrm{rank}(\mathfrak{s}')$. The decomposition $ad(\mathfrak{s}') = ad(\mathfrak{s}') \oplus \Gamma^{\mathrm{ch}}$ clearly implies that the characteristic representation contains no copy of the identity representation Γ_0, since otherwise this would contradict the fact that $\mathfrak{s}'$ is of maximal rank. Now let $\mathfrak{s}''$ be a subalgebra of $\mathfrak{s}'$ having $\mathrm{rank}(\mathfrak{s})-1$. The branching rule for the chain $\mathfrak{s} \supset \mathfrak{s}' \supset \mathfrak{s}''$ equals

$$ad(\mathfrak{s}) = ad\left(\mathfrak{s}''\right) \oplus \Gamma'^{\mathrm{ch}} \oplus \Gamma^{\mathrm{ch}}|_{\mathfrak{s}''}.$$

   Suppose that there exists a contraction $\mathfrak{s} \rightsquigarrow \mathfrak{s}''\overrightarrow{\oplus}_{\Gamma'^{\mathrm{ch}}\oplus\Gamma^{\mathrm{ch}}}\mathfrak{h}_N$. It follows from our previous analysis that $\mathfrak{s}'' \oplus \mathfrak{u}(1)$ must be a subalgebra of $\mathfrak{s}'$, we thus conclude that Γ'^{ch} must contain a copy of Γ_0. Two possibilities are given for this situation:

   (i) If Γ^{ch} is compatible, the restriction to $\mathfrak{s}''$ is still compatible. Therefore, $\Gamma'^{\mathrm{ch}} - \Gamma_0$ must be a compatible representation.

   (ii) If Γ^{ch} is not compatible, then the contraction exists whenever $\Gamma^{\mathrm{ch}}|_{\mathfrak{s}''} \oplus \left(\Gamma'^{\mathrm{ch}} - \Gamma_0\right)$ is a compatible representation of $\mathfrak{s}''$.

2. Now let $\mathrm{rank}(\mathfrak{s}') = \mathrm{rank}\,(\mathfrak{s}) - 1$.

   (i) If $\mathit{\Gamma}^{\mathrm{ch}}$ contains a copy of $\mathit{\Gamma}_0$ and $\left(\mathit{\Gamma}^{\mathrm{ch}} - \mathit{\Gamma}_0\right)$ is compatible, we have finished. If $\mathfrak{s}''$ is a maximal rank subalgebra of $\mathfrak{s}'$ such that $\mathit{\Gamma}'^{\mathrm{ch}}$ is also compatible, then we obtain the contraction $\mathfrak{s} \rightsquigarrow \mathfrak{s}''\overrightarrow{\oplus}_{\mathit{\Gamma}'^{\mathrm{ch}} \oplus \mathit{\Gamma}^{\mathrm{ch}}} \mathfrak{h}_N$.

   (ii) If $\mathit{\Gamma}^{\mathrm{ch}}$ contains a copy of $\mathit{\Gamma}_0$ and $\left(\mathit{\Gamma}^{\mathrm{ch}} - \mathit{\Gamma}_0\right)$ is not compatible, we have to consider a maximal rank subalgebra $\mathfrak{s}''$ of $\mathfrak{s}'$. We obtain the branching rule $ad\,(\mathfrak{s}) = ad\,(\mathfrak{s}'') \oplus \mathit{\Gamma}'^{\mathrm{ch}} \oplus \mathit{\Gamma}^{\mathrm{ch}}|_{\mathfrak{s}''}$, and it suffices that $\mathit{\Gamma}'^{\mathrm{ch}} \oplus \left(\mathit{\Gamma}^{\mathrm{ch}} - \mathit{\Gamma}_0\right)|_{\mathfrak{s}''}$ is a compatible representation of $\mathfrak{s}''$.

   (iii) If $\mathit{\Gamma}^{\mathrm{ch}}$ does not contain a copy of $\mathit{\Gamma}_0$, then its reduction $\mathit{\Gamma}^{\mathrm{ch}}|_{\mathfrak{s}''}$ must contain a copy of the identity representation. In fact, since the branching rule for $\mathfrak{s}' \supset \mathfrak{s}''$ is $ad\,(\mathfrak{s}') = ad\,(\mathfrak{s}'') \oplus \mathit{\Gamma}'^{\mathrm{ch}}$, the characteristic representation cannot contain copies of $\mathit{\Gamma}_0$ because both Lie algebras have the same rank. In addition $\left(\mathit{\Gamma}^{\mathrm{ch}}|_{\mathfrak{s}''} - \mathit{\Gamma}_0\right) \oplus \mathit{\Gamma}'^{\mathrm{ch}}$ must be a compatible representation of $\mathfrak{s}''$.

Observe that the argument is independent on the maximality of the embedded subalgebras. Therefore, an iteration process allows us to generalize this result to chains of subalgebras

$$\mathfrak{s} \supset \mathfrak{s}' \supset \mathfrak{s}'' \supset \ldots \supset \mathfrak{s}^{(k)}\,,$$

where $\mathrm{rank}(\mathfrak{s}^{(k)}) = \mathrm{rank}(\mathfrak{s}) - 1$. The iteration is performed by reducing each step with respect to a maximal subalgebra [32].

In addition to the general branching rules, we have to add some criteria that will allow us to decide whether the induces representations that appear with odd multiplicity satisfy the condition 3 of (11). To this extent, we have to analyze the wedge products of tensor products of representations.

Lemma 2 *Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be semisimple Lie algebras and V_1, V_2 be self-dual irreducible representations of $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively. Then $V_1 \otimes V_2$ is a compatible representation of $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ if and only if one of the following conditions is satisfied*

1. $\mathrm{Sym}^2 V_1 \supseteq \mathit{\Gamma}_0$ *and* $\overset{2}{\bigwedge} V_2 \supseteq \mathit{\Gamma}_0$,

2. $\overset{2}{\bigwedge} V_1 \supseteq \mathit{\Gamma}_0$ *and* $\mathrm{Sym}^2 V_2 \supseteq \mathit{\Gamma}_0$.

The proof follows easily from the properties of tensor products. For $i = 1, 2$ we have the well known decomposition

$$V_i \otimes V_i = \mathrm{Sym}^2 V_i \oplus \overset{2}{\bigwedge} V_i\,,$$

where $\mathrm{Sym}^2 V_i$ denotes the symmetric tensor product, and $\bigwedge^2 V_i$ the wedge product of the representation V_i [33]. Now, considering the double tensor product $(V_1 \otimes V_2) \otimes (V_1 \otimes V_2)$, it is straightforward to verify that

$$\bigwedge^2 (V_1 \otimes V_2) = \left(\mathrm{Sym}^2 V_1 \otimes \bigwedge^2 V_2 \right) \oplus \left(\bigwedge^2 V_1 \otimes \mathrm{Sym}^2 V_2 \right). \tag{16}$$

Thus, if $V_1 \otimes V_2$ is a compatible representation, it must satisfy condition 3 of (11). The latter holds if and only if the identity representation is contained in one of the summands of $\bigwedge^2 (V_1 \otimes V_2)$, thus if either one of the possibilities above occurs.

This technical result can also be generalized to an arbitrary number of factors $V_1 \otimes \dots \otimes V_n$. In this case, the components of $\bigwedge^2 (V_1 \otimes \dots \otimes V_n)$ will be the products of an even number of spaces $\mathrm{Sym}^2 V_i$ with an odd number of spaces $\bigwedge^2 V_j$.

5. Classification of contractions of exceptional algebras

In this section we classify all semidirect products $\mathfrak{s} \overrightarrow{\oplus}_{\Gamma \oplus \Gamma_0} \mathfrak{h}_N$ that arise as contraction of one of the complex exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8. By the preceding sections, we only have to consider the subalgebras that either have the same rank as $\mathfrak{s}$, or $\mathrm{rank}(\mathfrak{s}) - 1$. We use the same notations as in [34] to denote the irreducible representations $[a_1, \dots, a_l]$ of Lie algebras. In addition, we use the abbreviation $\left[0^k, a_{k+1}, \dots, a_l\right]$ for the representation $\left[0, \dots^{(k \text{ times})} \dots, 0, a_{k+1}, \dots, a_l\right]$, *etc.* In similar way, tensor products $[a_1, \dots, a_l] \otimes [b_1, \dots, b_k]$ are abbreviated by $(a_1, \dots, a_l)\,(b_1, \dots, b_k)$. All branching rules and reductions have been computed using the tables of [34] and [32].

We will analyze the case of E_6 in full detail. For the remaining exceptional algebras, the procedure will be almost the same, and the results will be presented in tabular form. In order to simplify the reading of the branching rules in the tables, the adjoint representation of the subalgebras will always be written in bold face, to distinguish it from the characteristic representation describing the semidirect product.

5.1. The Lie algebra E_6

As follows from Table II, the maximal semisimple algebras[8] of E_6 having either rank 5 or 6 are $D_5, A_1 \times A_5$ and A_2^3. For D_5, all reductions with respect to maximal rank subalgebras have also to be considered, while for

[8] By maximal semisimple subalgebra we mean that the subalgebra is maximal among the semisimple, not necessarily among the reductive subalgebras.

TABLE II

Maximal semisimple subalgebras of exceptional Lie algebras.

$\mathfrak{g}$	Subalgebras satisfying $\mathrm{Rang}(\mathfrak{s})-1\leq \mathrm{Rang}\,(\mathfrak{s}')\leq \mathrm{Rang}(\mathfrak{s})^{\mathrm{a}}$	$ad\,(\mathfrak{g})^{\mathrm{b}}$
G_2	$A_2,\, A_1\times A_1,\, A_1$	$[1,0]$
F_4	$B_4,\, A_2\times A_2,\, C_3\times A_1,\, G_2\times A_1$	$\left[1,0^3\right]$
E_6	$A_5\times A_1,\, A_2\times A_2\times A_2,\, D_5$	$\left[0^5,1\right]$
E_7	$A_7,\, D_6\times A_1,\, A_5\times A_2,\, E_6$	$\left[1,0^6\right]$
E_8	$A_8,\, D_8,\, E_7\times A_1,\, D_6\times B_2,\, A_4\times A_4$	$\left[1,0^7\right]$

[a]The correction to Dynkin's rule on maximality have already been considered [32].
[b]Adjoint representation of $\mathfrak{g}$ in the notation of [32].

$A_1\times A_5$ and A_2^3 we have to determine the reduction with respect to rank five subalgebras. The cases that must be analyzed can be schematized as follows:

$$E_6\supset\left\{\begin{array}{ll} D_5\supset & A_1^2\times A_3 \\ A_1\times A_5\supset & \left\{\begin{array}{l} A_1\times A_4\\ A_1^2\times A_3\\ A_1\times A_2^2\\ A_5\end{array}\right. \\ A_2^3\supset & \left\{\begin{array}{l} A_1\times A_2^2\\ A_1'\times A_2^2\end{array}\right. \end{array}\right. \tag{17}$$

We study the branching rules beginning always the reduction chain with one of the maximal semisimple algebras listed above.

1. $E_6\supset D_5$
   The branching rule for the adjoint representation $\left[0^5,1\right]$ of E_6 is

$$[000001]\supset(\mathbf{01000})+(00010)+(00001)+(00000)\,.$$

Since the spinor representations (00010) and (00001) are mutually dual, the branching rule satisfies the requirements of (11), and the representation $R=(00010)+(00001)$ is compatible with a Heisenberg algebra. This reduction gives rise to the contraction

$$E_6\rightsquigarrow D_5\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_{16}\,. \tag{18}$$

We can further reduce this chain by considering the rank 5 subalgebras of D_5. As follows from the classification of subalgebras of semisimple Lie algebras [31,34], there is only one possibility, namely the reduction $D_5\supset A_1^2\times A_3$. By proposition 2, we only have to analyze the branching rule for the adjoint representation of D_5, since the representation R of D_5 is compatible. The decomposition of the adjoint representation of E_6 with respect to the chain $E_6 \supset D_5 \supset A_1^2 \times A_3$ therefore equals:

$$\begin{aligned}[000001] \supset\ & (\mathbf{0})\,(\mathbf{0})\,(\mathbf{101}) + (\mathbf{2})\,(\mathbf{0})\,(\mathbf{000}) + (\mathbf{0})\,(\mathbf{2})\,(\mathbf{000}) \\ & + (1)\,(1)\,(010) + (0)\,(1)\,(100) + (0)\,(1)\,(001) \\ & + (0)\,(0)\,(000)\ .\end{aligned}$$

It is straightforward to verify that for representation $R = (1)\,(1)\,(010) + (0)\,(1)\,\{(100) + (001)\} + (1)\,(0)\,\{(100) + (001)\} + (0)\,(0)\,(000)$, the constituents satisfy the requirements on the multiplicities. However, in order to be compatible, for the self-dual representation $(1)\,(1)\,(010)$ the wedge product $\bigwedge^2 (1)\,(1)\,(010)$ should contain a copy of the trivial representation, according to condition 3 of (11). Using formula (16) we compute this space and obtain:

$$\begin{aligned}\bigwedge^2 (1)\,(1)\,(010)\ =\ & (2)\,(2)\,(101) + (2)\,(0)\,(020) + (2)\,(0)\,\left(0^3\right) \\ & + (0)\,(2)\,(020) + (0)\,(2)\,\left(0^3\right) + (0)\,(0)\,(101)\ .\end{aligned}$$

Since no copy of the identity representation is contained, this representation is not compatible, and therefore, no contraction of E_6 can exist.

2. $E_6 \supset A_1 \times A_5$
Since the subalgebra is of maximal rank, the reduction of the adjoint representation of E_6 cannot contain a copy of the identity representation Γ_0. The branching rule is given by:

$$[000001] \supset (1)\,(00100) + (\mathbf{0})\,(\mathbf{10001}) + (\mathbf{2})\,(\mathbf{00000}).$$

We now reduce with respect to rank 5 subalgebras of $A_1 \times A_5$.

(i) $E_6 \supset A_1 \times A_5 \supset A_1 \times A_4$
The adjoint representation decomposes as:

$$\begin{aligned}[000001]\ \supset\ & (\mathbf{2})\,(\mathbf{0000}) + (\mathbf{0})\,(\mathbf{1001}) + (1)\,(0010) + (1)\,(0100) \\ & + (0)\,(0001) + (0)\,(1000) + (0)\,(0)\,(000)\ .\end{aligned}$$

It is easily verified that the representation $R = (1)\,(0010) + (1)\,(0100) + (0)\,(0001) + (0)\,(1000) + (0)\,(0000)$ satisfies the conditions of (11), thus is compatible. This contraction associated to this reduction is

$$E_6 \rightsquigarrow (A_1 \times A_4) \overrightarrow{\oplus}_{R\oplus\Gamma_0} \mathfrak{h}_{25}\ . \tag{19}$$

(ii) $E_6 \supset A_1 \times A_5 \supset A_1^2 \times A_3$.
In this case, the branching rule reads

$$\begin{aligned}[000001] \supset\ & (\mathbf{2})\,(\mathbf{0})\,(\mathbf{000}) + (\mathbf{0})\,(\mathbf{2})\,(\mathbf{000}) + (\mathbf{0})\,(\mathbf{0})\,(\mathbf{101}) \\ & + (1)\,(1)\,(010) + (1)\,(0)\,(001) + (1)\,(0)\,(100) \\ & + (0)\,(1)\,(001) + (0)\,(1)\,(100) + (0)\,(0)\,(000)\ .\end{aligned}$$

We see that $(1)\,(1)\,(010)$ appears with multiplicity one. To be compatible, it must satisfy condition 3 of (11). As already seen before, $\bigwedge((1)\,(1)\,(010))$ does not contain a copy of Γ_0, thus no compatibility is given.

(iii) $E_6 \supset A_1 \times A_5 \supset A_1 \times A_2^2$

$$\begin{aligned}[000001] \supset\ & (\mathbf{2})\,(\mathbf{00})\,(\mathbf{00}) + (\mathbf{0})\,(\mathbf{11})\,(\mathbf{00}) + (\mathbf{0})\,(\mathbf{00})\,(\mathbf{11}) \\ & + (1)\,(10)\,(01) + (1)\,(01)\,(10) + 2\,(1)\,(00)\,(00) \\ & + (0)\,(10)\,(01) + (0)\,(01)\,(10) + (0)\,(0)\,(000)\ .\end{aligned}$$

Now we have the isomorphisms $((1)\,(10)\,(01))^* \simeq (1)\,(01)\,(10)$ and $((0)\,(10)\,(01))^* \simeq (0)\,(01)\,(10)$, which imply that the representation is compatible. In this case, we obtain the contraction

$$E_6 \rightsquigarrow \left(A_1 \times A_2^2\right) \overrightarrow{\oplus}_{R\oplus\Gamma_0} \mathfrak{h}_{29}. \tag{20}$$

(iv) $E_6 \supset A_1 \times A_5 \supset A_5$
This case is entirely trivial, since the corresponding branching rule is simply

$$[000001] \supset (\mathbf{10001}) + 2(00100) + 3(00000)\,.$$

The contraction provided by this reduction is

$$E_6 \rightsquigarrow A_5 \overrightarrow{\oplus}_{R\oplus 3\Gamma_0} \mathfrak{h}_{21}\,. \tag{21}$$

3. $E_6 \supset A_2^3$
Again the first reduction contains no copy of the identity representation, for being a maximal rank subalgebra. The adjoint representation gives the branching rule

$$\begin{aligned}[000001] \supset\ & (\mathbf{11})\,(\mathbf{00})\,(\mathbf{00}) + (\mathbf{00})\,(\mathbf{11})\,(\mathbf{00}) + (\mathbf{00})\,(\mathbf{00})\,(\mathbf{11}) \\ & + (10)\,(01)\,(10) + (01)\,(10)\,(01)\ .\end{aligned}$$

It follows from this decomposition that all copies of A_2 involve the same representations. This means that the reductions will give rise to the same branching rules. It, therefore, suffices to reduce with respect to one of the A_2 copies. We reduce with respect to the last copy of A_2.

(i) $E_6 \supset A_2^3 \supset A_2^2 \times A_1$

$$\begin{aligned}[000001] \supset\ & (\mathbf{11})\,(\mathbf{00})\,(\mathbf{0}) + (\mathbf{00})\,(\mathbf{11})\,(\mathbf{0}) + (\mathbf{00})\,(\mathbf{00})\,(\mathbf{2}) \\ & + (10)\,(01)\,(1) + (10)\,(01)\,(0) + (01)\,(10)\,(1) \\ & + (01)\,(10)\,(0) + 2\,(00)\,(00)\,(1) + (0)\,(0)\,(000)\ .\end{aligned}$$

All irreducible components of $R = (10)\,(01)\,(1) + (10)\,(01)\,(0) + (01)\,(10)\,(1) + (01)\,(10)\,(0) + 2\,(00)\,(00)\,(1)$ have either even multiplicity or the same multiplicity as its contragredient representation. Observe further that the branching rule is identical to that of case 2*(iii)* above. As a consequence, the contracted Lie algebra will be isomorphic:

$$E_6 \rightsquigarrow \left(A_1 \times A_2^2\right) \overrightarrow{\oplus}_{R \oplus \Gamma_0} \mathfrak{h}_{29}\,. \tag{22}$$

(ii) $E_6 \supset A_2^3 \supset A_2^2 \times A_1'$
For the singular embedding of A_1 into A_2, no copy of the trivial representation is obtained, since the adjoint representation (11) of A_2 branches as

$$(11) \supset (4) + (2)\ .$$

In absence of a copy of the trivial representation, no contraction can be obtained for this embedding.

This finishes the analysis of E_6. The contractions are reproduced, together with the scalar N, in Table III. The embedding $E_6 \supset A_5$ has the particularity that Γ_0 has multiplicity three. In addition to the listed contraction, another one onto a Lie algebra with a three dimensional centre will be possible. We also observe that the third and last cases give rise to

TABLE III

Contractions of E_6.

	Reduction chain	Branching rule	N
1	$E_6 \supset D_5$	$(\mathbf{01000}) + (00010) + (00001) + (00000)$	16
2	$E_6 \supset A_1 \times A_5 \supset A_1 \times A_4$	$(\mathbf{2})\left(\mathbf{0^4}\right) + (\mathbf{0})\left(\mathbf{10^21}\right) + (1)\left(0^210\right) + (1)\left(010^2\right)$ $+(0)\left(10^3\right) + (0)\left(0^31\right) + (0)\left(0^4\right)$	25
3	$E_6 \supset A_1 \times A_5 \supset A_1 \times A_2^2$	$(\mathbf{2})\,(\mathbf{00})\,(\mathbf{00}) + (\mathbf{0})\,(\mathbf{11})\,(\mathbf{00}) + (\mathbf{0})\,(\mathbf{00})\,(\mathbf{11})$ $+ (0)\,(10)\,(01) + (0)\,(01)\,(10) + (1)\,(10)\,(01)$ $+ (1)\,(01)\,(10) + 2\,(1)\,(00)\,(00) + (0)\,(00)\,(00)$	29
4	$E_6 \supset A_1 \times A_5 \supset A_5$	$(\mathbf{10001}) + 2\,(00100) + 3\left(0^5\right)$	21
5	$E_6 \supset A_2^3 \supset A_1 \times A_2^2$	Branching rule identical to case 3	29

the same branching rule, in consequence, to equivalent contractions of E_6, but following different reductions chains. Such a pattern is by no means an exception, and will also appear often in the analysis of the remaining exceptional Lie algebras, up to the rank 2 Lie algebra G_2.

5.2. The Lie algebra E_7

For the rank seven exceptional algebra we have three maximal semisimple subalgebras of rank seven and one of rank six (see Table II). Reducing further the former to rank six subalgebras, we get 45 different reductions chains to be analyzed. These embeddings are schematically reproduced in the following diagram:

$$
E_7 \supset \left\{
\begin{array}{ll}
A_7 \supset & \left\{ \begin{array}{c} A_6 \\ A_1 \times A_5 \\ A_2 \times A_4 \\ A_3^2 \end{array} \right. \\
E_6 \supset & \left\{ \begin{array}{c} A_1 \times A_5 \\ A_2^3 \end{array} \right. \\
A_1 \times D_6 \supset & \left\{ \begin{array}{ll}
A_1 \times D_5 & \\
A_1 \times B_5 \supset & \left\{ \begin{array}{l} A_1 \times D_5 \supset A_1^3 \times A_3 \\ A_1^2 \times B_3 \supset A_1^3 \times A_3 \\ A_1^2 \times D_4 \supset A_1^6 \\ A_1 \times A_3 \times C_2 \supset A_1^3 \times A_3 \end{array} \right. \\
A_1^3 \times D_4 \supset & \left\{ \begin{array}{l} A_1^2 \times B_3 \supset A_1^3 \times A_3 \\ A_1^4 \times C_2 \supset A_1^6 \\ A_1^7 \supset A_1^6 \end{array} \right. \\
A_1 \times A_3^2 \supset & \left\{ \begin{array}{l} A_1 \times A_3 \times A_2 \\ A_1 \times A_3 \times C_2 \supset A_1^3 \times A_3 \\ A_1^3 \times A_3 \end{array} \right. \\
A_1^2 \times B_4 \supset & \left\{ \begin{array}{l} A_1^2 \times D_4 \supset A_1^6 \\ A_1^4 \times C_2 \supset A_1^6 \\ A_1 \times C_2 \times A_3 \end{array} \right. \\
A_1 \times C_2 \times B_3 \supset & \left\{ \begin{array}{l} A_1 \times C_2 \times A_3 \supset A_1^3 \times A_3 \\ A_1^3 \times B_3 \supset A_1^3 \times A_3 \\ A_1^4 \times C_2 \supset A_1^6 \end{array} \right.
\end{array} \right. \\
A_2 \times A_5 \supset & \left\{ \begin{array}{c} A_1 \times A_5 \\ A_1' \times A_5 \\ A_2 \times A_4 \\ A_2 \times A_1 \times A_3 \\ A_2^3 \end{array} \right.
\end{array} \right.
\tag{23}
$$

Analyzing the branching rule for the adjoint representation of E_7 along each of these reduction chains, and proceeding exactly by the same procedure as done for E_6, we find that among these embeddings, only 13 of them will provide compatible representations, therefore contractions of the required type. These compatible reductions with their corresponding branching rules and the scalar N are given in Table IV.

TABLE IV

Contractions of E_7.

	Reduction chain	Branching rule	N
1	$E_7 \supset A_7 \supset A_6$	$(\mathbf{10^41})+(0^310^2)+(0^210^3)+(10^5)+(0^51)+(0^6)$	42
2	$E_7 \supset A_7 \supset A_1\times A_5$	$(\mathbf{0})(\mathbf{10^31})+(\mathbf{2})(\mathbf{0^5})+(1)(0^210^2)+(0)(010^3)$ $+(0)(0^310)+(1)(10^4)+(1)(0^41)+(0)(0^5)$	47
3	$E_7 \supset A_7 \supset A_2\times A_4$	$(\mathbf{11})(\mathbf{0^4})+(\mathbf{00})(\mathbf{10^21})+(10)(0^210)+(10)(0^31)$ $+(01)(010^2)+(01)(10^3)+(00)(0^31)$ $+(00)(10^3)+(00)(0^4)$	50
4	$E_7 \supset E_6$	$(\mathbf{0^51})+(0^410)+(10^5)+(0^6)$	27
5	$E_7 \supset E_6 \supset A_1\times A_5$	Branching rule identical to case 2	47
6	$E_7 \supset E_6 \supset A_2^3$	$(\mathbf{11})(\mathbf{00})(\mathbf{00})+(\mathbf{00})(\mathbf{11})(\mathbf{00})+(\mathbf{00})(\mathbf{00})(\mathbf{11})$ $+(10)(01)(10)+(01)(10)(01)+(10)(00)(01)$ $+(01)(00)(10)+(01)(01)(00)+(10)(10)(00)$ $+(00)(10)(10)+(00)(01)(01)+(00)(00)(00)$	54
7	$E_7 \supset A_1\times D_6 \supset A_1\times A_5$	$(\mathbf{2})(\mathbf{0^5})+(\mathbf{0})(\mathbf{10^31})+(0)(010^3)+(0)(0^310)$ $+(1)(010^3)+(1)(0^310)+2(1)(0^5)+(0)(0^5)$	47
8	$E_7 \supset A_1\times D_6 \supset A_1^3\times D_4$ $\supset A_1^3\times B_3 \supset A_1^3\times A_3$	$(\mathbf{2})(\mathbf{0})(\mathbf{0})(\mathbf{0^3})+(\mathbf{0})(\mathbf{2})(\mathbf{0})(\mathbf{0^3})+(\mathbf{0})(\mathbf{0})(\mathbf{2})(\mathbf{0^3})$ $+(\mathbf{0})(\mathbf{0})(\mathbf{0})(\mathbf{101})+(1)(0)(1)(0^21)$ $+(1)(0)(1)(10^2)+(1)(1)(0)(0^3)$ $+(0)(1)(1)(10^2)+(1)(1)(0)(10^2)$ $+2(0)(0)(0)(010)+(0)(1)(1)(0^21)$ $+(1)(1)(0)(0^21)+(0)(0)(0)(0^3)$	54
9	$E_7 \supset A_1\times D_6 \supset A_1\times A_3^2$ $+\supset A_1\times A_3\times A_2$	$(\mathbf{2})(\mathbf{0^3})(\mathbf{00})+(\mathbf{0})(\mathbf{101})(\mathbf{00})+(\mathbf{0})(\mathbf{0^3})(\mathbf{11})$ $(0)(010)(01)+(0)(010)(10)+(0)(0^3)(10)+$ $(0)(0^3)(01)+(1)(0^21)(10)+(1)(10^2)(01)$ $+(1)(10^2)(00)+(1)(0^21)(00)+(0)(0^3)(00)$	53
10	$E_7 \supset A_2\times A_5 \supset A_1\times A_5$	Branching rule identical to case 7	47
11	$E_7 \supset A_2\times A_5 \supset A_2\times A_4$	Branching rule identical to case 3	50
12	$E_7 \supset A_2\times A_5$ $\supset A_2\times A_1\times A_3$	Branching rule identical to case 9	53
13	$E_7 \supset A_2\times A_5 \supset A_2^3$	Branching rule identical to case 6	54

We see from the table that five of the contractions can be reached by different reduction chains, thus there are only eight non-isomorphic contractions of E_7.

5.3. The exceptional algebras G_2 and F_4

For G_2 the reduction scheme to rank one subalgebras is extremely simple:

$$G_2 \supset A_2 \supset \left\{ \begin{array}{c} A_1 \\ A_1' \end{array} \right. ; \quad G_2 \supset A_1^2 \supset \left\{ \begin{array}{c} A_1 \\ A_1' \end{array} \right. . \tag{24}$$

Of these four cases, three give rise to a contraction of a semidirect product $A_1 \overrightarrow{\oplus}_{R\oplus\Gamma_0} \mathfrak{h}_5$, while the fourth, corresponding to the singular embedding of A_1 in A_2, does not give a compatible representation. We also observe that for any of the obtained contractions, the identity representation has multiplicity three in the describing representation of the semidirect product — see Table V. This phenomenon only occurs for G_2, and is directly related to its low rank.

TABLE V

Contractions of G_2 and F_4.

	Reduction chain	Branching rule	N
1	$G_2 \supset A_2 \supset A_1$	$(\mathbf{2}) + 4\,(1) + 3\,(0)$	5
2	$G_2 \supset A_1^2 \supset A_1$	$(\mathbf{2}) + 2\,(3) + 3\,(0)$	5
3	$G_2 \supset A_1^2 \supset A_1$	$(\mathbf{2}) + 4\,(1) + 3\,(0)\,(0)$	5
1	$F_4 \supset B_4 \supset D_4 \supset B_3$	$(\mathbf{010}) + (100)^2 + (001)^2 + (000)$	15
2	$F_4 \supset B_4 \supset A_1 \times A_3 \supset A_1 \times A_2$	$(\mathbf{2})\,(\mathbf{00}) + (\mathbf{0})\,(\mathbf{11}) + (0)\,(01) + (0)\,(10)$ $+ (2)\,(01) + (2)\,(10) + (1)\,(10) + (1)\,(01)$ $+2\,(1)\,(00) + (0)\,(00)$	20
3	$F_4 \supset A_1 \times C_3 \supset C_3$	$(\mathbf{200}) + 2\,(001) + 3\,(000)$	15
4	$F_4 \supset A_1 \times C_3 \supset A_1 \times A_2$	$(\mathbf{2})\,(\mathbf{00}) + (\mathbf{0})\,(\mathbf{11}) + (1)\,(20) + (1)\,(02)$ $+2\,(1)\,(00) + (0)\,(20) + (0)\,(02) + (0)\,(00)$	20
5	$F_4 \supset A_2 \times A_2 \supset A_2 \times A_1$	Branching rule identical to case 2	20

For the rank four exceptional algebra F_4, 24 possible reductions to rank three subalgebras arise. The reduction scheme is represented graphically as follows:

$$F_4 \supset \left\{ \begin{array}{ll} A_2 \times A_2 \supset & A_1 \times A_2 \\ B_4 \supset & \left\{ \begin{array}{ll} D_4 \supset & \left\{ \begin{array}{ll} B_3 & \supset \left\{ \begin{array}{l} A_3 \\ A_1^3 \end{array} \right. \\ A_1 \times C_2 & \supset A_1^3 \\ A_1^4 & \supset A_1^3 \end{array} \right. \\ A_1^2 \times C_2 \supset & \left\{ \begin{array}{ll} A_1 \times C_2 & \supset A_1^3 \\ A_1^4 & \supset A_1^3 \\ A_1^3 & \end{array} \right. \\ A_1 \times A_3 \supset & \left\{ \begin{array}{l} A_3 \\ A_1 \times A_2 \supset A_1^3 \\ A_1 \times C_2 \supset A_1^3 \\ A_1^3 \end{array} \right. \end{array} \right. \\ A_1 \times C_3 \supset & \left\{ \begin{array}{l} C_3 \supset A_1 \times C_2 \supset A_1^3 \\ A_1 \times A_2 \\ A_1 \times C_2 \supset A_1^3 \\ A_1^3 \end{array} \right. \end{array} \right. . \tag{25}$$

Among these reductions, only five cases will provide compatible representations, thus only five contractions on semidirect products exist, two of which are equivalent. The contractions, with the corresponding branching rules, are also given in Table V. In addition, we observe that for the embedding $F_4 \supset C_3$, the identity representation has multiplicity three.

5.4. The exceptional algebra E_8

The analysis of reductions of E_8-representations with respect to rank seven subalgebras is by far the longest and most complicated case among the exceptional algebras, due to its rich structure of subalgebras and the high number of non-equivalent possibilities. We note that only for the embedding $E_8 \supset D_8$, 97 different reductions must be analyzed. For this reason we skip the detailed reduction scheme, and only indicate the resulting contractions with their corresponding branching rules in Table VI.

TABLE VI

Contractions of E_8.

	Reduction chain	Branching rule	N
1	2	3	4
1	$E_8 \supset A_8 \supset A_7$	$(\mathbf{10^51}) + (0^61) + (10^6) + (0^210^4) + (0^410^2) + (010^5)$ $+ (0^510) + (0^7)$	92
2	$E_8 \supset A_8 \supset A_1 \times A_6$	$(\mathbf{2})(\mathbf{0^6}) + (\mathbf{0})(\mathbf{10^41}) + (1)(0^51) + (1)(10^5) + (0)(0^210^3)$ $+ (0)(0^310^2) + (1)(010^4) + (0)(10^5) + (0)(0^51) + (0)(0^6)$	98
3	$E_8 \supset A_8 \supset A_2 \times A_5$	$(\mathbf{11})(\mathbf{0^5}) + (\mathbf{00})(\mathbf{10^31}) + (10)(0^41) + (01)(10^4)$ $+ (10)(010^3) + (01)(0^310) + 2(00)(0^210^2) + (01)(10^4)$ $+ (10)(0^41) + 3(00)(0^5)$	102
4	$E_8 \supset A_8 \supset A_3 \times A_4$	$(\mathbf{101})(\mathbf{0^4}) + (\mathbf{0^3})(\mathbf{10^21}) + (10^2)(0^31) + (0^21)(10^3)$ $+ (10^2)(010^2) + (0^21)(0^210) + (010)(10^3) + (010)(0^31)$ $+ (0^3)(0^210) + (0^3)(010^2) + (0^21)(0^4) + (10^2)(0^4)$ $+ (0^3)(0^4)$	104
5	$E_8 \supset D_8 \supset A_7$	Branching rule identical to case 1	92
6	$E_8 \supset D_8 \supset B_7 \supset D_7$	$(\mathbf{010^5}) + 2(10^6) + (0^510) + (0^61) + (0^7)$	
7	$E_8 \supset D_8 \supset A_1^2 \times D_6$ $\supset A_1^2 \times A_5$	$(\mathbf{2})(\mathbf{0})(\mathbf{0^5}) + (\mathbf{0})(\mathbf{2})(\mathbf{0^5}) + (\mathbf{0})(\mathbf{0})(\mathbf{10^31}) + (1)(0)(0^210^2)$ $+ (0)(0)(010^3) + (0)(1)(010^3) + (0)\{(0) + (1)\}(0^310)$ $+2(0)(1)(0^5) + (1)\{(0)+(1)\}\{(10^4) + (0^41)\} + (0)(0)(0^5)$	103
8	$E_8 \supset D_8 \supset A_3 \times D_5$ $\supset A_3 \times A_4$	Branching rule identical to case 4	104
9	$E_8 \supset D_8 \supset A_3 \times D_5$ $\supset A_2 \times D_5$	$(\mathbf{11})(\mathbf{0^5}) + (\mathbf{02})(\mathbf{010^3}) + (10)(0^41) + (01)(0^310)$ $+(10)(10^4) + (01)(10^4) + (00)(0^41) + (00)(0^310)$ $+(10)(0^5) + (01)(0^5) + (00)(0^5)$	97
10	$E_8 \supset A_4 \times A_4$ $\supset A_4 \times A_3$	$(\mathbf{101})(\mathbf{0^4}) + (\mathbf{0^3})(\mathbf{10^21}) + (10^2)(0^31) + (0^21)(10^3)$ $+(0^21)(010^2) + (10^2)(0^210) + (010)(10^3)$ $+(010)(0^31) + (0^3)(0^210) + (0^3)(010^2) + (0^21)(0^4)$ $+(10^2)(0^4) + (0^3)(0^4)$	104
11	$E_8 \supset A_4 \times A_4$ $\supset A_4 \times A_1 \times A_2$	$((\mathbf{10^21})(\mathbf{0}) + (\mathbf{0^4})(\mathbf{2}))(\mathbf{00}) + (\mathbf{0^4})(\mathbf{0})(\mathbf{11})$ $+(0^31)(0)(0^2) + (0^4)(1)\{(01) + (10)\}$ $+(0^31)\{(1)(10) + (0)(01)\} + (0^210)\{(0)(10)+(1)(00)\}$ $+(010^2)\{(0)(01) + (1)(00)\}$ $+(10^3)\{(1)(01) + (0)(10) + (0)(00)\} + (0^4)(0)(00)$	106
12	$E_8 \supset A_1 \times E_7 \supset$ $A_1 \times A_7 \supset A_1 \times A_6$	Branching rule identical to case 2	98
13	$E_8 \supset A_1 \times E_7 \supset$ $A_1 \times A_7 \supset A_1^2 \times A_5$	$(\mathbf{2})(\mathbf{0})(\mathbf{0^5}) + (\mathbf{0})(\mathbf{2})(\mathbf{0^5}) + (\mathbf{0})(\mathbf{0})(\mathbf{10^31})$ $+(0)(1)(0^210^2) + (0)(0)(010^3)$ $+\{(0) + (1)\}(0)(0^310)$ $+2(1)(0)(0^5) + (1)(0)(010^3)$ $+\{(0) + (1)\}(1)\{(10^4) + (0^41)\} + (0)(0)(0^4)$	102

1	2	3	4
14	$E_8 \supset A_1 \times E_7 \supset A_1 \times A_7 \supset A_4 \times A_1 \times A_2$	$(\mathbf{10^2 1})(\mathbf{0})(\mathbf{00}) + (\mathbf{0^4})(\mathbf{2})(\mathbf{00}) + (\mathbf{0^4})(\mathbf{0})(\mathbf{11})$ $+\left(010^2\right)(1)(00) + (0^4)(1)\{(01) + (10)\}$ $+(010^2)(0)(01) + (0^2 10)(1)(00)$ $+(10^3)\{(1)(10) + (0)(01) + (0)(00)\} + (0^2 10)(0)(10)$ $+(0^3 1)\{(1)(01) + (0)(10) + (0)(00)\} + (0^4)(0)(00)$	106
15	$E_8 \supset A_1 \times E_7 \supset A_1 \times E_6$	$(\mathbf{2})\left(\mathbf{0^6}\right) + (\mathbf{0})\left(\mathbf{0^5 1}\right) + \{(1) + (0)\}\left\{\left(0^4 10\right) + \left(10^5\right)\right\}$ $+2(1)\left(0^6\right) + (0)\left(0^6\right)$	83
16	$E_8 \supset A_1 \times E_7 \supset A_1 \times E_6 \supset A_1 \times A_2^3$	$(\mathbf{2})(\mathbf{00})(\mathbf{00})(\mathbf{00}) + (\mathbf{0})(\mathbf{11})(\mathbf{00})(\mathbf{00}) + (\mathbf{0})(\mathbf{00})(\mathbf{11})(\mathbf{00})$ $+(\mathbf{0})(\mathbf{00})(\mathbf{00})(\mathbf{11}) + 2(1)(00)(00)(00)$ $+(0)(10)(01)(10)$ $+\{(0) + (1)\}(00)\{(01)(01) + (10)(10)\}$ $+(0)(01)(10)(01) + \{(0) + (1)\}$ $\times((10)(10)(00) + (10)(00)(01) + (01)(00)(10))$ $+(0)(01)(01)(00) + (1) + (01)(01)(00) + (00)(00)(00)$	110
17	$E_8 \supset A_1 \times E_7 \supset A_1 \times E_6 \supset A_1^2 \times A_5$	Branching rule identical to case 7	103
18	$E_8 \supset A_2 \times E_6 \supset A_1 \times E_6$	Branching rule identical to case 15	83
19	$E_8 \supset A_2 \times E_6 \supset A_2 \times D_5$	Branching rule identical to case 9	97
20	$E_8 \supset A_2 \times E_6 \supset A_1 \times A_2 \times A_5 \supset A_1^2 \times A_5$	Branching rule identical to case 7	103
21	$E_8 \supset A_2 \times E_6 \supset A_1 \times A_2 \times A_5 \supset A_1 \times A_2 \times A_4$	Branching rule identical to case 14	106
22	$E_8 \supset A_2 \times E_6 \supset A_1 \times A_2 \times A_5 \supset A_1 \times A_2^3$	Branching rule identical to case 17	110
23	$E_8 \supset A_2 \times E_6 \supset A_2^4 \supset A_1 \times A_2^3$	Branching rule identical to case 17	110

6. Remarks on the Casimir operators of contractions

As already follows from the general properties of semidirect products $\mathfrak{s}' \overrightarrow{\oplus}_{R\oplus\Gamma_0} \mathfrak{h}_N$, the degrees of the rank $(\mathfrak{s}')+1$ Casimir operators are $1, 2\deg C_i$, $i = 1, \dots, \operatorname{rank}(\mathfrak{s})$, where the C_i are the primitive Casimir operators[9] of $\mathfrak{s}'$ [7, 11]. In particular, only the generator of the centre is an invariant of odd degree.

Since in addition the semidirect products obtained here are contractions of exceptional algebras $\mathfrak{s}'$, one may ask how the invariants of the contraction can be recovered from the Casimir operators of $\mathfrak{s}$. Unfortunately, there is

[9] With primitive we mean that the operator cannot be written as a polynomial of lower order operators [2].

no direct and obvious procedure to construct a basis of invariants, as the contraction of the primitive invariants, using formula (9), will generally lead to dependent functions of $\mathfrak{s}'$ [17]. In particular, the quadratic Casimir operator will always contract onto the square of the centre generator of the Heisenberg algebra. This happens because the generator of the centre after the contraction belongs to the Cartan subalgebra of $\mathfrak{s}$, as shown in section 4. Thus, in order to obtain the Casimir operators of $\mathfrak{s}'\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_N$ as a contraction of the invariants of $\mathfrak{s}$, we have to consider functions of the primitive Casimir operators. This problem is deeply related to the labelling problem [35,36], which is far from being completely solved. In order to illustrate the difficulties of deriving the invariants of the contraction from those of the contracted Lie algebra, take for instance the embedding $E_6 \supset D_5$ and the associated contraction $E_6 \rightsquigarrow D_5\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_{16}$. E_6 has primitive Casimir operators $\{C_2, C_5, C_6, C_8, C_9, C_{12}\}$ of degrees $2, 5, 6, 8, 9$ and 12, respectively. The Casimir operators of D_5 have degrees $2, 4, 5, 6$ and 8, so that the degrees of the invariants $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $D_5\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_{16}$ are $1, 4, 8, 10, 12$ and 16, respectively. It is, therefore, clear that contracting the operators C_5 and C_9 of E_6 will lead to dependent functions on the contraction. For example, to recover the invariant F_4 of degree 10, we should consider first a Casimir operator of E_6 having this degree. This amounts to analyze the generic linear combinations

$$a_1C_2C_8 + a_2C_2^2C_6 + a_3C_5^2 + a_4C_2^5$$

and determine for which values of the a_i the limit (9) provides an invariant of $D_5\overrightarrow{\oplus}_{R\oplus\Gamma_0}\mathfrak{h}_{16}$ that is primitive. This example shows the sharp limitation of the expansion method to determine the Casimir operators of Lie algebras from those of a contraction, at least for this class of algebras.

7. Applications to Schrödinger algebras

It follows at once from the results in Tables III–VI that the Schrödinger algebras $\widehat{S}(N)$ cannot appear as a contraction of an exceptional simple Lie algebra. Actually this result can be extended to classical Lie algebras, without any need to apply the cohomological or geometrical tools, as usually done [8]. The branching rule obtained from the the Levi decomposition of $\widehat{S}\,(N)$ and the properties of indices of representations, which we recall briefly, are enough to prove the assertion.

Consider an embedding $f : \mathfrak{s}' \longrightarrow \mathfrak{s}$ of a Lie algebra $\mathfrak{s}'$ into a simple Lie algebra $\mathfrak{s}$. Any embedding of Lie algebras determines an integer factor j_f given by the relation

$$\big(f(x), f(x') = j_f\,\big(x, x'\big)\ , \tag{26}$$

where $(.,.)$ is the usual scalar product defined on $\mathfrak{s}$. This scalar is generally called the embedding index of $\mathfrak{s}'$ in the Lie algebra $\mathfrak{s}$. Given disjoint subalgebras $\mathfrak{s}'_j$ of $\mathfrak{s}$, the direct sum of the subalgebras defines an embedding $f=\sum f_i$, the index of which is simply the sum of the various indices j_{f_i}. Further, for reduction chains $\mathfrak{s}\supset\mathfrak{s}'\supset\mathfrak{s}''$, the index of the last algebra in $\mathfrak{s}$ is the product of the corresponding indices of the chain members [31]. The most important property used here concerns the representations. Given $f:\mathfrak{s}'\rightarrow\mathfrak{s}$ and a linear representation Φ of $\mathfrak{s}$, then the embedding index is determined by the formula:

$$j_f=\frac{l_{f\Phi}}{l_{\Phi}}\,, \tag{27}$$

where l_{Φ} and $l_{f\Phi}$ denote the index of Φ and the induced representation $f\Phi$ on the subalgebra, respectively. We recall that the index l_{Φ} of any representation is obtained from the formula $l_{\Phi}=(\dim\Phi/\dim\mathfrak{s})C_2(\Phi)$, where C_2 is the quadratic Casimir operator.

Now the Schrödinger algebra $\widehat{S}\,(N)$ has the Levi decomposition

$$\widehat{S}\,(N)=(\mathfrak{so}\,(N)\oplus\mathfrak{sl}\,(2,R))\,\overrightarrow{\oplus}_R\mathfrak{h}_N\,,$$

where $R=\left[10^{N-1}\right]\otimes D_{\frac{1}{2}}\oplus\Gamma_0$ is the tensor product of the standard representations. If we suppose that there exists a simple Lie algebra $\mathfrak{s}$ that contracts onto $\widehat{S}\,(N)$, then the adjoint representation of $\mathfrak{s}$ must satisfy the branching rule

$$ad\,(\mathfrak{s})\supset\left[010^{N-2}\right]\otimes[0]\oplus\left[0^N\right]\otimes[2]\oplus\left[10^{N-1}\right]\otimes D_{\frac{1}{2}}\oplus\Gamma_0\,. \tag{28}$$

By Proposition 1, the rank of $\mathfrak{s}$ must be $l=2+\left[\frac{N}{2}\right]$. In addition, for the embedding $(\mathfrak{so}(N)\oplus\mathfrak{sl}(2,R))\subset\mathfrak{s}$, the index j_f must be a positive integer. By formula (27), the index of the induced representation $j_{\phi f}$ can be easily computed using the properties above, and equals

$$j_{\phi f}=3N+4$$

for any N. We now distinguish two cases, according to the parity of N.

1. N is odd. In this case, the rank of $\mathfrak{s}$ is $l=\frac{1}{2}\,(N+3)$. We argue considering the different classical algebras:

    (i) $\mathfrak{s}\simeq A_l$. The index of the adjoint representation is $j_{\phi}=2l+2=N+5$. It follows from (27) that

    $$j_f=\frac{3N+4}{N+5}<3$$

for all N. This means that either $j_f = 1, 2$. For $j_f = 1$ we get no integer N, while for $j_f = 2$ we obtain $N = 6$, contradicting that it is odd.

(ii) $\mathfrak{s} \simeq B_l$. Then $j_\phi = 4l - 2 = 2N + 4$. We obtain that $j_f = \frac{3N+4}{2N+4} < \frac{3}{2}$. The only possibility $j_f = 1$ would imply that $N = 0$, which does not provide a solution.

(iii) $\mathfrak{s} \simeq C_l$. This case is identical to *(i)* since the index of the adjoint representation is identical $j_\phi = 2l + 2$.

(iv) $\mathfrak{s} \simeq D_l$. Here $j_\phi = 4l - 4 = 2N + 2$ and $j_f = \frac{3N+4}{2N+2} < \frac{3}{2}$. The only possibility $j_f = 1$ leads to $N = 2$, contradicting the fact that N is odd.

2. N is even. The rank of $\mathfrak{s}$ must be, therefore, $l = N/2 + 2$.

(i) $\mathfrak{s} \simeq A_l$. The index of the adjoint representation is $j_\phi = 2l + 2 = N + 6$. Thus

$$j_f = \frac{3N + 4}{N + 6} < 3\,.$$

If $j_f = 2$, then $N = 8$ and $l = 6$. Here $\dim A_6 = \dim \widehat{S}\,(8) = 48$. However, it is straightforward to verify that $D_4 \times A_1$ is not a rank five subalgebra of A_6 [31,34]. Therefore, no contraction of A_6 onto $\widehat{S}\,(8)$ can exist. For $j_f = 1$ no integer rank l is obtained, thus this case cannot appear.

(ii) $\mathfrak{s} \simeq B_l$. Then $j_\phi = 4l - 2 = 2N + 6$ and $j_f = \frac{3N+4}{2N+6} < \frac{3}{2}$. For the only possibility $j_f = 1$ we would obtain that $N = 2$ and $l = 3$, but $\dim \widehat{S}\,(2) = 9$ and $\dim B_3 = 21$, which contradicts the assumption.

(iii) $\mathfrak{s} \simeq C_l$. This case is similar to *(i)*. Again, the Lie algebra $D_4 \times A_1$ is not a subalgebra of C_6.

(iv) $\mathfrak{s} \simeq D_l$. Here $j_\phi = 4l - 4 = 2N + 4$ and $j_f = \frac{3N+4}{2N+4} < \frac{3}{2}$. The only possibility $j_f = 1$ leads to $N = 0$, contradicting that $N > 0$.

A similar argument can be applied to the sum of two simple Lie algebras, but the number of possibilities increases rapidly, and in order to see that the Schrödinger algebras are not contractions of semisimple algebras is better carried out with the usual methods [8,30].

8. Concluding remarks

We have classified all contractions of complex simple exceptional Lie algebras onto semidirect products of semisimple and Heisenberg Lie algebras. An analogous procedure holds for the real forms of the exceptional algebras, with the necessary modifications on their representations [23]. As for the classical algebras, the corresponding contractions can be analyzed up to any fixed rank, using the classification of subalgebras [31] and the general properties of branching rules [32, 34]. At least for low ranks, various interesting cases appear, like the contraction $\mathfrak{su}(5) \rightsquigarrow (\mathfrak{su}(3) \oplus \mathfrak{su}(2)) \overrightarrow{\oplus}_R \mathfrak{h}_6$, which implies the embedding $\mathfrak{su}(5) \supset \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, well known from the Georgi–Glashow model [4], or the reduction chain $A_3 \supset C_2 \supset A_1 \times A_1$, the real forms of which provide the contractions $\mathfrak{so}(5-k, k) \rightsquigarrow \mathfrak{so}(1,3) \overrightarrow{\oplus}_{2\Gamma_4 \oplus \Gamma_0} \mathfrak{h}_4$ $(k = 1, 2, 3)$ used in quantum relativistic kinematics [8]. The procedure has also been shown to be sufficient to prove that the class of Schrödinger do not arise as contractions of simple Lie algebras. This is an important fact concerning the stability of models described by this type of symmetry [9].

On the other hand, the scalar N giving the number of boson operator pairs in the contraction $\mathfrak{s} \overrightarrow{\oplus}_{\Gamma \oplus \Gamma_0} \mathfrak{h}_N$ can be interpreted as an upper rank for the number of additional labelling operators necessary to determine the set of elementary multiplets for the reduction chain $\mathfrak{s} \supset \mathfrak{s}' \oplus \mathfrak{u}(1)$. The actual number of required labelling operators is $N - \mathrm{rank}(\mathfrak{s})$, following [35]. The high values obtained provide further an estimation of the difficulty in finding these elementary multiplets, as already observed in [6]. Instead of the expansion method, which is not the most suitable procedure for obtaining the labelling operators, as follows from our remarks on the invariants of the contractions, a direct approach seems to be computationally more feasible.

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