CLASSIFICATION OF THE TRACELESS RICCI TENSOR IN 4-DIMENSIONAL PSEUDO-RIEMANNIAN SPACES OF NEUTRAL SIGNATURE

Adam Chudecki

Center of Mathematics and Physics, Lodz University of Technology
Al. Politechniki 11, 90-924 Łódź, Poland
adam.chudecki@p.lodz.pl

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The traceless Ricci tensor C_{ab} in 4-dimensional pseudo-Riemannian spaces equipped with the metric of the neutral signature is analyzed. Its algebraic classification is given. This classification uses the properties of C_{ab} treated as a matrix. The Petrov–Penrose types of *Plebański spinors* associated with the traceless Ricci tensor are given. Finally, the classification is compared with a similar classification in the complex case.

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1. Introduction

The algebraic structure of Ricci tensor in general relativity was investigated by many authors (see, *e.g.*, Petrov [1], Plebański [2], Penrose [3], Hall [4], Plebański and Stachel [5]). In particular, in 1964, an algebraic classification of the traceless Ricci tensor C_{ab} in real 4-dimensional Lorentzian manifolds was given by Plebański in his distinguished work in *Acta Physica Polonica* [2]. Investigation of this problem was motivated by the obvious relation between traceless Ricci tensor and the tensor of matter T_{ab}. Plebański proved that there were exactly 15 different types of the tensor of matter. In [2], the algebraic structure of C_{ab} is investigated from several points of view. First, C_{ab} is considered as a matrix. Then the structure of the so-called *Plebański spinors* has been investigated. It appeared that any tensor of matter can be represented as a superposition of three energy-momentum tensors of the electromagnetic type. Careful analysis of this fact was the third line of studies on the properties of C_{ab} presented in [2].

In the seventies, a great deal of interest was devoted to the complex 4-dimensional spaces. It appeared that the Plebański algebraic classification of the traceless Ricci tensor could be easily carried over to the complex spacetimes [6]. Since it does not make any sense to distinguish between spacelike and timelike vectors in complex spaces, one could expect that the structure of C_{ab} in complex spaces should be less complicated then the analogous structure in the Lorentzian case. Surprisingly, there appeared 17 different types of the traceless Ricci tensor in complex spacetime. Some of these types do not have their counterparts in real Lorentzian case. Results from [6] allowed one to understand better the complex relativity and the differences between complex and real manifolds.

It is worth to note that in both papers [2, 6], the spinorial formalism has been intensively used [7–9]. It helped to simplify the calculations and allowed to define spinorial objects (like the Plebański spinors), which appeared to be essential in further analysis.

Recently, the real 4-dimensional spaces equipped with the metric of the neutral (ultrahyperbolic) signature $(+ + - -)$ has attracted the great deal of interest. The Walker and Osserman spaces, integrable systems, self-dual and anti-self-dual structures, para-Hermite and para-Kähler structures — these all concepts are related to the real 4-dimensional, neutral spaces. Especially interesting are recently discovered relations between real 4-dimensional, neutral Einstein spaces equipped with the para-Kähler structure and the 5-dimensional spaces equipped with the $(2, 3, 5)$-distributions [10, 11]. Thus, it seems that the 4-dimensional pseudo-Riemannian spaces with neutral signature will play more and more important role in the theoretical physics.

Our paper is devoted to such spaces. We investigate the algebraic structure of traceless Ricci tensor C_{ab} in the real 4-dimensional, neutral spaces. To classify C_{ab}, we follow the works by Plebański and Przanowski using the same techniques. Our approach uses discrete classification (the number and type of eigenvectors of C_{ab}) and the continuous classification (the number and type of eigenvalues of the characteristic polynomial of C_{ab}). Moreover, we distinguish spacelike, timelike and null eigenvectors. The Plebański spinors have the same structure as a self-dual or anti-self-dual Weyl spinors and in neutral signature case, they can be divided into 10 different Petrov–Penrose types. This way, we obtain another criteria helpful in classification of the traceless Ricci tensor. Finally, we arrive at 33 different types of C_{ab}. We realize that the structure of the traceless Ricci tensor is much richer that we could suspect.

It is well-known that real analytic spaces can be obtained from the complex spaces as the *real slices*. In many cases, real analytic spaces with the metric of the neutral signature can be obtained from the complex ones particularly simple. It is enough to replace complex variables by real ones and holomorphic functions by real analytic ones. However, in classification of the C_{ab}, there appear subtle differences between complex spaces and real neutral spaces. Single complex type of C_{ab} splits in a few subtypes in the real case. It is related to the existence of the spacelike and timelike vectors in real spaces. In Section 3, we point all these differences by listing generic complex types and real types into which these complex types split.

We believe that our work fills the gap left by the works of Plebański and Przanowski published in *Acta Physica Polonica B* and will be helpful in analysis of non-Einsteinian para-Hermite and para-Kähler spaces. Some applications of ideas presented here have been already used in our work [12].

The paper is organized as follows. In Section 2, a portion of basic facts about the null and orthonormal tetrad in both complex and real neutral spaces is presented. Then we discuss the different types of the roots of the 4^{th} order polynomial and the criteria which allow to distinguish these types. The polynomials with the complex and real coefficients are both discussed. The essential difference between Petrov–Penrose classification of the 4-index, dotted and undotted totally symmetric spinors in complex and real neutral spaces are also sketched. Finally, the new symbol of the type of traceless Ricci tensor is introduced (2.17). At the first glance, this symbol is more complicated then the symbols used by Plebański and Przanowski in [2, 6]. We believe however, that the great number of different types of C_{ab} in real neutral spaces and the complexity of the degeneration schemes (like Scheme 2) justify using such a symbol.

Section 3 is devoted to the detailed classification of the traceless Ricci tensor. We present the canonical forms of C_{ab} and we discuss its possible degenerations. Also, the Petrov–Penrose type of the Plebański spinors is analyzed. The results are gathered in the tables and also the graphs of possible degenerations are presented. Concluding remarks end the paper.

2. Preliminaries

2.1. Formalism

In this section, we present the foundations of the formalism used in this paper. For more detailed treatment, see [7–9].

We consider 4-dimensional manifold $\mathcal{M}$ equipped with the metric tensor ds^2. $\mathcal{M}$ could be complex analytic differentiable manifold endowed with a holomorphic metric ds^2 or a real 4-dimensional smooth differentiable manifold endowed with a real smooth metric ds^2 of the neutral signature $(++--)$. Thus, one deals with *complex relativity* (**CR**) or with *real ultrahyperbolic (neutral) relativity* (**UR**).

The metric of $\mathcal{M}$ in *null tetrad* (e^1, e^2, e^3, e^4) reads

$$ds^2 = g_{ab}\, e^a e^b = 2e^1e^2 + 2e^3e^4\,, \qquad (g_{ab}) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.1}$$

In *orthonormal tetrad* $(E^{1'}, E^{2'}, E^{3'}, E^{4'})$, the metric takes the form of

$$\begin{aligned} ds^2 &= g_{a'b'} E^{a'} E^{b'} = E^{1'}E^{1'} + E^{2'}E^{2'} - E^{3'}E^{3'} - E^{4'}E^{4'}\,, \\ (g_{a'b'}) &:= \mathrm{diag}(1, 1, -1, -1)\,. \end{aligned} \tag{2.2}$$

The relation between null and orthonormal tetrad is

$$\begin{cases} \sqrt{2}\, E^{1'} = e^1 + e^2 \\ \sqrt{2}\, E^{2'} = e^3 + e^4 \\ \sqrt{2}\, E^{3'} = e^1 - e^2 \\ \sqrt{2}\, E^{4'} = e^3 - e^4 \end{cases} \iff \begin{cases} \sqrt{2}\, e^1 = E^{1'} + E^{3'} \\ \sqrt{2}\, e^2 = E^{1'} - E^{3'} \\ \sqrt{2}\, e^3 = E^{2'} + E^{4'} \\ \sqrt{2}\, e^4 = E^{2'} - E^{4'} \end{cases}. \tag{2.3}$$

In the spinorial formalism, the metric reads

$$ds^2 = -\tfrac{1}{2}\, g_{A\dot{B}} g^{A\dot{B}}\,, \qquad A = 1, 2\,, \qquad \dot{B} = \dot{1}, \dot{2}\,, \tag{2.4}$$

where $g_{A\dot{B}}$ are given by

$$\left(g^{A\dot{B}}\right) := \sqrt{2} \begin{bmatrix} e^4 & e^2 \\ e^1 & -e^3 \end{bmatrix}. \tag{2.5}$$

Let us consider now the pair of normalized undotted and dotted spinors, (k_A, l_B), $(k_{\dot{A}}, l_{\dot{B}})$ $k^A l_A = 1$ and $k^{\dot{A}} l_{\dot{A}} = 1$. They generate the new null tetrad $(\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^3, \widetilde{e}^4)$ according to the formulas:

$$\begin{aligned} \sqrt{2}\, \widetilde{e}^1 &:= k_A l_{\dot{B}}\, g^{A\dot{B}}\,, \\ \sqrt{2}\, \widetilde{e}^2 &:= l_A k_{\dot{B}}\, g^{A\dot{B}}\,, \\ -\sqrt{2}\, \widetilde{e}^3 &:= k_A k_{\dot{B}}\, g^{A\dot{B}}\,, \\ \sqrt{2}\, \widetilde{e}^4 &:= l_A l_{\dot{B}}\, g^{A\dot{B}}\,. \end{aligned} \tag{2.6}$$

Define the matrix $g_{aA\dot{B}}$ by the relation $g^{A\dot{B}} = g_a^{A\dot{B}} e^a$. The following identities hold

$$g_{aA\dot{B}} g^{bA\dot{B}} = -2\delta_a^b\,, \qquad g_{aA\dot{B}} g^{aC\dot{D}} = -2\delta_A^C \delta_{\dot{B}}^{\dot{D}}\,. \tag{2.7}$$

Thus, we have

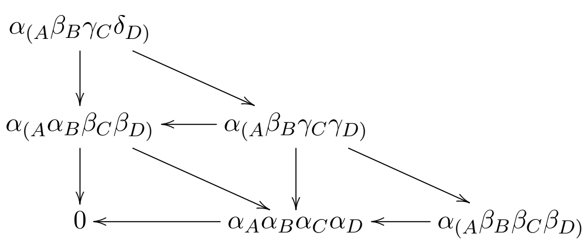
$$\begin{cases} \sqrt{2}\,\widetilde{e}^{1}{}_{a} = k_A l_{\dot{B}}\, g_a{}^{A\dot{B}} \\ \sqrt{2}\,\widetilde{e}^{2}{}_{a} = l_A k_{\dot{B}}\, g_a{}^{A\dot{B}} \\ -\sqrt{2}\,\widetilde{e}^{3}{}_{a} = k_A k_{\dot{B}}\, g_a{}^{A\dot{B}} \\ \sqrt{2}\,\widetilde{e}^{4}{}_{a} = l_A l_{\dot{B}}\, g_a{}^{A\dot{B}} \end{cases} \iff \begin{cases} \widetilde{e}_{2a}\, g^{a}{}_{A\dot{B}} = -\sqrt{2}\, k_A l_{\dot{B}} \\ \widetilde{e}_{1a}\, g^{a}{}_{A\dot{B}} = -\sqrt{2}\, l_A k_{\dot{B}} \\ \widetilde{e}_{4a}\, g^{a}{}_{A\dot{B}} = \sqrt{2}\, k_A k_{\dot{B}} \\ \widetilde{e}_{3a}\, g^{a}{}_{A\dot{B}} = -\sqrt{2}\, l_A l_{\dot{B}} \end{cases} . \tag{2.8}$$

2.2. The Petrov–Penrose classification of totally symmetric 4-index spinors

Algebraic classification of totally symmetric 4-index spinors has been presented in classical paper [7], see also [2]. It can be applied for SD (or ASD) part of the Weyl spinor C_{ABCD} ($C_{\dot{A}\dot{B}\dot{C}\dot{D}}$, respectively). We use these results to classify the Plebański spinors (2.11). First, we consider complex undotted Plebański spinor V_{ABCD} and its contraction with the arbitrary 1-index spinor ξ^A: $\Omega := V_{ABCD}\xi^A\xi^B\xi^C\xi^D$. Clearly, Ω has the form of $\Omega = (\xi^2)^4\,\mathcal{V}(z)$, where $\mathcal{V}(z)$ is a 4^{th} order polynomial in $z := \xi^1/\xi^2$. Due to the fundamental theorem of algebra, Ω can be always brought to the factorized form $\Omega = (\alpha_A\xi^A)(\beta_B\xi^B)(\gamma_C\xi^C)(\delta_D\xi^D)$. Because of the arbitrariness of ξ^A, we find

$$V_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}\,. \tag{2.9}$$

In general, 1-index spinors α_A, β_A, γ_A and δ_A are mutually linearly independent. Such case corresponds to the case when the polynomial $\mathcal{V}(z)$ has four different roots. The possible coincidences between spinors α_A, β_A, γ_A and δ_A brought us to the well-known *Petrov–Penrose diagram* (Scheme 1).



Scheme 1: The Petrov–Penrose diagram.

In the complex case, there are 6 different Petrov–Penrose types of the spinor V_{ABCD}. On the other hand, if we consider real totally symmetric 4-index spinor V_{ABCD}, then the scheme of the roots of $\mathcal{V}(z)$ is more complicated. There appear 10 different Petrov–Penrose types. The symbols which are usually used as abbreviations of the corresponding Petrov–Penrose types of spinor V_{ABCD} and the scheme of the roots of the polynomial $\mathcal{V}(z)$ are gathered in Table I. (In Table I, Z means that the root is complex, while R

stands for the real root, the power denotes the multiplicity of corresponding root, spinors α_A, β_A, γ_A and δ_A are complex, spinors μ_A, ν_A, ξ_A and ζ_A are real, bar stands for the complex conjugation).

TABLE I

The Petrov–Penrose types of complex or real totally symmetric 4-index spinor.

Complex case			Real case		
Type	$V_{ABCD} =$	Roots of $\mathcal{V}(z)$	Type	$V_{ABCD} =$	Roots of $\mathcal{V}(z)$
[I]	$\alpha_{(A}\beta_B\gamma_C\delta_{D)}$	$Z_1Z_2Z_3Z_4$	$[\mathrm{I}]_{\mathrm{r}}$	$\mu_{(A}\nu_B\xi_C\zeta_{D)}$	$R_1R_2R_3R_4$
			$[\mathrm{I}]_{\mathrm{rc}}$	$\mu_{(A}\nu_B\alpha_C\bar{\alpha}_{D)}$	$R_1R_2Z\bar{Z}$
			$[\mathrm{I}]_{\mathrm{c}}$	$\alpha_{(A}\bar{\alpha}_B\beta_C\bar{\beta}_{D)}$	$Z_1\bar{Z}_1Z_2\bar{Z}_2$
[II]	$\alpha_{(A}\beta_B\gamma_C\gamma_{D)}$	$Z_1Z_2Z_3^2$	$[\mathrm{II}]_{\mathrm{r}}$	$\mu_{(A}\nu_B\xi_C\xi_{D)}$	$R_1R_2R_3^2$
			$[\mathrm{II}]_{\mathrm{rc}}$	$\mu_{(A}\mu_B\alpha_C\bar{\alpha}_{D)}$	$R^2Z\bar{Z}$
[D]	$\alpha_{(A}\alpha_B\beta_C\beta_{D)}$	$Z_1^2Z_2^2$	$[\mathrm{D}]_{\mathrm{r}}$	$\mu_{(A}\mu_B\nu_C\nu_{D)}$	$R_1^2R_2^2$
			$[\mathrm{D}]_{\mathrm{c}}$	$\alpha_{(A}\alpha_B\bar{\alpha}_C\bar{\alpha}_{D)}$	$Z^2\bar{Z}^2$
[III]	$\alpha_{(A}\beta_B\beta_C\beta_{D)}$	$Z_1Z_2^3$	$[\mathrm{III}]_{\mathrm{r}}$	$\mu_{(A}\nu_B\nu_C\nu_{D)}$	$R_1R_2^3$
[N]	$\alpha_A\alpha_B\alpha_C\alpha_D$	Z^4	$[\mathrm{N}]_{\mathrm{r}}$	$\mu_A\mu_B\mu_C\mu_D$	R^4
[–]	0	–	[–]	0	–

It is clear that the Petrov–Penrose types of both real and complex totally symmetric 4-index spinors V_{ABCD} are related to the nature of roots of the corresponding polynomial $\mathcal{V}(z)$. It is well-known that such a 4^{th} order polynomial can be always brought to the *canonical form*. The criteria which allow to distinguish the scheme of roots of the 4^{th} order polynomial in the canonical form are discussed in the next subsection.

Of course, similar classification can be applied for the dotted 4-index spinors $V_{\dot{A}\dot{B}\dot{C}\dot{D}}$ and for the "dotted" polynomial $\dot{\mathcal{V}}(\dot{z})$.

2.3. Traceless Ricci tensor

The relation between traceless Ricci tensor C_{ab} and its spinorial image $C_{AB\dot{C}\dot{D}}$ reads

$$C_{ab} = g_a{}^{A\dot{C}} g_b{}^{B\dot{D}} C_{AB\dot{C}\dot{D}} \iff C_{AB\dot{C}\dot{D}} = \tfrac{1}{4} C_{ab}\, g^a{}_{A\dot{C}} g^b{}_{B\dot{D}} \tag{2.10}$$

(compare (2.7)). Using spinorial image $C_{AB\dot{C}\dot{D}}$ of the traceless Ricci tensor C_{ab}, one defines the undotted and dotted *Plebański spinors* by the relations [2, 8]

$$V_{ABCD} := 4\, C_{(AB}{}^{\dot{M}\dot{N}} C_{AC)\dot{M}\dot{N}}\,, \qquad V_{\dot{A}\dot{B}\dot{C}\dot{D}} = 4\, C_{MN(\dot{A}\dot{B}} C^{MN}{}_{\dot{C}\dot{D})}\,. \tag{2.11}$$

The dotted and undotted Plebański spinors are totally symmetric $V_{ABCD} = V_{(ABCD)}$ and $V_{\dot{A}\dot{B}\dot{C}\dot{D}} = V_{(\dot{A}\dot{B}\dot{C}\dot{D})}$.

The characteristic polynomial of the matrix $C^a{}_b$ of the traceless Ricci tensor reads

$$\mathcal{W}(x) := \det\left(C^a{}_b - x\delta^a{}_b\right) = \sum_{i=0}^{4}(-1)^i \underset{[i]}{\mathbb{C}}\, x^{4-i} \equiv \underset{[0]}{\mathbb{C}}x^4 - \underset{[1]}{\mathbb{C}}x^3 + \underset{[2]}{\mathbb{C}}x^2 - \underset{[3]}{\mathbb{C}}x + \underset{[4]}{\mathbb{C}}\,, \tag{2.12}$$

where the coefficients $\underset{[i]}{\mathbb{C}}$ are given by

$$\underset{[0]}{\mathbb{C}} := 1\,, \qquad \underset{[k]}{\mathbb{C}} := C^{a_1}{}_{[a_1} \ldots C^{a_k}{}_{a_k]}\,, \qquad k = 1,2,3,4\,. \tag{2.13}$$

Since the matrix $C^a{}_b$ is traceless, we find that $\underset{[1]}{\mathbb{C}} := C^a{}_a = 0$ so, finally, the characteristic polynomial $\mathcal{W}(x)$ takes the form

$$\mathcal{W}(x) = x^4 + \underset{[2]}{\mathbb{C}}x^2 - \underset{[3]}{\mathbb{C}}x + \underset{[4]}{\mathbb{C}}\,. \tag{2.14}$$

In **UR** coefficients, $\underset{[i]}{\mathbb{C}} \in \boldsymbol{R}$. Criteria which allow us to distinguish the properties of the roots of $\mathcal{W}(x)$ have been widely discussed in [8, 13–15]. Define

$$\begin{aligned} -8J &:= \tfrac{1}{2}\underset{[3]}{\mathbb{C}}^2 - \tfrac{4}{3}\underset{[2]}{\mathbb{C}}\underset{[4]}{\mathbb{C}} + \tfrac{1}{27}\underset{[2]}{\mathbb{C}}^3\,, \qquad I := \underset{[4]}{\mathbb{C}} + \tfrac{1}{12}\underset{[2]}{\mathbb{C}}^2\,, \qquad K := \tfrac{1}{4}\underset{[3]}{\mathbb{C}} \\ L &:= \tfrac{1}{6}\underset{[2]}{\mathbb{C}}\,, \qquad N := \tfrac{1}{4}\underset{[2]}{\mathbb{C}}^2 - \underset{[4]}{\mathbb{C}}\,, \qquad P := -9\underset{[3]}{\mathbb{C}}^2 - 2\underset{[2]}{\mathbb{C}}\left(\underset{[2]}{\mathbb{C}}^2 - 4\underset{[4]}{\mathbb{C}}\right)\,. \end{aligned} \tag{2.15}$$

Then, the discriminant of polynomial (2.14) reads

$$\varDelta = 256\left(I^3 - 27J^2\right)\,. \tag{2.16}$$

As it was mentioned in the previous subsection, there are exactly 9 cases which should be distinguished using the criteria from Table II.

In **CR**, the coefficients $\underset{[i]}{\mathbb{C}}$ are complex. There are only 5 distinct cases (see Table III).

TABLE II

Roots of the quartic equation with real coefficients.

Criteria			Roots
$\Delta < 0$			$R_1 R_2 Z \bar{Z}$
$\Delta > 0$	$L < 0$ and $N > 0$		$R_1 R_2 R_3 R_4$
	$L \geq 0$ or $N < 0$		$Z_1 \bar{Z}_1 Z_2 \bar{Z}_2$
$\Delta = 0$	$I \neq 0,\ J \neq 0$	$P > 0$	$R_1 R_2 R_3^2$
	$(K \neq 0$ or $N \neq 0)$	$P < 0$	$R^2 Z \bar{Z}$
	$I \neq 0,\ J \neq 0$	$J < 0$	$R_1^2 R_2^2$
	$K = N = 0$	$J > 0$	$Z^2 \bar{Z}^2$
	$I = J = 0$	$N \neq 0$ and $K \neq 0$	$R_1 R_2^3$
		$N = K = 0$	R^4

TABLE III

Roots of the quartic equation with complex coefficients.

Criteria			Roots
$\Delta \neq 0$			$Z_1 Z_2 Z_3 Z_4$
$\Delta = 0$	$I \neq 0,\ J \neq 0$	$P \neq 0$	$Z_1 Z_2 Z_3^2$
		$P = 0$	$Z_1^2 Z_2^2$
	$I = J = 0$	$L \neq 0$	$Z_1 Z_2^3$
		$L = 0$	Z^4

2.4. Terminology and symbols

To classify traceless Ricci tensor in **UR**, we use the notation similar to Plebański's notation from [2] and the Plebański–Przanowski notation from [6]. The number of eigenvectors are considered as a main criterion, while the properties of the eigenvalues and the form of the minimal polynomial serve as subcriteria.

The complete information about the type of the matrix, $(C^a{}_b)$ is gathered in the symbol

$$^{[\mathrm{A}]_j \otimes [\mathrm{B}]_k}[n_1 E_1 - n_2 E_2 - \ldots]^{v}_{(q_1 q_2 \ldots)} \,. \tag{2.17}$$

Inside the square bracket, all different eigenvalues E_i, $i = 1, 2, \ldots, N_0$ together with their multiplicities n_i are listed. Of course,

$$\begin{aligned} n_1 + n_2 + \cdots + n_{N_0} &= 4\,, \\ n_1 E_1 + n_2 E_2 + \cdots + n_{N_0} E_{N_0} &= 0\,. \end{aligned} \tag{2.18}$$

The last equality follows from the fact that the matrix $(C^a{}_b)$ is traceless. The characteristic polynomial takes the form of

$$\mathcal{W}(x) = \prod_{i=1}^{N_0} (x - E_i)^{n_i}\,. \tag{2.19}$$

A complex eigenvalue is denoted by Z and a real one by R. Real eigenvalues have additional superscript which denotes the type of the corresponding eigenvector. $R^{\rm s}$ means that the eigenvector which corresponds to the eigenvalue R is spacelike, $R^{\rm t}$ — timelike, $R^{\rm n}$ — null, $R^{\rm ns}$ — null or spacelike, $R^{\rm nt}$ — null or timelike and, finally, $R^{\rm nst}$ means that the eigenvector can be of the arbitrary type. [With respect to the orthonormal tetrad (2.2), the definitions of spacelike, timelike and null vectors are as follows: $V^a V_a > 0$ means that V^a is spacelike, $V^a V_a < 0$ stands for a timelike vector and, finally, $V^a V_a = 0$ means that the vector is null].

Superscript v denotes the number of eigenvectors. Numbers q_i in the round bracket determine the form of the minimal polynomial, *i.e.*, the polynomial of the lowest possible order with the leading term equal to 1 such that $\mathcal{W}_{\min}(C^a{}_b) = 0$. Namely, the minimal polynomial of the matrix $(C^a{}_b)$ has the form of

$$\mathcal{W}_{\min}(x) = \prod_{i=1}^{N_0} (x - E_i)^{q_i}\,. \tag{2.20}$$

Finally, the symbol $[\mathrm{A}]_j \otimes [\mathrm{B}]_k$ defines the Petrov–Penrose types of the Plebański spinors, V_{ABCD} and $V_{\dot{A}\dot{B}\dot{C}\dot{D}}$, respectively (2.11). For example, $[\mathrm{III}]_{\rm r} \otimes [\mathrm{N}]_{\rm r}$ means that V_{ABCD} is of the type $[\mathrm{III}]_{\rm r}$, while $V_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is of the type $[\mathrm{N}]_{\rm r}$.

3. Classification of the traceless Ricci tensor in UR

3.1. Parent Types

The eigenvalue criteria (Table II), the number and the type of eigenvectors and the Petrov–Penrose type of the undotted and dotted Plebański spinors allow to distinguish exactly 33 types of the traceless Ricci tensor. They appear as the degenerations of 9 *parent Types* (according to Plebański's terminology, "Types" by capital "T"). Each of these parent Types has the minimal equation being exactly the Hamilton–Cayley equation. The symbols of the Types are quite similar to the symbols of Petrov–Penrose types of the Plebański spinors. To distinguish them, we do not put the symbol of the Types into the square bracket (as we do in the case of the Petrov–Penrose types of the Plebański spinors). Types I and II have subscripts "r" (all eigenvectors real), "c" (all eigenvectors complex) or "rc" (two eigenvectors complex, one or two eigenvectors real). Types III and IV have only real eigenvectors. However, Type III admits two null eigenvectors (subscript "n"), one null eigenvector and one timelike (subscript "t") or one null eigenvector and one spacelike (subscript "s"). We use the symbols of the parent Types in the **CR** like in [6] (I, II, III_C, III_N and IV).

TABLE IV

Parent Types of C_{ab}.

Type	Symbols of the parent Types
$\mathrm{I_c}$	${}^{[\mathrm{I}]_\mathrm{c}\otimes[\mathrm{I}]_\mathrm{r}}\big[Z_1-\bar{Z}_1-Z_2-\bar{Z}_2\big]^4_{(1111)}$, ${}^{[\mathrm{I}]_\mathrm{r}\otimes[\mathrm{I}]_\mathrm{c}}\big[Z_1-\bar{Z}_1-Z_2-\bar{Z}_2\big]^4_{(1111)}$
$\mathrm{I_{rc}}$	${}^{[\mathrm{I}]_\mathrm{rc}\otimes[\mathrm{I}]_\mathrm{rc}}\big[Z-\bar{Z}-R^\mathrm{s}_1-R^\mathrm{t}_2\big]^4_{(1111)}$
$\mathrm{I_r}$	${}^{[\mathrm{I}]_\mathrm{c}\otimes[\mathrm{I}]_\mathrm{c}}[R^\mathrm{s}_1-R^\mathrm{s}_2-R^\mathrm{t}_3-R^\mathrm{t}_4]^4_{(1111)}$, ${}^{[\mathrm{I}]_\mathrm{r}\otimes[\mathrm{I}]_\mathrm{r}}[R^\mathrm{s}_1-R^\mathrm{s}_2-R^\mathrm{t}_3-R^\mathrm{t}_4]^4_{(1111)}$
$\mathrm{II_{rc}}$	${}^{[\mathrm{II}]_\mathrm{rc}\otimes[\mathrm{II}]_\mathrm{r}}\big[Z-\bar{Z}-2R^\mathrm{n}\big]^3_{(112)}$, ${}^{[\mathrm{II}]_\mathrm{r}\otimes[\mathrm{II}]_\mathrm{rc}}\big[Z-\bar{Z}-2R^\mathrm{n}\big]^3_{(112)}$
$\mathrm{II_r}$	${}^{[\mathrm{II}]_\mathrm{rc}\otimes[\mathrm{II}]_\mathrm{rc}}[R^\mathrm{s}_1-R^\mathrm{t}_2-2R^\mathrm{n}_3]^3_{(112)}$, ${}^{[\mathrm{II}]_\mathrm{r}\otimes[\mathrm{II}]_\mathrm{r}}[R^\mathrm{s}_1-R^\mathrm{t}_2-2R^\mathrm{n}_3]^3_{(112)}$
$\mathrm{III_n}$	${}^{[\mathrm{D}]_\mathrm{r}\otimes[\mathrm{II}]_\mathrm{r}}[2R^\mathrm{n}_1-2R^\mathrm{n}_2]^2_{(22)}$, ${}^{[\mathrm{II}]_\mathrm{r}\otimes[\mathrm{D}]_\mathrm{r}}[2R^\mathrm{n}_1-2R^\mathrm{n}_2]^2_{(22)}$
$\mathrm{III_s}$	${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[R^\mathrm{s}_1-3R^\mathrm{n}_2]^2_{(13)}$
$\mathrm{III_t}$	${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[R^\mathrm{t}_1-3R^\mathrm{n}_2]^2_{(13)}$
IV	${}^{[\mathrm{N}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[4R^\mathrm{n}]^1_{(4)}$, ${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{N}]_\mathrm{r}}[4R^\mathrm{n}]^1_{(4)}$

In the next section, we present the canonical forms of the parent Types and Tables of possible degenerations together with the continuous characteristics of the matrix $(C^a{}_b)$. For the canonical forms, we use both null and orthonormal tetrads. The classification of the traceless Ricci tensor in complex spaces can be treated as a "generic" classification for the **UR**. This is why we list the Plebański–Przanowski types described in details in [6] (we keep the original symbols of types used in [6]). For the Plebański–Przanowski classification, we use the abbreviation *PP classification*.

3.2. Type I (4 eigenvectors)

3.2.1. Type $\mathrm{I_r}$ (4 real eigenvectors; 2 spacelike and 2 timelike eigenvectors)

The canonical form of C_{ab} for the parent Type $\mathrm{I_r}$ reads

$$\begin{aligned} C_{ab} &= R^{\mathrm{s}}_1 E_{1'a}E_{1'b} + R^{\mathrm{s}}_2 E_{2'a}E_{2'b} - R^{\mathrm{t}}_3 E_{3'a}E_{3'b} - R^{\mathrm{t}}_4 E_{4'a}E_{4'b} \\ &= \tfrac{1}{2}\left(R^{\mathrm{s}}_1 - R^{\mathrm{t}}_3\right)\left(e_{1a}e_{1b} + e_{2a}e_{2b}\right) + \tfrac{1}{2}\left(R^{\mathrm{s}}_1 + R^{\mathrm{t}}_3\right)\left(e_{1a}e_{2b} + e_{2a}e_{1b}\right) \\ &\quad + \tfrac{1}{2}\left(R^{\mathrm{s}}_2 - R^{\mathrm{t}}_4\right)\left(e_{3a}e_{3b} + e_{4a}e_{4b}\right) + \tfrac{1}{2}\left(R^{\mathrm{s}}_2 + R^{\mathrm{t}}_4\right)\left(e_{3a}e_{4b} + e_{4a}e_{3b}\right) . \end{aligned} \tag{3.1}$$

The eigenvectors and corresponding eigenvalues are:

$$E_{1'} \longleftrightarrow R^{\mathrm{s}}_1 , \qquad E_{2'} \longleftrightarrow R^{\mathrm{s}}_2 , \qquad E_{3'} \longleftrightarrow R^{\mathrm{t}}_3 , \qquad E_{4'} \longleftrightarrow R^{\mathrm{t}}_4 .$$

The eigenvalues have to satisfy

$$R^{\mathrm{s}}_1 + R^{\mathrm{s}}_2 + R^{\mathrm{t}}_3 + R^{\mathrm{t}}_4 = 0 .$$

Using (2.8), one finds the form of the Plebański spinors

$$\begin{aligned} V_{ABCD} &= \tfrac{1}{2}\left(R^{\mathrm{s}}_1 - R^{\mathrm{t}}_3\right)\left(R^{\mathrm{s}}_2 - R^{\mathrm{t}}_4\right)\left(k_A k_B k_C k_D + l_A l_B l_C l_D\right) \\ &\quad + \tfrac{1}{2}\Big(\left(R^{\mathrm{s}}_2 - R^{\mathrm{t}}_4\right)^2 - \left(3R^{\mathrm{s}}_1 + R^{\mathrm{t}}_3\right)\left(R^{\mathrm{s}}_1 + 3R^{\mathrm{t}}_3\right)\Big)\, k_{(A}k_B l_C l_{D)} , \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= \tfrac{1}{2}\left(R^{\mathrm{s}}_1 - R^{\mathrm{t}}_3\right)\left(R^{\mathrm{s}}_2 - R^{\mathrm{t}}_4\right)\left(k_{\dot{A}} k_{\dot{B}} k_{\dot{C}} k_{\dot{D}} + l_{\dot{A}} l_{\dot{B}} l_{\dot{C}} l_{\dot{D}}\right) \\ &\quad + \tfrac{1}{2}\Big(\left(R^{\mathrm{s}}_2 - R^{\mathrm{t}}_4\right)^2 - \left(3R^{\mathrm{s}}_1 + R^{\mathrm{t}}_3\right)\left(R^{\mathrm{s}}_1 + 3R^{\mathrm{t}}_3\right)\Big)\, k_{(\dot{A}}k_{\dot{B}} l_{\dot{C}} l_{\dot{D})} . \end{aligned} \tag{3.2}$$

Investigation of the polynomials $\mathcal{V}(z)$ and $\dot{\mathcal{V}}(\dot{z})$ (defined in Subsection 2.2) proves that both Plebański spinors are, in general, of the Petrov–Penrose types $[\mathrm{I}]_{\mathrm{r}}$ or $[\mathrm{I}]_{\mathrm{c}}$. Define the quantity σ_1 by the formula

$$\sigma_1 := \left(R^{\mathrm{t}}_3 - R^{\mathrm{s}}_1\right)\left(R^{\mathrm{t}}_3 - R^{\mathrm{s}}_2\right)\left(R^{\mathrm{t}}_3 + R^{\mathrm{s}}_1 + 2R^{\mathrm{s}}_2\right)\left(R^{\mathrm{t}}_3 + 2R^{\mathrm{s}}_1 + R^{\mathrm{s}}_2\right) . \tag{3.3}$$

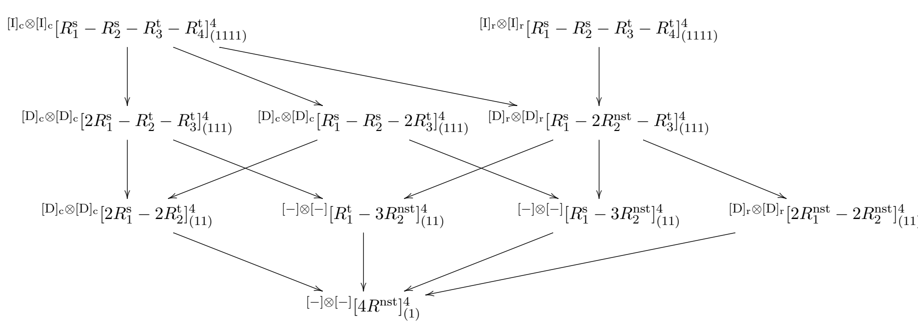
Then we get the criterion

$$\begin{aligned} \sigma_1 < 0 &\iff \text{both } V_{ABCD} \text{ and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ are of the type } [\mathrm{I}]_{\mathrm{r}} , \\ \sigma_1 > 0 &\iff \text{both } V_{ABCD} \text{ and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ are of the type } [\mathrm{I}]_{\mathrm{c}} . \end{aligned} \tag{3.4}$$

TABLE V

Subtypes of the Type $\mathrm{I_r}$.

PP classification		Neutral signature classification	
Eigenvalues	Type I	Eigenvalues	Type $\mathrm{I_r}$
$Z_1Z_2Z_3Z_4$	$[C_1 - C_2 - C_3 - C_4]_4$	$R_1R_2R_3R_4$	${}^{[\mathrm{I}]_\mathrm{r}\otimes[\mathrm{I}]_\mathrm{r}}[R_1^\mathrm{s} - R_2^\mathrm{s} - R_3^\mathrm{t} - R_4^\mathrm{t}]^4_{(1111)}$
			${}^{[\mathrm{I}]_\mathrm{c}\otimes[\mathrm{I}]_\mathrm{c}}[R_1^\mathrm{s} - R_2^\mathrm{s} - R_3^\mathrm{t} - R_4^\mathrm{t}]^4_{(1111)}$
$Z_1Z_2Z_3^2$	$[C_1 - C_2 - 2N]_3$	$R_1R_2R_3^2$	${}^{[\mathrm{D}]_\mathrm{c}\otimes[\mathrm{D}]_\mathrm{c}}[2R_1^\mathrm{s} - R_2^\mathrm{t} - R_3^\mathrm{t}]^4_{(111)}$
			${}^{[\mathrm{D}]_\mathrm{r}\otimes[\mathrm{D}]_\mathrm{r}}[R_1^\mathrm{s} - 2R_2^\mathrm{nst} - R_3^\mathrm{t}]^4_{(111)}$
			${}^{[\mathrm{D}]_\mathrm{c}\otimes[\mathrm{D}]_\mathrm{c}}[R_1^\mathrm{s} - R_2^\mathrm{s} - 2R_3^\mathrm{t}]^4_{(111)}$
$Z_1^2Z_2^2$	$[2N_1 - 2N]_2$	$R_1^2R_2^2$	${}^{[\mathrm{D}]_\mathrm{c}\otimes[\mathrm{D}]_\mathrm{c}}[2R_1^\mathrm{s} - 2R_2^\mathrm{t}]^4_{(11)}$
			${}^{[\mathrm{D}]_\mathrm{r}\otimes[\mathrm{D}]_\mathrm{r}}[2R_1^\mathrm{nst} - 2R_2^\mathrm{nst}]^4_{(11)}$
$Z_1Z_2^3$	$[C_1 - 3N]_2$	$R_1R_2^3$	${}^{[-]\otimes[-]}[R_1^\mathrm{s} - 3R_2^\mathrm{nst}]^4_{(11)}$
			${}^{[-]\otimes[-]}[R_1^\mathrm{t} - 3R_2^\mathrm{nst}]^4_{(11)}$
Z^4	$[4N]_1$	R^4	${}^{[-]\otimes[-]}[4R^\mathrm{nst}]^4_{(1)}$



Scheme 2: Degeneration scheme of the Type $\mathrm{I_r}$.

3.2.2. Type $\mathrm{I_c}$ (4 complex eigenvectors)

The canonical form of the C_{ab} for the parent Type $\mathrm{I_c}$ has the form of

$$\begin{aligned} C_{ab} &= \frac{1}{2}\left(Z_1+\bar{Z}_1\right)(E_{2'a}E_{2'b}-E_{4'a}E_{4'b})+\frac{i}{2}\left(Z_1-\bar{Z}_1\right)(E_{2'a}E_{4'b}+E_{4'a}E_{2'b}) \\ &\quad +\frac{1}{2}\left(Z_2+\bar{Z}_2\right)(E_{1'a}E_{1'b}-E_{3'a}E_{3'b})+\frac{i}{2}\left(Z_2-\bar{Z}_2\right)(E_{1'a}E_{3'b}+E_{3'a}E_{1'b}) \\ &= \frac{1}{2}\left(Z_1+\bar{Z}_1\right)(e_{3a}e_{4b}+e_{4a}e_{3b})+\frac{i}{2}\left(Z_1-\bar{Z}_1\right)(e_{3a}e_{3b}-e_{4a}e_{4b}) \\ &\quad +\frac{1}{2}\left(Z_2+\bar{Z}_2\right)(e_{1a}e_{2b}+e_{2a}e_{1b})+\frac{i}{2}\left(Z_2-\bar{Z}_2\right)(e_{1a}e_{1b}-e_{2a}e_{2b}) . \end{aligned} \tag{3.5}$$

The eigenvectors and corresponding eigenvalues are:

$$\begin{aligned} &\frac{1}{\sqrt{2}}(E_{2'}+iE_{4'}) \longleftrightarrow Z_1 , \qquad \frac{1}{\sqrt{2}}(E_{2'}-iE_{4'}) \longleftrightarrow \bar{Z}_1 , \\ &\frac{1}{\sqrt{2}}(E_{1'}+iE_{3'}) \longleftrightarrow Z_2 , \qquad \frac{1}{\sqrt{2}}(E_{1'}-iE_{3'}) \longleftrightarrow \bar{Z}_2 . \end{aligned}$$

The constraints for the eigenvalues read

$$\mathrm{Im}(Z_1)\neq 0 , \qquad \mathrm{Im}(Z_2)\neq 0 , \qquad \mathrm{Re}(Z_1)+\mathrm{Re}(Z_2)=0 .$$

The Plebański spinors are

$$\begin{aligned} V_{ABCD} &= 2\mathrm{Im}(Z_1)\mathrm{Im}(Z_2)(k_Ak_Bk_Ck_D+l_Al_Bl_Cl_D) \\ &\quad -\left(2(\mathrm{Im}(Z_1))^2+2(\mathrm{Im}(Z_2))^2+8(\mathrm{Re}(Z_1))^2\right)\, k_{(A}k_Bl_Cl_{D)} , \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= -2\mathrm{Im}(Z_1)\mathrm{Im}(Z_2)(k_{\dot{A}}k_{\dot{B}}k_{\dot{C}}k_{\dot{D}}+l_{\dot{A}}l_{\dot{B}}l_{\dot{C}}l_{\dot{D}}) \\ &\quad -\left(2(\mathrm{Im}(Z_1))^2+2(\mathrm{Im}(Z_2))^2+8(\mathrm{Re}(Z_1))^2\right)\, k_{(\dot{A}}k_{\dot{B}}l_{\dot{C}}l_{\dot{D})} \end{aligned} \tag{3.6}$$

and they both are, in general, of the type $[\mathrm{I}]_\mathrm{r}$ and $[\mathrm{I}]_\mathrm{c}$. To distinguish these two types, we have the following criterion

$$\begin{aligned} \mathrm{Im}(Z_1)\mathrm{Im}(Z_2)<0 &\iff V_{ABCD} \text{ is of the type } [\mathrm{I}]_\mathrm{c} \\ &\qquad \text{and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ is of the type } [\mathrm{I}]_\mathrm{r} , \\ \mathrm{Im}(Z_1)\mathrm{Im}(Z_2)>0 &\iff V_{ABCD} \text{ is of the type } [\mathrm{I}]_\mathrm{r} \\ &\qquad \text{and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ is of the type} [\mathrm{I}]_\mathrm{c} . \end{aligned} \tag{3.7}$$

It is interesting to note that only the subtype ${}^{[\mathrm{I}]_\mathrm{r}\otimes[\mathrm{I}]_\mathrm{c}}[Z_1-\bar{Z}_1-Z_2-\bar{Z}_2]^4_{(1111)}$ allows the degeneration into the type ${}^{[\mathrm{D}]_\mathrm{r}\otimes[\mathrm{D}]_\mathrm{c}}[2Z-2\bar{Z}]^4_{(11)}$.

TABLE VI

Subtypes of the Type $\mathrm{I_c}$.

PP classification		Neutral signature classification	
Eigenvalues	Type I	Eigenvalues	Type $\mathrm{I_c}$
$Z_1 Z_2 Z_3 Z_4$	$[C_1 - C_2 - C_3 - C_4]_4$	$Z_1 \bar{Z}_1 Z_2 \bar{Z}_2$	$^{[\mathrm{I}]_\mathrm{c} \otimes [\mathrm{I}]_\mathrm{r}} \left[Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2\right]^4_{(1111)}$
			$^{[\mathrm{I}]_\mathrm{r} \otimes [\mathrm{I}]_\mathrm{c}} \left[Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2\right]^4_{(1111)}$
$Z_1^2 Z_2^2$	$[2N_1 - 2N]_2$	$Z^2 \bar{Z}^2$	$^{[\mathrm{D}]_\mathrm{r} \otimes [\mathrm{D}]_\mathrm{c}} \left[2Z - 2\bar{Z}\right]^4_{(11)}$

$^{[\mathrm{I}]_\mathrm{c} \otimes [\mathrm{I}]_\mathrm{r}}[Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]^4_{(1111)}$ $\qquad$ $^{[\mathrm{I}]_\mathrm{r} \otimes [\mathrm{I}]_\mathrm{c}}[Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]^4_{(1111)}$

$\downarrow$

$^{[\mathrm{D}]_\mathrm{r} \otimes [\mathrm{D}]_\mathrm{c}}[2Z - 2\bar{Z}]^4_{(11)}$

Scheme 3: Degeneration scheme of the Type $\mathrm{I_c}$.

3.2.3. Type $\mathrm{I_{rc}}$ (4 eigenvectors; two complex, one timelike and one spacelike eigenvectors)

The canonical form of the C_{ab} for the parent Type $\mathrm{I_{rc}}$ is:

$$\begin{aligned} C_{ab} &= R_1^{\mathrm{s}} E_{1'a} E_{1'b} - R_2^{\mathrm{t}} E_{3'a} E_{3'b} \\ &\quad + \frac{1}{2}\left(Z + \bar{Z}\right)\left(E_{2'a}E_{2'b} - E_{4'a}E_{4'b}\right) + \frac{i}{2}\left(Z - \bar{Z}\right)\left(E_{2'a}E_{4'b} + E_{4'a}E_{2'b}\right) \\ &= \frac{1}{2}\left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right)\left(e_{1a}e_{1b} + e_{2a}e_{2b}\right) + \frac{1}{2}\left(R_1^{\mathrm{s}} + R_2^{\mathrm{t}}\right)\left(e_{1a}e_{2b} + e_{2a}e_{1b}\right) \\ &\quad + \frac{1}{2}\left(Z + \bar{Z}\right)\left(e_{3a}e_{4b} + e_{4a}e_{3b}\right) + \frac{i}{2}\left(Z - \bar{Z}\right)\left(e_{3a}e_{3b} - e_{4a}e_{4b}\right). \end{aligned} \tag{3.8}$$

The eigenvectors and corresponding eigenvalues are:

$$E_{1'} \longleftrightarrow R_1^{\mathrm{s}}, \qquad E_{3'} \longleftrightarrow R_2^{\mathrm{t}},$$

$$\frac{1}{\sqrt{2}}(E_{2'} + iE_{4'}) \longleftrightarrow Z, \qquad \frac{1}{\sqrt{2}}(E_{2'} - iE_{4'}) \longleftrightarrow \bar{Z}.$$

The relations between eigenvalues read

$$\mathrm{Im}(Z) \neq 0, \qquad R_1^{\mathrm{s}} + R_2^{\mathrm{t}} + 2\mathrm{Re}(Z) = 0.$$

The Plebański spinors have the following form:

$$
\begin{aligned}
V_{ABCD} &= \left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right) \mathrm{Im}(Z)(k_A k_B k_C k_D - l_A l_B l_C l_D) \\
&\quad + \tfrac{1}{2} \left(\left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right)^2 - 4(\mathrm{Im}(Z))^2 - 16(\mathrm{Re}(Z))^2 \right) k_{(A} k_B l_C l_{D)}\,, \\
V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= \left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right) \mathrm{Im}(Z)(k_{\dot{A}} k_{\dot{B}} k_{\dot{C}} k_{\dot{D}} - l_{\dot{A}} l_{\dot{B}} l_{\dot{C}} l_{\dot{D}}) \\
&\quad + \tfrac{1}{2} \left(\left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right)^2 - 4(\mathrm{Im}(Z))^2 - 16(\mathrm{Re}(Z))^2 \right) k_{(\dot{A}} k_{\dot{B}} l_{\dot{C}} l_{\dot{D})}\,.
\end{aligned}
\tag{3.9}
$$

Both Plebański spinors for the nondegenerate Type $\mathrm{I_{rc}}$ are of the Petrov–Penrose type $[\mathrm{I}]_{\mathrm{rc}}$.

TABLE VII

Subtypes of the Type $\mathrm{I_{rc}}$.

PP classification		Neutral signature classification	
Eigenvalues	Type I	Eigenvalues	Type $\mathrm{I_{rc}}$
$Z_1 Z_2 Z_3 Z_4$	$[C_1 - C_2 - C_3 - C_4]_4$	$Z\bar{Z}R_1R_2$	${}^{[\mathrm{I}]_{\mathrm{rc}}\otimes[\mathrm{I}]_{\mathrm{rc}}}\left[Z - \bar{Z} - R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right]^4_{(1111)}$
$Z_1 Z_2 Z_3^2$	$[C_1 - C_2 - 2N]_2$	$Z\bar{Z}R^2$	${}^{[\mathrm{D}]_{\mathrm{r}}\otimes[\mathrm{D}]_{\mathrm{r}}}\left[Z - \bar{Z} - 2R^{\mathrm{nst}}\right]^4_{(111)}$

$$
{}^{[\mathrm{I}]_{\mathrm{rc}}\otimes[\mathrm{I}]_{\mathrm{rc}}}[Z - \bar{Z} - R_1^{\mathrm{s}} - R_2^{\mathrm{t}}]^4_{(1111)}
$$

$$
\downarrow
$$

$$
{}^{[\mathrm{D}]_{\mathrm{r}}\otimes[\mathrm{D}]_{\mathrm{r}}}[Z - \bar{Z} - 2R^{\mathrm{nst}}]^4_{(111)}
$$

Scheme 4: Degeneration scheme of the Type $\mathrm{I_{rc}}$.

3.3. Type II (3 eigenvectors)

3.3.1. Type $\mathrm{II_r}$ (3 eigenvectors; one timelike, one spacelike and one null eigenvectors)

The canonical form of the C_{ab} for the parent Type $\mathrm{II_r}$ is:

$$
\begin{aligned}
C_{ab} &= R_1^{\mathrm{s}} E_{1'a} E_{1'b} - R_2^{\mathrm{t}} E_{3'a} E_{3'b} + R_3^{\mathrm{n}} (E_{2'a} E_{2'b} - E_{4'a} E_{4'b}) \\
&\quad + \tfrac{1}{2} (E_{2'a} E_{2'b} + E_{4'a} E_{4'b} - E_{2'a} E_{4'b} - E_{4'a} E_{2'b}) \\
&= \tfrac{1}{2} \left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right) (e_{1a} e_{1b} + e_{2a} e_{2b}) + \tfrac{1}{2} \left(R_1^{\mathrm{s}} + R_2^{\mathrm{t}}\right) (e_{1a} e_{2b} + e_{2a} e_{1b}) \\
&\quad + R_3^{\mathrm{n}} (e_{3a} e_{4b} + e_{4a} e_{3b}) + e_{4a} e_{4b}\,.
\end{aligned}
\tag{3.10}
$$

The eigenvectors and corresponding eigenvalues are:

$$E_{1'} \longleftrightarrow R_1^{\mathrm{s}}\,, \qquad E_{3'} \longleftrightarrow R_2^{\mathrm{t}}\,, \qquad \frac{1}{\sqrt{2}}(E_{2'} - E_{4'}) \longleftrightarrow R_3^{\mathrm{n}}\,.$$

The eigenvalues have to satisfy the relation

$$R_1^{\mathrm{s}} + R_2^{\mathrm{t}} + 2R_3^{\mathrm{n}} = 0\,.$$

The Plebański spinors for the Type $\mathrm{II_r}$ can be brought to the form

$$\begin{aligned} V_{ABCD} &= \tfrac{1}{2}\left(2\left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right) k_{(A}k_B - \left(3R_1^{\mathrm{s}} + R_2^{\mathrm{t}}\right)\left(R_1^{\mathrm{s}} + 3R_2^{\mathrm{t}}\right) l_{(A}l_B\right) k_C k_{D)}\,, \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= \tfrac{1}{2}\Big(2\left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right) k_{(\dot{A}}k_{\dot{B}} - \left(3R_1^{\mathrm{s}} + R_2^{\mathrm{t}}\right)\left(R_1^{\mathrm{s}} + 3R_2^{\mathrm{t}}\right) l_{(\dot{A}}l_{\dot{B}}\Big) k_{\dot{C}} k_{\dot{D})}\,. \end{aligned} \tag{3.11}$$

Consider the quantity σ_2

$$\sigma_2 := \left(R_1^{\mathrm{s}} - R_2^{\mathrm{t}}\right)\left(3R_1^{\mathrm{s}} + R_2^{\mathrm{t}}\right)\left(R_1^{\mathrm{s}} + 3R_2^{\mathrm{t}}\right)\,. \tag{3.12}$$

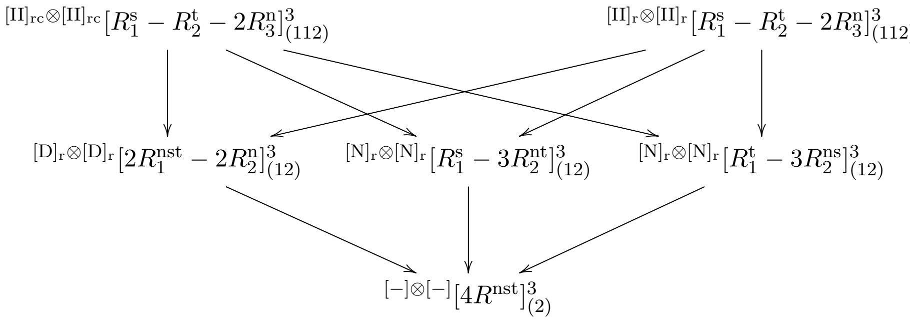
Then we find the following criterion

$$\begin{aligned} \sigma_2 > 0 &\Longleftrightarrow V_{ABCD} \text{ and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ are of the type } [\mathrm{II}]_{\mathrm{r}}\,, \\ \sigma_2 < 0 &\Longleftrightarrow V_{ABCD} \text{ and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ are of the type } [\mathrm{II}]_{\mathrm{rc}}\,. \end{aligned} \tag{3.13}$$

TABLE VIII

Subtypes of the Type $\mathrm{II_r}$.

PP classification		Neutral signature classification	
Eigenvalues	Type II	Eigenvalues	Type $\mathrm{II_r}$
$Z_1Z_2Z_3^2$	$[C_1 - C_2 - 2N]_4$	$R_1R_2R_3^2$	$^{[\mathrm{II}]_{\mathrm{rc}}\otimes[\mathrm{II}]_{\mathrm{rc}}}[R_1^{\mathrm{s}} - R_2^{\mathrm{t}} - 2R_3^{\mathrm{n}}]^3_{(112)}$
			$^{[\mathrm{II}]_{\mathrm{r}}\otimes[\mathrm{II}]_{\mathrm{r}}}[R_1^{\mathrm{s}} - R_2^{\mathrm{t}} - 2R_3^{\mathrm{n}}]^3_{(112)}$
$Z_1^2Z_2^2$	$[2N_1 - 2N]_{(1-2)}$	$R_1^2R_2^2$	$^{[\mathrm{D}]_{\mathrm{r}}\otimes[\mathrm{D}]_{\mathrm{r}}}[2R_1^{\mathrm{nst}} - 2R_2^{\mathrm{n}}]^3_{(12)}$
$Z_1Z_2^3$	$[C_1 - 3N]_3$	$R_1R_2^3$	$^{[\mathrm{N}]_{\mathrm{r}}\otimes[\mathrm{N}]_{\mathrm{r}}}[R_1^{\mathrm{s}} - 3R_2^{\mathrm{nt}}]^3_{(12)}$
			$^{[\mathrm{N}]_{\mathrm{r}}\otimes[\mathrm{N}]_{\mathrm{r}}}[R_1^{\mathrm{t}} - 3R_2^{\mathrm{ns}}]^3_{(12)}$
Z^4	$^{(3)}[4N]_2$	R^4	$^{[-]\otimes[-]}[4R^{\mathrm{nst}}]^3_{(2)}$



Scheme 5: Degeneration scheme of the Type $\mathrm{II_r}$.

3.3.2. Type $\mathrm{II_{rc}}$ (3 eigenvectors; two complex and one null eigenvectors)

The canonical form of the C_{ab} for the parent Type $\mathrm{II_{rc}}$ has the form of

$$\begin{aligned}
C_{ab} = {} & \frac{1}{2}\left(Z + \bar{Z}\right)\left(E_{1'a}E_{1'b} - E_{3'a}E_{3'b}\right) + \frac{i}{2}\left(Z - \bar{Z}\right)\left(E_{1'a}E_{3'b} + E_{3'a}E_{1'b}\right) \\
& + R^{\mathrm{n}}(E_{2'a}E_{2'b} - E_{4'a}E_{4'b}) + \frac{1}{2}(E_{2'a}E_{2'b} + E_{4'a}E_{4'b} - E_{2'a}E_{4'b} - E_{4'a}E_{2'b}) \\
= {} & \frac{1}{2}\left(Z + \bar{Z}\right)\left(e_{1a}e_{2b} + e_{2a}e_{1b}\right) + \frac{i}{2}\left(Z - \bar{Z}\right)\left(e_{1a}e_{1b} - e_{2a}e_{2b}\right) \\
& + R^{\mathrm{n}}(e_{3a}e_{4b} + e_{4a}e_{3b}) + e_{4a}e_{4b}\,.
\end{aligned} \tag{3.14}$$

The eigenvectors and corresponding eigenvalues are given by

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(E_{1'} + iE_{3'}) \longleftrightarrow Z\,, \qquad \frac{1}{\sqrt{2}}(E_{1'} - iE_{3'}) \longleftrightarrow \bar{Z}\,, \\
&\frac{1}{\sqrt{2}}(E_{2'} - E_{4'}) \longleftrightarrow R^{\mathrm{n}}\,.
\end{aligned}$$

The conditions for eigenvalues are:

$$\mathrm{Im}(Z) \neq 0\,, \qquad R^{\mathrm{n}} + \mathrm{Re}(Z) = 0\,.$$

The Plebański spinors read

$$\begin{aligned}
V_{ABCD} &= 2\left(\mathrm{Im}(Z)\,k_{(A}k_B - \left((\mathrm{Im}(Z))^2 + 4(\mathrm{Re}(Z))^2\right)\, l_{(A}l_{B}\right)\, k_C k_{D)}\,, \\
V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= 2\left(-\mathrm{Im}(Z)\,k_{(\dot{A}}k_{\dot{B}} - \left((\mathrm{Im}(Z))^2 + 4(\mathrm{Re}(Z))^2\right)\, l_{(\dot{A}}l_{\dot{B}}\right)\, k_{\dot{C}}k_{\dot{D})}\,.
\end{aligned} \tag{3.15}$$

This time, we find the following criterion:

$$\begin{aligned}\mathrm{Im}(Z)>0 &\iff V_{ABCD} \text{ is of the type } [\mathrm{II}]_{\mathrm{r}} \\ &\quad\ \text{and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ is of the type } [\mathrm{II}]_{\mathrm{rc}}\,, \\ \mathrm{Im}(Z)<0 &\iff V_{ABCD} \text{ is of the type } [\mathrm{II}]_{\mathrm{rc}} \\ &\quad\ \text{and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ is of the type } [\mathrm{II}]_{\mathrm{r}}\,. \end{aligned} \tag{3.16}$$

TABLE IX

Subtypes of the Type $\mathrm{II}_{\mathrm{rc}}$.

PP classification		Neutral signature classification	
Eigenvalues	Type II	Eigenvalues	Type $\mathrm{II}_{\mathrm{rc}}$
$Z_1 Z_2 Z_3^2$	$[C_1 - C_2 - 2N]_4$	$Z\bar{Z}R^2$	$^{[\mathrm{II}]_{\mathrm{r}}\otimes[\mathrm{II}]_{\mathrm{rc}}}[Z-\bar{Z}-2R^{\mathrm{n}}]^3_{(112)}$
			$^{[\mathrm{II}]_{\mathrm{rc}}\otimes[\mathrm{II}]_{\mathrm{r}}}[Z-\bar{Z}-2R^{\mathrm{n}}]^3_{(112)}$

3.4. Type III (2 eigenvectors)

3.4.1. Types $\mathrm{III}_{\mathrm{s}}$ and $\mathrm{III}_{\mathrm{t}}$ (2 eigenvectors; one null and one spacelike or timelike eigenvectors)

The canonical form of the C_{ab} for the parent Type $\mathrm{III}_{\mathrm{t}}$ is:

$$\begin{aligned} C_{ab} &= -R_1^{\mathrm{t}} E_{3'a}E_{3'b} + R_2^{\mathrm{n}}(E_{1'a}E_{1'b} + E_{2'a}E_{2'b} - E_{4'a}E_{4'b}) \\ &\quad + E_{1'a}E_{2'b} + E_{2'a}E_{1'b} - E_{1'a}E_{4'b} - E_{4'a}E_{1'b} \\ &= \tfrac{1}{2}\left(R_2^{\mathrm{n}} + R_1^{\mathrm{t}}\right)(e_{1a}e_{2b} + e_{2a}e_{1b}) + \tfrac{1}{2}\left(R_2^{\mathrm{n}} - R_1^{\mathrm{t}}\right)(e_{1a}e_{1b} + e_{2a}e_{2b}) \\ &\quad + R_2^{\mathrm{n}}(e_{3a}e_{4b} + e_{4a}e_{3b}) + e_{1a}e_{4b} + e_{4a}e_{1b} + e_{2a}e_{4b} + e_{4a}e_{2b}\,. \end{aligned} \tag{3.17}$$

The eigenvectors and corresponding eigenvalues are

$$E_{3'} \longleftrightarrow R_1^{\mathrm{t}}\,, \qquad e_4 \longleftrightarrow R_2^{\mathrm{n}}\,.$$

The eigenvalues have to satisfy the condition

$$R_1^{\mathrm{t}} + 3R_2^{\mathrm{n}} = 0\,.$$

The Plebański spinors have the following form:

$$\begin{aligned} V_{ABCD} &= -2\left(k_{(A} + 8R_2^{\mathrm{n}}\, l_{(A}\right)\, k_B k_C k_{D)}\,, \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= -2\left(k_{(\dot{A}} + 8R_2^{\mathrm{n}}\, l_{(\dot{A}}\right)\, k_{\dot{B}} k_{\dot{C}} k_{\dot{D})}\,. \end{aligned} \tag{3.18}$$

For the nondegenerate Type $\mathrm{III_t}$, these spinors are both of the Petrov–Penrose type $[\mathrm{III}]_\mathrm{r}$.

The canonical form of the C_{ab} for the parent Type $\mathrm{III_s}$ reads

$$\begin{aligned} C_{ab} &= R_1^{\mathrm{s}} E_{1'a}E_{1'b} + R_2^{\mathrm{n}}(E_{2'a}E_{2'b} - E_{3'a}E_{3'b} - E_{4'a}E_{4'b}) \\ &\quad + E_{3'a}E_{2'b} + E_{2'a}E_{3'b} - E_{4'a}E_{3'b} - E_{3'a}E_{4'b} \\ &= \tfrac{1}{2}\left(R_1^{\mathrm{s}} - R_2^{\mathrm{n}}\right)(e_{1a}e_{1b} + e_{2a}e_{2b}) + \tfrac{1}{2}\left(R_1^{\mathrm{s}} + R_2^{\mathrm{n}}\right)(e_{1a}e_{2b} + e_{2a}e_{1b}) \\ &\quad + R_2^{\mathrm{n}}(e_{3a}e_{4b} + e_{4a}e_{3b}) + e_{1a}e_{4b} + e_{4a}e_{1b} - e_{2a}e_{4b} - e_{4a}e_{2b}\,. \end{aligned} \tag{3.19}$$

The eigenvectors and corresponding eigenvalues are:

$$E_{1'} \longleftrightarrow R_1^{\mathrm{s}}\,, \qquad e_4 \longleftrightarrow R_2^{\mathrm{n}}\,.$$

The eigenvalues satisfy the relation

$$R_1^{\mathrm{s}} + 3R_2^{\mathrm{n}} = 0\,.$$

The Plebański spinors read

$$\begin{aligned} V_{ABCD} &= -2\left(k_{(A} - 8R_2^{\mathrm{n}}\, l_{(A}\right)\, k_B k_C k_{D)} \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= -2\left(k_{(\dot{A}} + 8R_2^{\mathrm{n}}\, l_{(\dot{A}}\right)\, k_{\dot{B}} k_{\dot{C}} k_{\dot{D})} \end{aligned} \tag{3.20}$$

and they represent the Petrov–Penrose type $[\mathrm{III}]_\mathrm{r}$.

TABLE X

Subtypes of the Types $\mathrm{III_t}$ and $\mathrm{III_s}$.

PP classification		Neutral signature classification		
Eigenvalues	Type III_C	Eigenvalues	Type $\mathrm{III_t}$	Type $\mathrm{III_s}$
$Z_1 Z_2^3$	$[C_1 - 3N]_4$	$R_1 R_2^3$	${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[R_1^{\mathrm{t}} - 3R_2^{\mathrm{n}}]^2_{(13)}$	${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[R_1^{\mathrm{s}} - 3R_2^{\mathrm{n}}]^2_{(13)}$
Z^4	$[4N]_3$	R^4	${}^{[\mathrm{N}]_\mathrm{r}\otimes[\mathrm{N}]_\mathrm{r}}[4R^{\mathrm{nt}}]^2_{(3)}$	${}^{[\mathrm{N}]_\mathrm{r}\otimes[\mathrm{N}]_\mathrm{r}}[4R^{\mathrm{ns}}]^2_{(3)}$

$${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[R_1^{\mathrm{t}} - 3R_2^{\mathrm{n}}]^2_{(13)} \longrightarrow {}^{[\mathrm{N}]_\mathrm{r}\otimes[\mathrm{N}]_\mathrm{r}}[4R^{\mathrm{nt}}]^2_{(3)}$$

$${}^{[\mathrm{III}]_\mathrm{r}\otimes[\mathrm{III}]_\mathrm{r}}[R_1^{\mathrm{s}} - 3R_2^{\mathrm{n}}]^2_{(13)} \longrightarrow {}^{[\mathrm{N}]_\mathrm{r}\otimes[\mathrm{N}]_\mathrm{r}}[4R^{\mathrm{ns}}]^2_{(3)}$$

Scheme 6: Degeneration scheme of the Types $\mathrm{III_t}$ and $\mathrm{III_s}$.

3.4.2. Type $\mathrm{III_n}$ (2 eigenvectors; both null)

We find here two subtypes. The canonical form of the C_{ab} reads

$$C_{ab} = e_{1a}e_{1b} + e_{4a}e_{4b} + R_1^{\mathrm{n}}(e_{3a}e_{4b} + e_{4a}e_{3b}) + R_2^{\mathrm{n}}(e_{1a}e_{2b} + e_{2a}e_{1b})\,. \tag{3.21}$$

The eigenvectors and corresponding eigenvalues are:

$$e_4 \longleftrightarrow R_1^{\mathrm{n}}\,, \qquad e_1 \longleftrightarrow R_2^{\mathrm{n}}\,.$$

For the first subtype, the Plebański spinors read

$$\begin{aligned} V_{ABCD} &= -8(R_1^{\mathrm{n}})^2\, k_{(A}k_B l_C l_{D)}\,, \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= 2\,(k_{(\dot{A}} + 2R_1^{\mathrm{n}}\, l_{(\dot{A}})(k_{\dot{B}} - 2R_1^{\mathrm{n}}\, l_{\dot{B}})k_{\dot{C}}k_{\dot{D})}\,. \end{aligned} \tag{3.22}$$

Undotted Plebański spinor for the first subtype of the Type $\mathrm{III_n}$ is of the type $[\mathrm{D}]_{\mathrm{r}}$ and the dotted one is of the type $[\mathrm{II}]_{\mathrm{r}}$.

The second possibility is

$$C_{ab} = e_{2a}e_{2b} + e_{4a}e_{4b} + R_1^{\mathrm{n}}(e_{3a}e_{4b} + e_{4a}e_{3b}) + R_2^{\mathrm{n}}(e_{1a}e_{2b} + e_{2a}e_{1b})\,. \tag{3.23}$$

The eigenvectors and corresponding eigenvalues are given by

$$e_4 \longleftrightarrow R_1^{\mathrm{n}}\,, \qquad e_2 \longleftrightarrow R_2^{\mathrm{n}}\,.$$

The second subtype is characterized by the following Plebański spinors

$$\begin{aligned} V_{ABCD} &= 2\,(k_{(A} + 2R_1^{\mathrm{n}}\, l_{(A})(k_B - 2R_1^{\mathrm{n}}\, l_B)k_C k_{D)}\,, \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= -8(R_1^{\mathrm{n}})^2\, k_{(\dot{A}}k_{\dot{B}} l_{\dot{C}} l_{\dot{D})} \end{aligned} \tag{3.24}$$

and they are of the Petrov–Penrose types $[\mathrm{II}]_{\mathrm{r}}$ and $[\mathrm{D}]_{\mathrm{r}}$, respectively.

The eigenvalues in both subtypes have to satisfy the relation

$$R_1^{\mathrm{n}} + R_2^{\mathrm{n}} = 0\,.$$

$$\begin{array}{ccc} {}^{[\mathrm{D}]_{\mathrm{r}}\otimes[\mathrm{II}]_{\mathrm{r}}}[2R_1^{\mathrm{n}} - 2R_2^{\mathrm{n}}]^2_{(22)} & \qquad & {}^{[\mathrm{II}]_{\mathrm{r}}\otimes[\mathrm{D}]_{\mathrm{r}}}[2R_1^{\mathrm{n}} - 2R_2^{\mathrm{n}}]^2_{(22)} \\ \downarrow & & \downarrow \\ {}^{[-]\otimes[\mathrm{N}]_{\mathrm{r}}}[4R^{\mathrm{n}}]^2_{(2)} & & {}^{[\mathrm{N}]_{\mathrm{r}}\otimes[-]}[4R^{\mathrm{n}}]^2_{(2)} \end{array}$$

Scheme 7: Degeneration scheme of the Type $\mathrm{III_n}$.

TABLE XI

Subtypes of the Type $\mathrm{III_n}$.

PP classification		Neutral signature classification	
Eigenvalues	Type III_N	Eigenvalues	Type $\mathrm{III_n}$
$Z_1^2 Z_2^2$	$[2N_1 - 2N]_4^a$	$R_1^2 R_2^2$	${}^{[\mathrm{D}]_\mathrm{r}\otimes[\mathrm{II}]_\mathrm{r}}[2R_1^\mathrm{n} - 2R_2^\mathrm{n}]_{(22)}^2$
	$[2N_1 - 2N]_4^b$		${}^{[\mathrm{II}]_\mathrm{r}\otimes[\mathrm{D}]_\mathrm{r}}[2R_1^\mathrm{n} - 2R_2^\mathrm{n}]_{(22)}^2$
Z^4	${}^{(2)}[4N]_2^a$	R^4	${}^{[-]\otimes[\mathrm{N}]_\mathrm{r}}[4R^\mathrm{n}]_{(2)}^2$
	${}^{(2)}[4N]_2^b$		${}^{[\mathrm{N}]_\mathrm{r}\otimes[-]}[4R^\mathrm{n}]_{(2)}^2$

3.5. Type IV (1 null eigenvector)

Finally, for the Type IV, we find two subtypes with canonical forms given by

$$C_{ab} = e_{1a}e_{1b} + e_{2a}e_{4b} + e_{4a}e_{2b} \tag{3.25}$$

or

$$C_{ab} = e_{2a}e_{2b} + e_{1a}e_{4b} + e_{4a}e_{1b}\,. \tag{3.26}$$

The eigenvectors and corresponding eigenvalues are:

$$e_4 \longleftrightarrow R^\mathrm{n}\,, \qquad R^\mathrm{n} = 0\,.$$

The Plebański spinors for the first subtype read

$$\begin{aligned} V_{ABCD} &= -2\,k_A k_B k_C k_D\,, \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= -4\,k_{(\dot{A}}k_{\dot{B}}k_{\dot{C}}l_{\dot{D})}\,. \end{aligned} \tag{3.27}$$

The Petrov–Penrose types of these spinors are $[\mathrm{N}]_\mathrm{r}$ and $[\mathrm{III}]_\mathrm{r}$, respectively. For the second subtype, we find

$$\begin{aligned} V_{ABCD} &= -4\,k_{(A}k_B k_C l_{D)}\,, \\ V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= -2\,k_{\dot{A}}k_{\dot{B}}k_{\dot{C}}k_{\dot{D}} \end{aligned} \tag{3.28}$$

and the Petrov–Penrose types of Plebański spinors are $[\mathrm{III}]_\mathrm{r}$ and $[\mathrm{N}]_\mathrm{r}$.

TABLE XII

Subtypes of the Type IV.

PP classification		Neutral signature classification	
Eigenvalues	Type IV	Eigenvalues	Type IV
Z^4	$[4N]_4^a$	R^4	$^{[\mathrm{N}]_{\mathrm{r}}\otimes[\mathrm{III}]_{\mathrm{r}}}[4R^{\mathrm{n}}]^1_{(4)}$
	$[4N]_4^b$		$^{[\mathrm{III}]_{\mathrm{r}}\otimes[\mathrm{N}]_{\mathrm{r}}}[4R^{\mathrm{n}}]^1_{(4)}$

4. Concluding remarks

In this paper, we analyzed the algebraic structure of the traceless Ricci tensor in 4-dimensional spaces equipped with the metric of the neutral signature. Detailed considerations brought us to the conclusion that there are 33 essentially different types of C_{ab} in such spaces. Our classification is purely algebraic. The alternate way of classification of traceless Ricci tensor in the Lorentzian spaces was given by Penrose [3]. It is an interesting question how the Penrose approach can be used in our case. We are going to study this problem soon.

In our work [12], some of the types of C_{ab} have been related to the existence of the so-called, *congruences of the SD null strings*. Another way of further investigations is to find a more detailed classification of the congruences of SD null strings and relate such a classification with the types of C_{ab} presented here. This question will be investigated elsewhere.

As mentioned in Introduction, we hope that our present work fills the gap left by two papers by Plebański and Przanowski [2, 6] in *Acta Physica Polonica B*.

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