DEFINING CHAOS IN THE LOGISTIC MAP BY SHARKOVSKII'S THEOREM*

M. Howard Lee

Department of Physics and Astronomy, University of Georgia
Athens, GA 30602, USA†
and
Korea Institute for Advanced Study, Seoul 130-012, Korea
MHLee@uga.edu

(Received May 10, 2013)

The fixed points of 3-cycle in the logistic map are obtained by solving a sextic polynomial analytically. Therewith, the domain of chaos is established by Sharkovskii's theorem. A fix-point spectrum is then constructed in the chaotic domain. By Sharkovskii's theorem, a chaotic trajectory is shown to be a superposition of all finite cycles, termed an aleph cycle. An aleph cycle means chaos and it defines chaos in the logistic map in an absolute sense. In particular, a trajectory which is ergodic is aleph-cyclic, hence it is also chaotic.

DOI:10.5506/APhysPolB.44.925
PACS numbers: 05.30.–d, 05.40.–a, 05.45.–a

1. Introduction

Although the term chaos is used to describe a certain class of iterative trajectories in the logistic map, the precise meaning of this term seems elusive. A trajectory is said to be chaotic if the Lyapunov exponent for it is positive [1]. Since the exponent is of a phenomenological origin, it might not be all inclusive and it might even mislead. Trajectories are also said to be chaotic if they show initial-value sensitivity. This topological condition is also of a phenomenological origin. It is thus possible that a trajectory is initial-value sensitive but yet periodic. Given these possibilities, it is desirable to have a definition of chaos that might be derived from some fundamental principles of chaos.

* Presented at the XXV Marian Smoluchowski Symposium on Statistical Physics, "Fluctuation Relations in Nonequilibrium Regime", Kraków, Poland, September 10–13, 2012.

† Permanent address.

In math texts on chaos, one finds various definitions of chaos. In Devaney's book [2], for example, he gives a definition as: If V is a set of points, $f : V \to V$ (transl. function f transforms or maps V onto itself) is said to be chaotic on V if f has sensitive dependence on initial values, if f is topologically transitive, if periodic points are dense in V.

As far as we can determine, this definition and others do not derive from some higher principles of chaos. To no surprise, chaos thus defined is sometimes referred to as (in this instance) chaos in Devaney's sense. That would suggest that there may be chaos but not in Devaney's sense, or even possibly that chaos in Devaney's sense may not be chaos.

Is it possible to define chaos in an absolute sense? One possible way is: Establish unambiguously by some higher principles where chaos exists in a map and then extract therefrom its basic properties with which to arrive at a definition. If this program could be implemented, one might be able to say that chaos is now defined in an absolute sense or from first principles at least for this map.

Probably the most widely studied map in chaos is the logistic map, a 1d continuous non-invertible map. A fundamental principle of chaos which is applicable to this map is a theorem due to Sharkovskii [3]. Another applicable theorem is one due to Li and Yorke [4], once said to be the only rigorous theorem on chaos [5]. But it is now known that the Li–Yorke theorem, a later arrival but perhaps the better known, is a corollary of the Sharkovskii theorem [6].

Although Sharkovskii's theorem has existed for 5 decades and often been quoted "remarkable" in math texts, to our knowledge no math definitions have been constructed based on it. To see why, let us look at this theorem. It states that if there is 3-cycle (period 3) in a 1d continuous non-invertible map, there are all other cycles which imply chaos. More closely examined, the theorem really has two parts to it: (a) The existence of 3-cycle which establishes the domain of chaos. (b) The existence of a multitude of all cycles which characterizes chaos in that domain.

To apply this theorem, one must first prove that 3-cycle exists in a specific map to establish the domain of chaos. For the logistic map, it would mean solving a sextic polynomial, which no one seems to have taken the challenge of. Not having taken the first step, the second cannot be taken.

In this work, we shall try to establish the domain of chaos in the logistic map by proving 3-cycle and therewith to deduce from it a definition of chaos for the logistic map. It would be a definition for chaos in the logistic map which could be said to be in an absolute sense or from first principles.

2. Logistic map and 3-cycle equation

The logistic map is defined by $f(x) = ax(1-x)$, where a is the control parameter ranging from 0 to 4 and $0 < x < 1$ [1]. One defines 3-cycle by

$$f^3(x) - x = 0\,, \tag{1}$$

where $f^3(x) = f(f(f(x)))$. Equation (1) is a polynomial of some high degree in x, whose roots are the fixed points of f^3. The roots of f are among them, which are removed if we consider $Q_a(x) = 0$, where

$$Q_a(x) = \left(f^3 - x\right)/(f - x)\,. \tag{2}$$

It is straightforward if tedious to obtain

$$\begin{aligned} Q_a(t) &= t^6 - (3a+1)t^5 + \left(3a^2 + 4a + 1\right)t^4 - \left(a^3 + 5a^2 + 3a + 1\right)t^3 \\ &+ \left(2a^3 + 3a^2 + 3a + 1\right)t^2 - \left(a^3 + 2a^2 + 2a + 1\right)t + \left(a^2 + a + 1\right)\,, \end{aligned} \tag{3}$$

where $t = ax$. We find that t is a more natural variable than x for Q_a. It is a sextic polynomial for which there are no ready made solutions. That does not mean that it is not solvable. To solve it, we shall proceed as follows: Since $Q_a(t)$ is real if t is real, there are two possibilities:

1. If the roots are complex, they must be 3 complex conjugate pairs.
2. If roots are real, there are two sets of three unequal roots. There are no other possibilities *i.e.*, a combination of complex and real roots for 3-cycle.

To see the complex roots, let us take $a = 0$. (Although not allowed by the definition $x = at$, Q_a is still well-defined if $a = 0$.) If $a = 0$,

$$Q_0 = t^6 - t^5 + t^4 - t^3 + t^2 - t + 1\,. \tag{4}$$

The r.h.s. is easily summed to

$$Q_0 = \left(t^7 + 1\right)/(t+1)\,, \qquad t \neq -1. \tag{5}$$

The roots are: $\exp \pm i(\pi/7)$, $\exp \pm i(3\pi/7)$ and $\exp \pm i(5\pi/7)$, 3 complex conjugate pairs, all lying on the unit circle, $t = -1$ excluded. As a increases, at some value the three pairs must become 3 real roots each doubly degenerate. Let us denote the value of a, where it occurs by $a = \tilde{a}$. If $a > \tilde{a}$, we may assume that (3) is a product of two cubic equations

$$Q_a = q_a \times {q'}_a\,. \tag{6}$$

Factorizing the sextic equation into a product of two cubic equations appears daunting. At first, we shall look for special solutions to guide us.

3. $a = \tilde{a}$

When $a = \tilde{a}$, evidently (6) reduces to

$$Q_{\tilde{a}} = \tilde{q}^2 \,, \tag{7}$$

where at least formally,

$$\tilde{q} = \tilde{t}^3 - \tilde{\alpha}\tilde{t}^2 + \tilde{\beta}\tilde{t} - \tilde{\gamma} \,. \tag{8}$$

It has already been shown [7] that $\tilde{a} = 1 + \sqrt{8} = 3.828427\ldots$,

$$\tilde{\alpha} = 1/2\ (3\tilde{a} + 1)\,, \tag{9}$$
$$\tilde{\beta} = 2\tilde{a} + 3\,, \tag{10}$$
$$\tilde{\gamma} = 1/2\ (\tilde{a} + 5)\,. \tag{11}$$

Since the coefficients are given in terms of $\tilde{a}$, the cubic polynomial (8) is solvable in a standard way. We can do it more simply by a reflection property, deferred to a later section, see Section 5. But observe that $\tilde{\alpha} - \tilde{\beta} + \tilde{\gamma} = 0$.

4. $a = 4$

When $a = 4$ [8, 9], we obtain from (3)

$$Q_4 = t^6 - 13t^5 + 65t^4 - 157t^3 + 189t^2 - 105t + 21\,, \tag{12}$$

The above, indeed, is a product of two cubic equations: $Q_4 = q_4 \times q'_4$, where

$$q_4 = t^3 - 7t^2 + 14t - 7\,, \tag{13}$$
$$q'_4 = t^3 - 6t^2 + 9t - 3\,. \tag{14}$$

Both cubic polynomials yield cyclic solutions in the form (recalling that $x = t/a = t/4$)

$$x = \sin^2(\pi y/2) \tag{15}$$

with cyclic values $y/2 = 1/7, 2/7, 3/7$ for q_4 and $1/9, 2/9, 4/9$ for q'_4. In both cases $0 < y < 1$.

Now by the theorem of Sharkovskii, all other cycles exist at $a = 4$. That is, without solving for higher cycles, we may assert that y takes on all rational values in the interval from 0 to 1 continuously to accommodate all possible cycles, infinitely many of them. One can actually show by solving for several higher cycles, this assertion is borne out [10].

4.1. Aleph cycle

As the collection of n-cycles, $n = 1, 2, \ldots, N$, grows, their roots of cycles begin to coalesce. In the $N \to \infty$ limit, each finite root is pinched from both sides in a manner recalling Yang–Lee zeros pinching the real temperature axis from above and below in the complex temperature plane. As a result, y is continuous from 0 to 1. By (15), x is thus made continuous in its interval from 0 to 1. The y interval consists of rational and irrational points, giving measure 1. A trajectory which begins from a point x corresponding to a point y belonging to a set of points of measure 1 behaves as

$$x_1, f(x_1) = x_2, f(x_2) = x_3, \ldots f(x_N) = x_{N+1}, N \to \infty\,. \tag{16}$$

A trajectory having this behavior is termed an aleph cycle. An aleph cycle is clearly a superposition of all finite cycles and it represents a chaotic trajectory after Sharkovskii's theorem. An aleph cycle is a single trajectory, which contains almost all points in the interval, $(0, 1)$ for $a = 4$. It does not include a countable set of points since such a set has measure 0.

4.2. Periodic trajectories

If a trajectory starts from a point belonging to a set of points of measure 0, it ends up at some finite value of N by $x_{N+1} = x_1$, generating a periodic trajectory. We shall now see whether the results obtained from 3-cycle at $a = 4$ are contained in other values of a from $\tilde{a}$ up to 4.

5. Reflected solutions at $\tilde{\mathbf{a}}$

Although the cubic equation $\tilde{q} = 0$, see (8), could have been solved directly by the formula of Cardano and Tartaglia, we deferred it for the following remarkable reason. If we substitute $\tilde{t}$ in (8) by $\tilde{\gamma} - t$, $q_4 = 0$ is obtained. That is

$$\tilde{q}(\tilde{t} = \tilde{\gamma} - t) = -q_4(t)\,. \tag{17}$$

This means that if $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$ are the three roots of (8),

$$\tilde{t}_1 \;=\; \tilde{\gamma} - 4\sin^2(\pi/7)\,, \tag{18}$$

$$\tilde{t}_2 \;=\; \tilde{\gamma} - 4\sin^2(2\pi/7)\,, \tag{19}$$

$$\tilde{t}_1 \;=\; \tilde{\gamma} - 4\sin^2(3\pi/7)\,. \tag{20}$$

Thus 3-cycle at $a = \tilde{a}$ is also cyclic. Now, we see how Sharkovskii's theorem makes the cyclic value y continuous in the interval from 0 to 1. The spectrum is not completely filled because $\tilde{\gamma}$ is greater than 4 by about 10%. Thus a trajectory starting from this gap is not chaotic. Outside the gap, there can be an aleph cycle as when $a = 4$.

6. General solution

The reduction of the sextic polynomial into a product of two cubic polynomials was rather easily achieved at $a = \tilde{a}$ and 4. It gives us an impetus to look for the same reduction, which evidently is not simple nor elementary. In deriving the value $a = \tilde{a} = 1 + \sqrt{8}$ [7], we found the condition that $\sigma = 0$, where

$$\sigma = \left(a^2 - 2a - 7\right)^{1/2} . \tag{21}$$

Observe that if $a = 4$, $\sigma = \pm\, 1$. The two values are like two values of some parity. The two solutions for 3-cycle at $a = 4$ could mean that they represent $+$ and $-$ parity states. In fact, if there are two solutions for 3-cycle in the interval between $\tilde{a}$ and 4, they could differ only in their parity. Thinking along these lines and using the results for $a = \tilde{a}$ and 4, we are able to find the reduction form. The analysis is rather lengthy, thus it will not be given here. We find that it is possible to write: If $q_a = q_a(+\sigma)$, $q'_a = q_a(-\sigma)$, where

$$q_a(+\sigma) = t^3 - \alpha t^2 + \beta t - \gamma\,, \tag{22}$$

then

$$\alpha = \tfrac{1}{2}\,(3a + 1 + \sigma)\,, \tag{23}$$

$$\beta = \tfrac{1}{2}\,\left(a^2 + 2a - 1 + (a+1)\sigma\right)\,, \tag{24}$$

$$\gamma = \tfrac{1}{2}\,\left(a^2 - a - 2 + a\sigma\right)\,. \tag{25}$$

Observe that $\alpha - \beta + \gamma = 0$. Given the above coefficients, (22) is now solvable. We put it in the reduced form by taking $t = \alpha/3 + \tau$,

$$\tau^3 - u\tau + v = 0\,, \tag{26}$$

where

$$u = \alpha^2/3 - \beta\,, \tag{27}$$

$$v = -2(\alpha/3)^3 + (\alpha/3)\beta - \gamma/27\,. \tag{28}$$

Before proceeding for a solution, we simplify the problem by introducing

$$r = \left(7 - \sigma + \sigma^2\right)^{1/2} . \tag{29}$$

In terms of r, u and v are simplified to read

$$u = r^2/3\,, \tag{30}$$

$$v = -(1 - 2\sigma)r^2/27 \tag{30a}$$

$$= -(1 - 2\sigma)(u/9)\,. \tag{30b}$$

Instead of following Cardanao and Tartaglia, we deviate by changing the variable τ to θ defined by

$$\tau = (4u/3)^{1/2}\cos(\theta/3)\,. \tag{31}$$

Then, (26) can be expressed as

$$4\cos^3\theta/3 - 3\cos\theta/3 - A = 0\,, \tag{32}$$

where

$$A = -(v/2)/(u/3)^{3/2} = (1-2\sigma)/2r\,. \tag{33}$$

If $|A| < 1$ provided that $u > 0$, $A = \cos\theta$, θ real. If $|\sigma| \le 1$ and σ real, that is if $a \ge \tilde{a}$, $|A| < 1$. Thus at once,

$$\begin{aligned}
\tau_1 &= 2r/3\cos(\theta/3)\,, & (34)\\
\tau_2 &= 2r/3\cos\{\theta/3 + 2\pi/3\}\,, & (35)\\
\tau_3 &= 2r/3\cos\{\theta/3 - 2\pi/3\}\,, & (36)
\end{aligned}$$

where

$$\theta = \cos^{-1}\{(1-2\sigma)/2r\}\,. \tag{37}$$

One can easily verify the sum rule that $\tau_1+\tau_2+\tau_3 = 0$. Since $t = \alpha/3+\tau$, the cubic equation (22) and the sextic equation (3) are both completely solved. The solutions for when $a = 4$ previously obtained (15) are special solutions. More important for chaos is that the general solutions are also in the cyclic form. It means that the theorem of Sharkovskii can come in now to sweep the angle, making it a continuous spectrum.

7. General reflection relations

Recall the remarkable reflection property between the cubic solutions of $a = \tilde{a}$ and $a = 4$ (Section 5). We now show that it is a special case of a more general property. In (32), if $A \to -A$, the structure of the cubic equation does not change if $\cos(\theta/3) \to -\cos(\theta/3)$. For $\sigma \ge 0$ (*i.e.*, positive phase only), let

$$\sigma = \eta - 1/2\,, \tag{38}$$

where $-1/2 \le \eta \le 1/2$. In terms of η, A is antisymmetric

$$A(-\eta) = -A(\eta)\,. \tag{39}$$

Observe that $\eta = -1/2$ corresponds to $\sigma = 0(a = \tilde{a})$ and $\eta = 1/2$ to $\sigma = 1(a = 4)$. This explains the reflection we have used unawares of the general reflection property in the cubic solution. There is a whole range of reflection about $\eta = 0$ or $\sigma = 1/2$, which corresponds to $a = a* = 1 + \sqrt{33/4} = 3.872281323\ldots$, evidently another special point of the control parameter.

8. 3-cycle window

An n-cycle window is defined by $|df^n/dx| \leq 1$. Thus the width of the window begins at some value of a, where $df^n/dx = +1$ and ends at another value of a, where $df^n/dx = -1$. The window is centered on the superstable point defined by $df^n/dx = 0$. We consider the case of $n = 3$. Since $df/dx = a(1-2x)$, by chain rule $df^3/dx = a^3(1-2x)(1-2f)(1-2f^2)$. If evaluated at any fixed point of f^3,

$$m_3 \equiv df^3/dx = a^3(1-2x_1)(1-2x_2)(1-2x_3)\,, \tag{40}$$

where x_1, x_2, x_3 are fixed points of f^3. Using the three symmetric coefficients α', β' and γ',

$$\begin{aligned} \alpha' &= x_1 + x_2 + x_3\,, && (41)\\ \beta' &= x_1x_2 + x_2x_3 + x_3x_1\,, && (42)\\ \gamma' &= x_1x_2x_3\,, && (43) \end{aligned}$$

we obtain

$$m_3 = a^3\left(1 - 2\alpha' + 4\beta' - 8\gamma'\right)\,. \tag{44}$$

If we write $\alpha' = \alpha/a$, $\beta' = \beta/a^2, \gamma' = \gamma/a^3$,

$$m_3 = a^3 - 2a^2\alpha + 4a\alpha\beta - 8\gamma, \tag{45}$$

where α, β, and γ are given by (23), (24), and (25), respectively. Using $\sigma = (a^2 - 2a - 7)^{1/2}$, after some straightforward algebra, we obtain

$$m_3(\sigma \geq 0) = 1 + 7\sigma - \sigma^2 + \sigma^3\,, \tag{45a}$$

$$m_3(\sigma < 0) = 1 - 7s - s^2 - s^3\,, \qquad s \equiv |\sigma|\,. \tag{45b}$$

Clearly, $m_3(\sigma \geq 0) \geq 1$. It is 1 when $\sigma = 0$ or $a = \tilde{a}$. To obtain $m_3 = -1$, we need to use one for $\sigma < 0$.

8.1. $m_3 = -1$

$$s^3 + s^2 + 7s - 2 = 0\,. \tag{46}$$

By solving the cubic equation, $\sigma = -.272243659\ldots$ Since $a = 1+(8+\sigma^2)^{1/2}$, we obtain $a = 3.841499008\ldots$, in agreement with Gordon [11] obtained in a different way, the only known solution.

8.2. $m_3 = 0$ (superstability)

$$s^3 + s^2 + 7a - 1 = 0\,. \tag{47}$$

By solving the cubic equation, $\sigma = -.139680582\ldots$ It yields $a = 3.831874055\ldots$ in agreement with our value obtained in a different way [12].

8.3. Significance of 3-cycle window

The 3-cycle window is bounded by $a = 3.828427125\ldots$ and $3.841499008\ldots$, centered on the superstable point $3.83187480\ldots$ For any value of a which lies in this interval, a trajectory starting from any point in $x = (0, 1)$ save the fixed points of f and f^3 are initial value insensitive. By the standard math definitions of chaos, this 3-cycle window cannot be chaotic. But it is chaotic according to Sharkovskii and Li–Yorke. We must conclude that the math definitions are at fault.

We can also establish a 4-cycle window deep in the chaotic interval of a. It is sufficient to show that there is a superstable 4 cycle since a window is centered on it. In Appendix A, we show that there are two superstable 4 cycle values at $a = 3.4985616999\ldots$ and $3.960270127\ldots$ The first one is in the bifurcation domain. Thus the first one is not in a chaotic window. The second one lies in the heart of the chaotic domain. In spite of the fact that trajectories are initial value insensitive, this window is chaotic as the 3-cycle window by Sharkovskii and Li–Yorke. This is because in this window is 3-cycle since it exists from $\tilde{a}$ to 4, in the interval in which this second 4-cycle window is contained.

9. Concluding remarks

To define chaos in an absolute sense, we have followed the two parts of Sharkovskii's theorem: First, we have solved the 3-cycle problem. This means that chaos exist from $a = \tilde{a}$ to $a = 4$, where the solutions are real. Second, we have deduced all cycles that are implied by 3-cycle. By doing so, a definition of chaos has emerged in the form of an aleph cycle. It is a continuous trajectory starting from a point which belongs to a set of points of measure 1. A non-aleph cycle is a periodic trajectory which starts from a point from the same interval but which belongs to a set of points of measure zero. Both trajectories are initial-value sensitive since these points are all fixed points of some cycles. Thus the initial-value sensitivity is not a good indicator of a chaotic trajectory as has been believed.

The interval from which an aleph cycle starts is like an unbroken string of irrational points. It implies an invariant measure and transitivity of space, which are precisely the requirements of an ergodic trajectory according to

Birkhoff [13, 14]. Thus a chaotic trajectory defined by an aleph cycle is also ergodic. Since an aleph cycle is a linear superposition of all finite cycles, another aleph cycle is merely a rearrangement or re-ordering of the same cycles. Thus, the time average is the same for all aleph cycles under the same condition. This is also what one finds in the classical theory of ergodicity: The time average is over any one of trajectories starting from a point belonging to a set of phase points of measure 1. Our work implies that if a trajectory is ergodic, it is also chaotic. The converse may not necessary be always true.

Appendix A

Superstable 4-cycle

The superstable 4-cycle is defined by

$$f^4(1/2) - 1/2 = 0\,. \tag{A.1}$$

Since one of the fixed points is at $x = 1/2$, the others can be expressed in terms of it leaving only one variable a. Using $y = a/2$,

$$\begin{aligned}\left(f^4 - 1/2\right) / \left(f^2 - 1/2\right) \;=\; & y^{12} - 6y^{11} + 12y^{10} - 5y^9 - 12y^8 + 12y^7 + y^6 \\ & -4y^4 - y^3 + 2y^2 + 1 = 0\,.\end{aligned} \tag{A.2}$$

Now, if we let $y = 1/2(t^{1/2} + 1)$ or $t = (2y-1)^2 = (a-1)^2$, (A.2) may be expressed as

$$t^6 - 18t^5 + 12t^4 - 524t^3 + 1511t^2 - 1858t + 4861 = 0\,. \tag{A.3}$$

There are only two positive real roots for (A.3), which may be obtained by a fixed point analysis. The larger root is attractive, and the smaller one is repulsive. The repulsive one may also be obtained by a reforming method due to the author [15]. The roots are: $t_1 = 6.2428105653\ldots$ $t_2 = 8.7631992261\ldots$ They correspond to $a_1 = 3.4985616999\ldots$ $a_2 = 3.960270127\ldots$ Since $\tilde{a} = 3.828427125\ldots$, $a_1 < \tilde{a}$ and $a_2 > \tilde{a}$, meaning that a_1 is not in the chaotic interval but a_2 is. Observe that the onset value of 4-cycle in the bifurcation domain is $1 + \sqrt{6} = 3.449489743\ldots$ It is clear that a_1 is the superstable value in the bifurcation domain.

A portion of this work was completed at the Korea Institute for Advanced Study, Seoul, South Korea. I thank Profs. Doochul Kim and Hyunggyu Park for their warm hospitality and support during my visits. I thank Ms Mayuri Perera for doing the numerical work of Appendix A. The work is supported by the Franklin College at the University of Georgia.

REFERENCES

[1] B. Hu, *Phys. Rep.* **91**, 233 (1982).

[2] R.L. Devaney, *Introd. to Chaotic Dynamical Systems*, second edition, Addison-Wesley, Redwood City 1989, p. 50.

[3] A.N. Sharkovskii, *Ukr. Math. Z.* **16**, 61 (1964).

[4] T.-Y. Li, J.A. Yorke, *Am. Math. Mon.* **82**, 985 (1975).

[5] F. Dyson, *Phys. Usp.* **53**, 825 (2010).

[6] D. Gulick, *Encounters with Chaos*, McGraw-Hill, New York 1992, p. 68.

[7] M.H. Lee, *Acta Phys. Pol. B* **42**, 1071 (2011).

[8] A. Fulinski, E. Gudowska-Nowak, *Acta Phys. Pol. B* **22**, 457 (1991).

[9] L. Skowronek, P.F. Gora, *Acta Phys. Pol. B* **38**, 1909 (2007); M. Ciesla, L. Longa, *Acta Phys. Pol. B* **38**, 1719 (2007).

[10] M.H. Lee, *Acta Phys. Pol. B* **43**, 1053 (2012).

[11] W.B. Gordon, *Math. Mag.* **69**, 118 (1996).

[12] M.H. Lee, *J. Math. Phys.* **50**, 122702 (2009); also see N.S. Ananikian, L.N. Ananikyan, L.A. Chakhmakhchyan, *JETP Lett.* **94**, 39 (2011).

[13] G.D. Birkhoff, *Proc. Nat. Acad. Sci. (USA)* **17**, 656 (1931).

[14] M.H. Lee, *Phys. Rev. Lett.* **98**, 110403 (2007); *Int. J. Mod. Phys.* **B22**, 4572 (2008).

[15] M.H. Lee, to be published, 2013.