THE FOUR-GROUP $Z_2 \times Z_2$ AS A DISCRETE INVARIANCE GROUP OF EFFECTIVE NEUTRINO MASS MATRIX*

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Two sets of four 3×3 matrices $\mathbf{1}^{(3)}, \varphi_1, \varphi_2, \varphi_3$ and $\mathbf{1}^{(3)}$, μ_1, μ_2, μ_3 are constructed, forming two unitarily isomorphic reducible representations $\underline{3}$ of the group $Z_2 \times Z_2$ called often the four-group. They are related to each other through the effective neutrino mixing matrix U with $s_{13} = 0$ and generate four discrete transformations of flavor and mass active neutrinos, respectively. If and only if $s_{13} = 0$, the generic form of effective neutrino mass matrix M becomes invariant under the subgroup Z_2 of $Z_2 \times Z_2$ represented by the matrices $\mathbf{1}^{(3)}$ and φ_3. In the approximation of $m_1 = m_2$, the matrix M becomes invariant under the whole $Z_2 \times Z_2$ represented by the matrices $\mathbf{1}^{(3)}$, φ_1, φ_2, φ_3. The effective neutrino mixing matrix U with $s_{13} = 0$ is always invariant under the whole $Z_2 \times Z_2$ represented in two ways, by the matrices $\mathbf{1}^{(3)}, \varphi_1,\ \varphi_2,\ \varphi_3$ and $\mathbf{1}^{(3)}$, μ_1, μ_2, μ_3.

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1. Introduction

As is well known, the neutrino experiments with solar ν_e's [1], atmospheric ν_μ's [2], long-baseline accelerator ν_μ's [3] and long-baseline reactor $\bar{\nu}_e$'s [4] are very well described by oscillations of three active neutrinos ν_e , ν_μ , ν_τ, where the mass-squared splittings of the related neutrino mass states ν_1 , ν_2 , ν_3 are estimated to be $\Delta m^2_{\rm sol} \equiv \Delta m^2_{21} \sim 8\times 10^{-5}$ eV^2 and $\Delta m^2_{\rm atm} \equiv \Delta m^2_{32} \sim 2.5\times 10^{-3}$ eV^2 [5]. The effective neutrino mixing matrix $U = (U_{\alpha i})$ ($\alpha = e, \mu, \tau$ and $i = 1, 2, 3$), responsible for the unitary transformation

$$\nu_\alpha = \sum_i U_{\alpha i}\, \nu_i\,, \tag{1}$$

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is experimentally consistent with the global bilarge form

$$U = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -c_{23}s_{12} & c_{23}c_{12} & s_{23} \\ s_{23}s_{12} & -s_{23}c_{12} & c_{23} \end{pmatrix} , \tag{2}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ with the estimations $\theta_{23} \sim 45^\circ$ and $\theta_{12} \sim 32^\circ$ (*i.e.*, $c_{23} \sim 1/\sqrt{2} \sim s_{23}$), while the matrix element $U_{e3} = s_{13}\exp(-i\delta)$ is neglected due to the non-observation of neutrino oscillations for short-baseline reactor $\bar{\nu}_e$'s, especially in the Chooz experiment [6] giving for s^2_{13} the upper limit $s^2_{13} < 0.04$. We assume here that $0 \leq \theta_{13} \leq \pi/2$, thus $s_{13} = 0$ implies $c_{13} = 1$.

However, the mixing matrix (1) (involving two experimentally fitted mass-squared scales Δm^2_{21} and Δm^2_{32}) cannot explain the possible LSND effect for short-baseline accelerator $\bar{\nu}_\mu$'s [7] that should require the existence of a third independent neutrino mass-squared scale, say, $\Delta m^2_{\rm LSND} \sim 1\ {\rm eV}^2$. Unless the CPT invariance is seriously violated in neutrino oscillations [8] (leading to considerable mass splittings between neutrinos and antineutrinos), such a third scale cannot appear in the oscillations of three neutrinos. So, if the ongoing MiniBooNE experiment [9] confirmed the LSND result, we should need one, at least, light sterile neutrino in addition to three active neutrinos in order to introduce the third scale (in the case, when the serious CPT violation was excluded).

The effective neutrino mass matrix $M = (M_{\alpha\beta})$ $(\alpha, \beta = e, \mu, \tau)$ is connected with the neutrino mixing matrix U through the formula

$$M_{\alpha\beta} = \sum_i U_{\alpha i}\, m_i\, U^*_{\beta i} , \tag{3}$$

if the flavor representation is used, where the mass matrix of charged leptons e^-, μ^-, τ^- is diagonal and so, the mixing matrix U is at the same time the diagonalizing matrix for M

$$\sum_{\alpha,\beta} U^*_{\alpha i} M_{\alpha\,\beta} U_{\beta j} = m_i \delta_{ij} , \tag{4}$$

(we assume that $M^\dagger = M^* = M$ for simplicity). Applying the generic form of the effective neutrino mixing matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{5}$$

(without one Dirac and two Majorana CP-violating phases for simplicity) that is reduced to the form (2) if $s_{13} = 0$, we obtain from Eq. (3)

$$
\begin{aligned}
M_{ee} &= c_{13}^2 \left(c_{12}^2 m_1 + s_{12}^2 m_2\right) + s_{13}^2 m_3 \,, \\
M_{\mu\mu} &= c_{23}^2 \left(s_{12}^2 m_1 + c_{12}^2 m_2\right) + s_{23}^2 \left[s_{13}^2 \left(c_{12}^2 m_1 + s_{12}^2 m_2\right) + c_{13}^2 m_3\right] \\
&\quad + 2 c_{23} s_{23} s_{13} c_{12} s_{12} (m_1 - m_2) \,, \\
M_{\tau\tau} &= s_{23}^2 \left(s_{12}^2 m_1 + c_{12}^2 m_2\right) + c_{23}^2 \left[s_{13}^2 \left(c_{12}^2 m_1 + s_{12}^2 m_2\right) + c_{13}^2 m_3\right] \\
&\quad - 2 c_{23} s_{23} s_{13} c_{12} s_{12} (m_1 - m_2) \,, \\
M_{e\mu} &= - c_{23} c_{13} c_{12} s_{12} (m_1 - m_2) - s_{23} c_{13} s_{13} \left(c_{12}^2 m_1 + s_{12}^2 m_2 - m_3\right) , \\
M_{e\tau} &= s_{23} c_{13} c_{12} s_{12} (m_1 - m_2) - c_{23} c_{13} s_{13} \left(c_{12}^2 m_1 + s_{12}^2 m_2 - m_3\right) , \\
M_{\mu\tau} &= - c_{23} s_{23} \left[s_{12}^2 m_1 + c_{12}^2 m_2 - s_{13}^2 \left(c_{12}^2 m_1 + s_{12}^2 m_2\right) - c_{13}^2 m_3\right] \\
&\quad + c_{12} s_{12} s_{13} \left(c_{23}^2 - s_{23}^2\right) (m_1 - m_2) \,.
\end{aligned}
\tag{6}
$$

If $s_{13} = 0$, Eqs. (6) can be rewritten in the matrix form as follows:

$$
\begin{aligned}
M = \frac{m_1 + m_2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^2 & -c_{23} s_{23} \\ 0 & -c_{23} s_{23} & s_{23}^2 \end{pmatrix} + m_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{23}^2 & c_{23} s_{23} \\ 0 & c_{23} s_{23} & c_{23}^2 \end{pmatrix} \\
+ \frac{m_1 - m_2}{2} \begin{pmatrix} c_{\rm sol} & -c_{23} s_{\rm sol} & s_{23} s_{\rm sol} \\ -c_{23} s_{\rm sol} & -c_{23}^2 c_{\rm sol} & c_{23} s_{23} c_{\rm sol} \\ s_{23} s_{\rm sol} & c_{23} s_{23} c_{\rm sol} & -s_{23}^2 c_{\rm sol} \end{pmatrix} ,
\end{aligned}
\tag{7}
$$

where $c_{\rm sol} = c_{12}^2 - s_{12}^2 = \cos 2\theta_{12}$ and $s_{\rm sol} = 2 c_{12} s_{12} = \sin 2\theta_{12}$. In Eq. (7), all three 3×3 matrices on its rhs commute, while its third term is of the form

$$
\frac{m_1 - m_2}{2} \left[c_{\rm sol} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c_{23}^2 & c_{23} s_{23} \\ 0 & c_{23} s_{23} & -s_{23}^2 \end{pmatrix} + s_{\rm sol} \begin{pmatrix} 0 & -c_{23} & s_{23} \\ -c_{23} & 0 & 0 \\ s_{23} & 0 & 0 \end{pmatrix} \right]
\tag{8}
$$

involving two anticommuting 3×3 matrices. Diagonalizing both sides of Eq. (7), one gets consistently

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = \frac{m_1+m_2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + m_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$+ \frac{m_1-m_2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{9}$$

2. Invariance of effective mass matrix M in the case of $s_{13} = 0$

Introduce three discrete transformations of active neutrinos ν_e, ν_μ, ν_τ,

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix}_a = \varphi_a \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \qquad (a = 1, 2, 3)\,, \tag{10}$$

where

$$\begin{aligned}
\varphi_1 &\equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & c_{\rm atm} & -s_{\rm atm} \\ 0 & -s_{\rm atm} & -c_{\rm atm} \end{pmatrix} \\
&= \mathrm{diag}\,(-1, c_{\rm atm}\sigma_3 - s_{\rm atm}\sigma_1) \overset{s_{\rm atm}\to\pm 1}{\longrightarrow} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \mp 1 \\ 0 & \mp 1 & 0 \end{pmatrix} , \\
\varphi_2 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \mathrm{diag}\left(1\,,\, -\mathbf{1}^{(2)}\right) , \\
\varphi_3 &\equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -c_{\rm atm} & s_{\rm atm} \\ 0 & s_{\rm atm} & c_{\rm atm} \end{pmatrix} \\
&= \mathrm{diag}\,(-1, -c_{\rm atm}\sigma_3 + s_{\rm atm}\sigma_1) \overset{s_{\rm atm}\to\pm 1}{\longrightarrow} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix} ,
\end{aligned} \tag{11}$$

are 3×3 Hermitian matrices with $c_{\rm atm} = c_{23}^2 - s_{23}^2 = \cos 2\theta_{23}$ and $s_{\rm atm} = 2c_{23}s_{23} = \sin 2\theta_{23}$ (here, the approximation of $s_{\rm atm} = \pm 1$ *i.e.*, $c_{23} = 1/\sqrt{2} = \pm s_{23}$ is experimentally satisfactory). Similarly, the 3×3 unit matrix

$$\mathbf{1}^{(3)} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}(1, \mathbf{1}^{(2)})\,, \tag{12}$$

where $\mathbf{1}^{(2)} = \mathrm{diag}(1,1)$, describes the active-neutrino identity transformation. The matrices (11) were already used in Ref. [10], but in the limit of $s_{\rm atm} \to 1$.

It is easy to see that the four matrices $\mathbf{1}^{(3)}$, φ_1 , φ_2 , φ_3 satisfy for any $c_{\rm atm}$ and $s_{\rm atm}$ the following algebraic relations

$$\varphi_1\varphi_2 = \varphi_3 \ \text{(cyclic)}\,, \qquad \varphi_a^2 = \mathbf{1}^{(3)}\,, \qquad \varphi_a\varphi_b = \varphi_b\varphi_a \tag{13}$$

and also the constraint

$$\mathbf{1}^{(3)} + \varphi_1 + \varphi_2 + \varphi_3 = 0\,. \tag{14}$$

The Cayley table equivalent to the relations (13) gets the form

	$\mathbf{1}^{(3)}$	φ_1	φ_2	φ_3
$\mathbf{1}^{(3)}$	$\mathbf{1}^{(3)}$	φ_1	φ_2	φ_3
φ_1	φ_1	$\mathbf{1}^{(3)}$	φ_3	φ_2
φ_2	φ_2	φ_3	$\mathbf{1}^{(3)}$	φ_1
φ_3	φ_3	φ_2	φ_1	$\mathbf{1}^{(3)}$

The algebraic relations (13) with $\mathbf{1}^{(3)}$ replaced by the generic unit element $\mathbf{1}$ (and φ_1, φ_2, φ_3 — by three other generic group elements) characterize a finite group $Z_2 \times Z_2$ of the order four often called the four-group [11] (Z_2 is the cyclic group of the order two). It is isomorphic to the dihedral group [11] of the order four and also to the group of four special permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{15}$$

of four objects. As is known, all finite groups of the order four are isomorphic either to the four-group $Z_2 \times Z_2$ or to the cyclic group Z_4 of the order four, these two being not isomorphic to each other. Both are Abelian. Note that the dihedral group of the order six is isomorphic to the permutation group S_3 of three objects, and the dihedral group of the order eight is the group D_4 considered in Refs. [12] and [13]. They are non-Abelian.

Four 3×3 matrices (12) and (11) constitute a reducible representation $\underline{3} = \underline{1} + \underline{2}$ of the four-group, where its representations $\underline{1}$ and $\underline{2}$ consist of four numbers

$$1\,,\ -1\,,\ 1\,,\ -1 \tag{16}$$

and of four 2×2 matrices

$$\mathbf{1}^{(2)}\;,\; c_{\rm atm}\sigma_3 - s_{\rm atm}\sigma_1\;,\; -\mathbf{1}^{(2)}\;,\; -c_{\rm atm}\sigma_3 + s_{\rm atm}\sigma_1\,, \tag{17}$$

respectively. The representations $\underline{1}$ and $\underline{2}$ are, respectively, irreducible and reducible but the second is *not* reduced (to the sum diag$(\underline{1},\underline{1})$ of two irreducible representations $\underline{1}$ consisting of four numbers $1,1,-1,-1$ and $1,-1,-1,1$). The constraint (14) with $\mathbf{1}^{(3)}$ replaced by the generic unit element $\mathbf{1}$ (and $\varphi_1,\varphi_2,\varphi_3$ — by three other generic group elements) is satisfied in both cases. But this constraint is not included in the definition of the four-group.

The four permutations (15) of four objects $1,2,3,4$ can be represented as a reducible representation $\underline{4}=\underline{2}+\underline{2}$ of the four-group consisting of four 4×4 matrices $\mathbf{1}^{(\rm D)}$, $\sigma_1^{(\rm D)}$, γ_5, $\gamma_5\sigma_1^{(\rm D)}=\alpha_1$, where $\mathbf{1}^{(\rm D)}$, $\sigma_1^{(\rm D)}$, γ_5 are formal Dirac 4×4 matrices in the Dirac representation: $\sigma_1^{(\rm D)}={\rm diag}(\sigma_1,\sigma_1)$, $\gamma_5 =$ antidiag$(\mathbf{1}^{(2)}, \mathbf{1}^{(2)})$ and, as always, $\mathbf{1}^{(\rm D)}=$ diag $(\mathbf{1}^{(2)},\mathbf{1}^{(2)})$. After a unitary transformation of Dirac matrices, one can write $\gamma_5={\rm diag}(\mathbf{1}^{(2)}\,,\,-\mathbf{1}^{(2)})$ and still $\sigma_1^{(\rm D)}={\rm diag}(\sigma_1,\sigma_1)$, as in the chiral representation. Then, $\underline{4}=$ diag $(\underline{2},\underline{2})$, where the second of two not reduced representations $\underline{2}$ of the four-group is identical with its representation (17), when $c_{\rm atm}=0$ and $s_{\rm atm}=-1$ (or $s_{\rm atm}=1$, but then the elements $\sigma_1^{(\rm D)}$ and $\gamma_5\sigma_1^{(\rm D)}$ of $\underline{4}$ are interchanged). Putting formally $(1,2,3,4)^{\rm T}=(-\nu_e/\sqrt{2}\,,\,\nu_e/\sqrt{2}\,,\,\nu_\mu\,,\,\nu_\tau)^{\rm T}$, one would obtain from the reducible representation $\underline{4}$ of the four-group its reducible representation $\underline{1}+\underline{1}+\underline{2}=\underline{1}+\underline{3}$, where $\underline{1}=(1,-1,1,-1)$ and $\underline{3}=(\mathbf{1}^{(3)},\,\varphi_1\,,\,\varphi_2\,,\,\varphi_3)$ as given in Eqs. (16) and (12), (11) with $c_{\rm atm}=0$ and $s_{\rm atm}=-1$ (or $s_{\rm atm}=1$, but then the elements φ_1 and φ_3 of $\underline{3}$ are interchanged). In general, the representation $\underline{4}$ of the four-group might suggest the existence of a fourth light neutrino, ν_s, sterile in the gauge interactions of Standard Model. Then, the entries 1 and 2 in $(1,2,3,4)^{\rm T}$ could be expressed through ν_s and ν_e. Note, however, that even in this case the strile neutrino ν_s may be eliminated, if the constraint

$$\left(\mathbf{1}^{(\rm D)}+\sigma_1^{(\rm D)}+\gamma_5+\gamma_5\sigma_1^{(\rm D)}\right)(1,2,3,4)^{\rm T}=0\,,$$

is imposed on $(1,2,3,4)^{\rm T}$. In fact, with $\gamma_5={\rm diag}(\mathbf{1}^{(2)},-\mathbf{1}^{(2)})$ this constraint is split into two conditions

$$\left(\mathbf{1}^{(2)}+\sigma_1+\mathbf{1}^{(2)}+\sigma_1\right)(1,2)^{\rm T}=0\,,\;\text{and}\;\left(\mathbf{1}^{(2)}+\sigma_1-\mathbf{1}^{(2)}-\sigma_1\right)(3,4)^{\rm T}=0\,,$$

of which the second is satisfied for 3 and 4 identically, while the first with $\sigma_1=$ antidiag$(1,1)$ gives for 1 and 2 one condition $(1)+(2)=0$ implying that from two orthogonal superpositions $\frac{1}{\sqrt{2}}[(1)+(2)]$ and $\frac{1}{\sqrt{2}}[(2)-(1)]$ only the second survives. Then, identifying ν_s and ν_e with the first and the second superposition, respectively, one obtains $\nu_s = 0$ and $\nu_e = (2)\sqrt{2} = -(1)\sqrt{2}$. In such a way ν_s may be eliminated indeed, giving $(1,2,3,4)^{\rm T} = (-\nu_e/\sqrt{2}, \nu_e/\sqrt{2}, \nu_\mu, \nu_\tau)^{\rm T}$, just as was formally considered above. After a unitary transformation, where $(1) \to \frac{1}{\sqrt{2}}[(1)+(2)] = 0$ and $(2) \to \frac{1}{\sqrt{2}}[(2)-(1)] = (2)\sqrt{2}$, one gets $(-\nu_e/\sqrt{2}, \nu_e/\sqrt{2}, \nu_\mu, \nu_\tau)^{\rm T} \to (0, \nu_e, \nu_\mu, \nu_\tau)^{\rm T}$. Notice that, after this transformation, the first of two representations $\underline{2}$ in $\underline{4} = {\rm diag}(\underline{2}, \underline{2})$ becomes reduced to the sum ${\rm diag}(\underline{1}, \underline{1})$ of two irreducible representations $\underline{1}$ consisting of four numbers $1,1,1,1$ and $1,-1,1,-1$.

Making use of the matrices $\mathbf{1}^{(3)}$, φ_1 , φ_2 , φ_3 as given in Eqs. (12) and (11), we can rewrite the formula (7) for the effective neutrino mass matrix, valid in the case of $s_{13} = 0$, as follows

$$\begin{aligned} M \;=\; & \frac{m_1+m_2}{2}\frac{1}{2}\left(\mathbf{1}^{(3)} - \varphi_3\right) + m_3\frac{1}{2}\left(\mathbf{1}^{(3)} + \varphi_3\right) \\ & - \frac{m_1-m_2}{2}\left[c_{\rm sol}\frac{1}{2}\left(\varphi_1 - \varphi_2\right) + s_{\rm sol}\frac{1}{2}\left(c_{23}\lambda_1 - s_{23}\lambda_4\right)\right], \end{aligned} \tag{18}$$

where

$$\lambda_1 = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \qquad \lambda_4 = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right), \tag{19}$$

are two of the eight Gell-Mann 3×3 matrices (here, the approximation of $s_{\rm atm} = \pm 1$ *i.e.*, $c_{23} = 1/\sqrt{2} = \pm s_{23}$ is experimentally satisfactory). Note that $\frac{1}{2}(\mathbf{1}^{(3)} + \varphi_3) = -\frac{1}{2}(\varphi_1 + \varphi_2)$.

With the matrix φ_3 as given in the third Eq. (11), it is not difficult to show from Eq. (18) that in the case of $s_{13} = 0$ the effective neutrino mass matrix M is *invariant* under the third ($a = 3$) neutrino transformation (10),

$$\varphi_3 M \varphi_3 = M\,, \tag{20}$$

while for the first ($a = 1$) and second ($a = 2$) transformations (10) one gets

$$\varphi_{1,2} M \varphi_{1,2} = M + (m_1 - m_2) s_{\rm sol}(c_{23}\lambda_1 - s_{23}\lambda_4) \stackrel{m_1-m_2\to 0}{\longrightarrow} M\,, \tag{21}$$

i.e., in the limit of $m_1 - m_2 \to 0$ the effective mass matrix M with $s_{13} = 0$ is *invariant* also under the first and second transformations (10). So, in this limit, the matrix M with $s_{13} = 0$ is invariant under the whole four-group.

It is also not difficult to demonstrate that, inversely, the invariance (20), if imposed on M, implies for s_{13} the restriction $s_{13} = 0$. In fact, Eqs. (6)

for $M_{\alpha\beta}$ valid for generic s_{13}, when substituted into Eq. (20), lead *e.g.* to the equality

$$\begin{aligned} M_{e\mu} &= (\varphi_3 M \varphi_3)_{e\mu} = c_{\rm atm} M_{e\mu} - s_{\rm atm} M_{e\tau} \\ &= M_{e\mu} + 2 s_{23} c_{13} s_{13} \left(c_{12}^2 m_1 + s_{12}^2 m_2 - m_3 \right) . \end{aligned} \qquad (22)$$

This implies that $s_{13} = 0$ since $c_{13} \neq 0$. Then, with $c_{13}^2 = 1$ the matrix M must have the form (7) or (18), while with $c_{13} = 1$ the matrix U has to be reduced to the form (2).

The proof that the restriction $s_{13} = 0$ follows from the invariance of M described essentially by Eq. (20) (even if $c_{\rm atm} = \cos 2\theta_{23} \neq 0$) was presented previously in Ref. [12]. Such an invariance (with $c_{\rm atm} = \cos 2\theta_{23} = 0$ and $s_{13} = 0$) was considered also in Refs. [13,14] as well as in Ref. [10].

3. Duality of atmospheric and solar mixing angle in the case of $s_{13} = 0$

Four 3×3 matrices $\mathbf{1}^{(3)}$, μ_1, μ_2, μ_3, where μ_a are defined by the unitary transformations

$$\mu_a \equiv U^\dagger \varphi_a U \,, \qquad (a = 1, 2, 3)\,, \qquad (23)$$

constitute in the case of $s_{13} = 0$ another reducible representation $\underline{3} = \underline{2} + \underline{1}$ of the four-group, that is unitarily isomorphic to its previous representation $\underline{3} = \underline{1} + \underline{2}$, consisting of 3×3 matrices $\mathbf{1}^{(3)}$, φ_1, φ_2, φ_3 introduced in Eqs. (11), (12). In fact, with the use of Eqs. (11) for φ_a (with any $c_{\rm atm}$ and $s_{\rm atm}$) and the form (2) of U valid in the case of $s_{13} = 0$, we obtain

$$\begin{aligned} \mu_1 &= \begin{pmatrix} -c_{\rm sol} & -s_{\rm sol} & 0 \\ -s_{\rm sol} & c_{\rm sol} & 0 \\ 0 & 0 & -1 \end{pmatrix} = {\rm diag}(-c_{\rm sol}\sigma_3 - s_{\rm sol}\sigma_1, -1)\,, \\ \mu_2 &= \begin{pmatrix} c_{\rm sol} & s_{\rm sol} & 0 \\ s_{\rm sol} & -c_{\rm sol} & 0 \\ 0 & 0 & -1 \end{pmatrix} = {\rm diag}(c_{\rm sol}\sigma_3 + s_{\rm sol}\sigma_1, -1)\,, \\ \mu_3 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = {\rm diag}(-\mathbf{1}^{(2)}, 1)\,. \end{aligned} \qquad (24)$$

Recall that $c_{\rm sol} = c_{12}^2 - s_{12}^2 = \cos 2\theta_{12}$ and $s_{\rm sol} = 2 c_{12} s_{12} = \sin 2\theta_{12}$. Here, it is convenient to write $\mathbf{1}^{(3)} = {\rm diag}(\mathbf{1}^{(2)}\,, 1)$. Evidently, the four matrices

$\mathbf{1}^{(3)}$, μ_1 , μ_2 , μ_3 satisfy for any $c_{\rm sol}$ and $s_{\rm sol}$ the algebraic relations identical in form with Eqs. (13)

$$\mu_1\mu_2 = \mu_3 \text{ (cyclic)}\,, \qquad \mu_a^2 = \mathbf{1}^{(3)}\,, \qquad \mu_a\mu_b = \mu_b\mu_a\,, \tag{25}$$

and also the constraint identical in form with Eq. (14)

$$\mathbf{1}^{(3)} + \mu_1 + \mu_2 + \mu_3 = 0\,. \tag{26}$$

The matrices μ_a were already considered in Ref. [10], but in the formal limit of $s_{\rm sol} \to 1$ (in contrast to $s_{\rm atm} = \pm 1$ *i.e.*, $c_{23} = 1/\sqrt{2} = \pm s_{23}$, the approximation $s_{\rm sol} = \pm 1$ *i.e.*, $c_{12} = 1/\sqrt{2} = \pm s_{12}$ is experimentally not satisfactory).

From Eqs. (23), (1) and (10) we can infer that

$$\left(\begin{array}{c} \nu'_1 \\ \nu'_2 \\ \nu'_3 \end{array}\right)_a = \mu_a \left(\begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array}\right) = U^\dagger \varphi_a \left(\begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array}\right) = U^\dagger \left(\begin{array}{c} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{array}\right)_a \quad (a = 1, 2, 3)\,. \tag{27}$$

Thus, the four-group transformations (27) of mass neutrinos ν_i (produced by three matrices μ_a) are *covariant* under the neutrino mixing (1): they transit into the four-group transformations (10) of flavor neutrinos ν_α (generated by three matrices φ_a).

In addition, Eqs. (11) and (24) involving $\theta_{\rm atm} = 2\theta_{23}$ and $\theta_{\rm sol} = 2\theta_{12}$, respectively, being related through the unitary transformations (23), tell us that the atmospheric and solar mixing angles, $\theta_{\rm atm} = 2\theta_{23}$ and $\theta_{\rm sol} = 2\theta_{12}$ are in a way mutually *dual* in the process of neutrino mixing described in Eq. (1)

$$\left(\begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array}\right) = U^\dagger \left(\begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array}\right). \tag{28}$$

Beside the duality relations $U^\dagger \varphi_a\,(c_{\rm atm}, s_{\rm atm})\,U = \mu_a\,(c_{\rm sol}, s_{\rm sol})$ with φ_a and μ_a given in Eqs. (11) and (24), respectively, we can show that

$$U^\dagger\,(c_{23}\lambda_1 - s_{23}\lambda_4)\,U = c_{12}\lambda_1 - s_{12}\lambda_3\,, \tag{29}$$

where

$$\lambda_3 = \left(\begin{array}{rrr} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{30}$$

is the third Gell-Mann 3×3 matrix. The duality relation (29) follows from a direct calculation using the form (2) of U valid in the case of $s_{13} = 0$.

The formula (27) compared with Eq. (28) shows that the effective neutrino mixing matrix U transforming ν_i into ν_α is *invariant* under the four-group. The same conclusion follows as a tautology from Eqs. (23) rewritten in the equivalent form

$$\varphi_a U \mu_a = U \quad (a = 1, 2, 3)\,, \tag{31}$$

where φ_a and μ_a belong to two unitarily isomorphic representations $\underline{3}$ of the four-group.

4. Conclusion

Thus, if and only if $s_{13} = 0$, the generic form of the effective neutrino mass matrix M becomes invariant under the subgroup Z_2 of the four-group $Z_2 \times Z_2$, represented by $\mathbf{1^{(3)}}$ and φ_3. In the approximation of $m_1 = m_2$, the matrix M becomes invariant under the whole four-group represented by the matrices $\mathbf{1}^{(3)}$, φ_1 , φ_2 , φ_3.

In the case of $s_{13} = 0$, the atmospheric and solar mixing angles, $\theta_{\rm atm} = 2\theta_{23}$ and $\theta_{\rm sol} = 2\theta_{12}$, turn out to be mutually dual in the process of neutrino mixing, what means that $U^\dagger \varphi_a(c_{\rm atm}, s_{\rm atm}) U = \mu_a(c_{\rm sol}, s_{\rm sol})$ $(a = 1, 2, 3)$, where $c_{\rm atm} = \cos\theta_{\rm atm}$, $s_{\rm atm} = \sin\theta_{\rm atm}$ and $c_{\rm sol} = \cos\theta_{\rm sol}$, $s_{\rm sol} = \sin\theta_{\rm sol}$. Here, $\mathbf{1}^{(3)}$, φ_1 , φ_2 , φ_3 and $\mathbf{1}^{(3)}$, μ_1 , μ_2 , μ_3 constitute two unitarily isomorphic reducible representations $\underline{3}$ of the four-group (producing four-group transformations of three flavor and three mass neutrinos, respectively).

Appendix A

A reducible representation $\underline{4}$ of the group D_4

The dihedral group of the order eight, D_4 considered in Refs. [12] and [13], is isomorphic to the group of eight special permutations of four objects: four even permutations given in Eq. (15) and four odd permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}. \tag{A.1}$$

These eight permutations of four objects 1,2,3,4 can be represented by the following eight 4×4 matrices

$$\begin{gathered} \mathbf{1}^{(\mathrm{D})}\,, \quad \sigma_1^{(\mathrm{D})}\,, \quad \gamma_5\,, \quad \gamma_5 \sigma_1^{(\mathrm{D})}\,, \\ \frac{1}{2}\left(\sigma_1^{(\mathrm{D})} \pm i\sigma_2^{(\mathrm{D})}\right) + \gamma_5 \frac{1}{2}\left(\sigma_1^{(\mathrm{D})} \mp i\sigma_2^{(\mathrm{D})}\right)\,, \\ \frac{1}{2}\left(\mathbf{1}^{(\mathrm{D})} \pm \sigma_3^{(\mathrm{D})}\right) + \gamma_5 \frac{1}{2}\left(\mathbf{1}^{(\mathrm{D})} \mp \sigma_3^{(\mathrm{D})}\right), \end{gathered} \tag{A.2}$$

respectively, constituting a reducible representation $\underline{4}$ of the group D_4. Here, $\mathbf{1}^{(\mathrm{D})}$, $\sigma_1^{(\mathrm{D})}$, $\sigma_2^{(\mathrm{D})}$, $\sigma_3^{(\mathrm{D})}$, γ_5 are formal Dirac 4×4 matrices in the Dirac representation: $\vec{\sigma}^{(\mathrm{D})} = \mathrm{diag}(\vec{\sigma},\, \vec{\sigma})$, $\gamma_5 = \mathrm{antidiag}(\mathbf{1}^{(2)}, \mathbf{1}^{(2)})$ and, as always, $\mathbf{1}^{(\mathrm{D})} = \mathrm{diag}(\mathbf{1}^{(2)}, \mathbf{1}^{(2)})$. As can be easily seen, the first four matrices (A.2) represent the four-group as a subgroup of D_4, while the second four matrices (A.2) represent the coset of the four-group in D_4, so, they make all the difference between the group D_4 and the four-group.

After a unitary transformation leading to the chiral representation of Dirac matrices, one can write $\gamma_5 = \mathrm{diag}(\mathbf{1}^{(2)}, -\mathbf{1}^{(2)})$ and still $\vec{\sigma}^{(\mathrm{D})} = \mathrm{diag}(\vec{\sigma},\, \vec{\sigma})$. Then, the eight matrices (A.2) are reduced, respectively, to the forms

$$\begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & \mathbf{1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & -\mathbf{1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix},$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}, \begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & \pm\sigma_3 \end{pmatrix} \tag{A.3}$$

constituting a reduced representation $\underline{4} = \mathrm{diag}(\underline{2}, \underline{2})$ of the group D_4. The second of its two representations $\underline{2}$ involved in $\underline{4} = \mathrm{diag}(\underline{2}, \underline{2})$ is irreducible, while the first of them is reducible (but not yet reduced to $\underline{2} = \mathrm{diag}(\underline{1}, \underline{1})$). This is in contrast to the four-group, where both representations $\underline{2}$, involved in its reduced representation $\underline{4} = \mathrm{diag}(\underline{2}, \underline{2})$, are reducible (but not yet reduced to $\underline{2} = \mathrm{diag}(\underline{1}, \underline{1})$).

In fact, after a second unitary transformation leading to the changes $\sigma_1 \to \sigma_3$, $\sigma_2 \to \sigma_2$, $\sigma_3 \to -\sigma_1$, the eight matrices (A.3) take, respectively, the following maximally reduced forms,

$$\begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & \mathbf{1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & -\mathbf{1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$

$$\begin{pmatrix} \sigma_3 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}, \begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & \mp\sigma_1 \end{pmatrix}, \tag{A.4}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ is diagonal, but σ_1 and σ_2 are not (and cannot be diagonalized simultaneously with each other and with σ_3). Thus, in the case of group D_4 or of four-group, the maximal reduction of $\underline{4} = \mathrm{diag}(\underline{2}, \underline{2})$ gives $\underline{4} = \mathrm{diag}(\underline{1}, \underline{1}, \underline{2})$ or $\underline{4} = \mathrm{diag}(\underline{1}, \underline{1}, \underline{1}, \underline{1})$, respectively. As follows from Eq. (A.4), in the case of group D_4 two irreducible representations $\underline{1}$ are involved, consisting of eight numbers

$$1, 1, 1, 1, 1, 1, 1, 1 \quad \text{and} \quad 1, -1, 1, -1, -1, -1, 1, 1\,,$$

while in the case of four-group four irreducible representations $\underline{1}$ appear, consisting of four numbers

$$1, 1, 1, 1\ ;\ 1, -1, 1, -1\ ;\ 1, 1, -1, -1\ \text{ and }\ 1, -1, -1, 1\,.$$

If the second unitary transformation leading to $\sigma_1 \to \sigma_3$, $\sigma_2 \to \sigma_2$, $\sigma_3 \to -\sigma_1$ is applied upstairs, *i.e., only* to the first $\underline{2}$ in $\underline{4} = \mathrm{diag}(\underline{2}, \underline{2})$, the eight matrices (A.3) transit, respectively, into the following reduced forms

$$\begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & \mathbf{1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & -\mathbf{1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_1 \end{pmatrix},$$

$$\begin{pmatrix} \sigma_3 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}, \begin{pmatrix} \mathbf{1}^{(2)} & 0 \\ 0 & \pm\sigma_3 \end{pmatrix}. \qquad \text{(A.5)}$$

Thus, in such a case, both for the group D_4 and the four-group the reduction of $\underline{4} = \mathrm{diag}(\underline{2}, \underline{2})$ gives $\underline{4} = \mathrm{diag}(\underline{1}, \underline{1}, \underline{2})$. The eight matrices (A.5) can be rewritten, respectively, in the convenient forms

$$\mathrm{diag}(1, \mathbf{1}^{(3)}) \qquad \mathrm{diag}(1, \varphi_a) \;\; (a = 1, 2, 3, 4, 5, 6, 7) \qquad \text{(A.6)}$$

constituting a reduced representation $\underline{4} = \mathrm{diag}(\underline{1}, \underline{3})$ of the group D_4, where also $\underline{3} = \mathrm{diag}(\underline{1}, \underline{2})$ is its reduced representation. Here

$$\mathbf{1}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{(A.7)}$$

$$\varphi_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \varphi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \varphi_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

are the four 3×3 matrices given in Eqs. (11) and (12) with $s_{23} = -1$ (or $s_{23} = 1$, but then the matrices φ_1 and φ_3 are interchanged), while

$$\varphi_{4,5} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \mp 1 & 0 \end{pmatrix}, \qquad \varphi_{6,7} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix} \qquad \text{(A.8)}$$

stand for four new 3×3 matrices. In Eqs. (A.8), $\varphi_{4,5}$ are non-Hermitian: $\varphi^\dagger_{4,5} = \varphi^{\mathrm{T}}_{4,5} = \varphi_{5,4} \neq \varphi_{4,5}$, but still orthogonal: $\varphi^{\mathrm{T}}_{4,5} = \varphi^{-1}_{4,5}$ with $\varphi^2_{4,5} = \mathrm{diag}(1,\text{-}1,\text{-}1) = \varphi_2 \neq \mathbf{1}^{(3)}$ (giving $\varphi^4_{4,5} = \mathbf{1}^{(3)}$), though $\varphi^\dagger_{4,5}\varphi_{4,5} = \mathbf{1}^{(3)}$. In contrast, all other φ_a ($a \neq 4, 5$) are Hermitian: $\varphi^\dagger_a = \varphi^{\mathrm{T}}_a = \varphi_a$ and equal to square roots of $\mathbf{1}^{(3)}$: $\varphi^2_a = \mathbf{1}^{(3)}$.

Notice that the sum of eight 4×4 matrices (A.3) is equal to $4\,\mathrm{diag}(\mathbf{1}^{(2)} + \sigma_1, 0, 0)$ and, consequently, the sum of eight 4×4 matrices (A.4) or (A.5) is equal to $4\ \mathrm{diag}(\mathbf{1}^{(2)} + \sigma_3, 0, 0) = 8\ \mathrm{diag}(1,0,0,0)$. Also for eight 4×4 matrices (A.6) one gets

$$\mathrm{diag}(1, \mathbf{1}^{(3)}) + \sum_{a=1}^{7} \mathrm{diag}(1, \varphi_a) = 8\,\mathrm{diag}(1, 0, 0, 0)\,, \qquad \text{(A.9)}$$

where the (strong) constraint

$$\mathbf{1}^{(3)} + \sum_{a=1}^{7} \varphi_a = 0 \tag{A.10}$$

holds.

It is evident from Eq. (A.9) that the (weak) constraint of the form

$$\left[\mathrm{diag}(1, \mathbf{1}^{(3)}) + \sum_{a=1}^{7} \mathrm{diag}(1, \varphi_a)\right] \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 0\,, \tag{A.11}$$

if imposed on the state $(1,2,3,4)^{\mathrm{T}}$ of four objects, eliminates the object 1, giving $(1,0,0,0)^{\mathrm{T}} = 0$, while it leaves the objects 2,3,4 non-constrained, being identically satisfied for $(0,2,3,4)^{\mathrm{T}}$. Therefore, similarly as in the case of four-group, also in the case of group D_4 the "physical" objects 2,3,4 might be interpreted as three active neutrinos ν_e, ν_μ, ν_τ, while the eliminated "unphysical" object 1 as one light sterile neutrino ν_s. Then, the state $(2,3,4)^{\mathrm{T}} = (\nu_e, \nu_\mu, \nu_\tau)^{\mathrm{T}}$ of three active neutrinos would be acted on by the reduced representation $\underline{3} = \mathrm{diag}(\underline{1}, \underline{2})$ of group D_4, consisting of eight 3×3 matrices $\mathbf{1}^{(3)}, \varphi_a$ $(a = 1,2,3,4,5,6,7)$. This is similar to the case of four-group, where the reduced representation $\underline{3} = \mathrm{diag}(\underline{1}, \underline{2})$ consists of the first four of these matrices only.

We know from Eqs. (20) and (21) that in the limit of $m_1 - m_2 \to 0$ the effective neutrino mass matrix M (with $s_{13} = 0$) given in Eq. (18) is invariant under the four-group. In fact, it commutes then with $\mathbf{1}^{(3)}, \varphi_1, \varphi_2, \varphi_3$. It is not difficult to see that, even in the limit of $m_1 - m_2 \to 0$, the effective matrix M (with $s_{13} = 0$) does not commute with $\varphi_{4,5}$ and $\varphi_{6,7}$. This follows from the commutation relations

$$[\varphi_3, \varphi_{4,5}] = \pm(\varphi_6 - \varphi_7) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix} \neq 0\,,$$

$$[\varphi_3, \varphi_{6,7}] = \pm(\varphi_4 - \varphi_5) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \mp 1 & 0 \end{pmatrix} \neq 0\,. \tag{A.12}$$

Thus, the matrices representing the coset of four-group in D_4 spoil a part of the invariance of the *effective* neutrino mass matrix M (with $s_{13} = 0$) under the group D_4. This group is perhaps too large and so, the embedding of four-group into D_4 not necessary.

Of course, in the limit of $m_2 - m_3 \to 0$ following the previous limit of $m_1 - m_2 \to 0$, the invariance of M under D_4 holds trivially. In fact, Eq. (18) can be rewritten in the form

$$M = \frac{m_1 + m_2}{2}\mathbf{1}^{(3)} + \left(m_3 - \frac{m_1 + m_2}{2}\right)\frac{1}{2}\left(\mathbf{1}^{(3)} + \varphi_3\right)$$
$$+ \frac{m_2 - m_1}{2}\left[c_{\rm sol}\frac{1}{2}\left(\varphi_1 - \varphi_2\right) + s_{\rm sol}\frac{1}{2}\left(c_{23}\lambda_1 - s_{23}\lambda_4\right)\right], \tag{A.13}$$

giving $M \propto \mathbf{1}^{(3)}$ in the limit of $m_1 - m_2 \to 0$ and $m_2 - m_3 \to 0$. Recall that the limit of $m_1 - m_2 \to 0$ was shown to be necessary and sufficient to extend the invariance of M under the subgroup Z_2 represented by $\mathbf{1}^{(3)}$ and φ_3 to the invariance under the whole four-group $Z_2 \times Z_2$ represented by $\mathbf{1}^{(3)}$ and $\varphi_1, \varphi_2, \varphi_3$. In the case of $m_1 \neq m_2$, the coset of this subgroup in the four-group, represented by φ_1 and φ_2, spoils a part of the invariance of M under the four-group.

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