SUPERSYMMETRIC QUANTUM MECHANICS AND THE ATIYAH-SINGER INDEX THEOREM*, **

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A new proof of the Atiyah-Singer index theorem is given using simple techniques developed in the context of supersymmetric field theories.

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CONTENT

1. Introduction

These lectures present a new proof of the Atiyah–Singer (A–S) Index theorem using simple and elegant ideas developed in the context of supersymmetric (SUSY) field theory. The A–S theorem [1] demonstrates the equality of analytical indices (related to the solutions of partial differential equations on compact manifold) to purely topological invariants. The strategy that will be followed to prove this theorem is extremely simple. First the analytical index of an elliptic operator (in the most general case it will be the Dirac operator on a compact manifold without boundary coupled to an external Yang–Mills field) is identified with the index $\mathrm{Tr}(-)^F$ introduced recently by Witten [2] to characterize the ground state of SUSY field theories (Section 2). Second, a path integral representation of this

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index is built — supersymmetric by construction (Section 3). The explicit evaluation of this functional expression will yield the corresponding topological index (Section 4). Self-contained, this presentation does not require knowledge of the usually needed complicated mathematical apparatus.

2. *Ind* (D) *as* $Tr(-)^F$

This section reviews some general facts about Witten's index [2] of supersymmetric theory and draws the parallel between this index and the analytical index of an elliptic operator. The similarities between the two cases should clarify the strategy followed in the rest of these lectures.

Consider a theory in a finite volume with only one SUSY charge since it contains all the structure needed in what follows. Let Q be the generator of SUSY and H the Hamiltonian. Then

$$Q^2 = H. \tag{1}$$

The content of (1) is twofold: 1) SUSY states, i.e. those annihilated by Q, have zero energy — the lowest possible level of the spectrum. The converse being equally obvious. 2) Every non-zero eigenvalue λ is associated with a pair of eigenstates related to each other by the action of Q since $[H, Q] = 0$,

$$Q|b, \lambda\rangle = \sqrt{\lambda}\,|F, \lambda\rangle, \quad Q|F, \lambda\rangle = \sqrt{\lambda}\,|b, \lambda\rangle \tag{2}$$

and

$$Q^2|b, \lambda\rangle = \lambda|b, \lambda\rangle, \quad Q^2|F, \lambda\rangle = \lambda|F, \lambda\rangle, \tag{3}$$

Q has fermionic quantum number and transforms a fermionic state into a bosonic one and vice versa. From this it follows that under a change of parameters (coupling constant, volume, ...) the number of bosonic zero energy states n_B minus the number of zero energy fermionic states n_F is invariant since by 2) they will reach and leave the zero level by pair. This is the very observation which led Witten to introduce the relation

$$\mathrm{Tr}\,(-)^F e^{-\beta H} = \sum_{\left\{\substack{\text{bosonic}\\ \text{states } b}\right\}} e^{-\beta\lambda_b} - \sum_{\left\{\substack{\text{fermionic}\\ \text{states } f}\right\}} e^{-\beta\lambda_F} = n_B(\lambda_b = 0) - n_F(\lambda_f = 0), \tag{4}$$

where the second equality derives from 2). Here F is the fermion number and $\{(-)^F, Q\} = 0$.

The right-hand side of (4) has precisely the form of what is called an index in the mathematical literature [1] as will be seen now in the case of the Dirac operator. Let

$$\mathbf{D} = \begin{pmatrix} & \mathbf{D}_L \\ \mathbf{D}_R & \end{pmatrix} = \begin{pmatrix} & \mathbf{D}_L \\ -\mathbf{D}_L^\dagger & \end{pmatrix} \tag{5}$$

be the Dirac operator on some compact manifold with

$$\mathbf{D}_L : S_L \to S_R, \quad \mathbf{D}_R : S_R \to S_L,$$

where $S_{L,R}$ are left- and right-handed spinors respectively. By definition

$$\text{Ind}(\mathbf{D}) = n_0^{L} - n_0^{R}, \tag{6}$$

where $n_0^{R,L}$ is the number of solutions with zero eigenvalue of $\mathbf{D}_{R,L}$
or

$$\text{Ind}(\mathbf{D}) = \dim\ker(\mathbf{D}_L) - \dim\ker(\mathbf{D}_R) \tag{7}$$

with

$$\ker \mathbf{D}_{L,R} = \{\psi, \mathbf{D}_{L,R}\psi = 0\}. \tag{8}$$

The similarity between (4) and (6) rests on the fact that as for $\text{Tr}(-)^F$ one can prove that Ind (D) is an invariant. The proof goes as follows.

Define the two self-adjoint operators

$$\Delta_L = \mathbf{D}_L^+\mathbf{D}_L \quad \text{and} \quad \Delta_R = \mathbf{D}_R^+\mathbf{D}_R \tag{9}$$

1) $$\ker \Delta_L = \ker \mathbf{D}_L. \tag{10}$$

Indeed

$$\mathbf{D}_L\psi = 0 \Rightarrow \mathbf{D}_L^+\mathbf{D}_L\psi = 0$$

and $\Delta_L\psi = 0$ implies

$$(\psi, \Delta_L\psi) = (\psi, \mathbf{D}_L^+\mathbf{D}_L\psi) = (\mathbf{D}_L\psi, \mathbf{D}_L\psi) = 0 \Rightarrow \mathbf{D}_L\psi = 0 \quad [\text{cf.}(1)].$$

The same holds for $\ker \Delta_R = \ker \mathbf{D}_R$.

2) The eigenstates of Δ_L and Δ_R are paired. Indeed

$$\Delta_L\psi = \lambda\psi \Rightarrow \Delta_R\mathbf{D}_L\psi = \lambda\mathbf{D}_L\psi \tag{11}$$

since

$$\Delta_R\mathbf{D}_L\psi = (\mathbf{D}_R^+\mathbf{D}_R)\mathbf{D}_L\psi = (\mathbf{D}_L\mathbf{D}_L^+)\mathbf{D}_L\psi = \mathbf{D}_L(\Delta_L)\psi = \mathbf{D}_L\lambda\psi$$

and

$$(\psi, \Delta_L\psi) = \lambda(\psi, \psi) = (\mathbf{D}_L\psi, \mathbf{D}_L\psi)$$

so

$$\mathbf{D}_L\psi \neq 0 \quad [\text{cf. }(2)].$$

From the definition

$$H = \mathbf{D}^+\mathbf{D} = \begin{pmatrix} & \mathbf{D}_R^+ \\ \mathbf{D}_L^+ & \end{pmatrix}\begin{pmatrix} & \mathbf{D}_L \\ \mathbf{D}_R & \end{pmatrix} = \begin{pmatrix} \Delta_R & \\ & \Delta_L \end{pmatrix} \tag{12}$$

and $\gamma_5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ results

$$\begin{aligned} \text{Tr}\,\gamma_5 e^{-\beta H} &= \dim\ker\Delta_L - \dim\ker\Delta_R \\ &= \dim\ker\mathbf{D}_L - \dim\ker\mathbf{D}_R \\ &= \text{Ind}(\mathbf{D}) \end{aligned} \tag{13}$$

by definition where the first equality follows from (11) and the second from (10). Notice that $\{\gamma_5, \mathbf{D}\} = 0$. Then γ_5 plays in (13) the same role as $(-)^F$ in (4) and $\mathbf{D}$ the role of Q.

3. *Supersymmetric path integral and* $Tr(-)^F e^{-\beta H}$

A path integral supersymmetric by construction will be associated with Tr $\gamma_5 e^{-\beta H}$. This construction will be done first for a free Dirac operator in four dimensions using a discrete proper time formulation. What is lost in elegance, compared to the canonical formalism [4], will hopefully be gained in understanding the subtle way Grassman variables do their job. The emphasis will be on fermionic degrees of freedom since the treatment of the bosonic ones is completely standard. More details about this construction can be found in Faddeev [3].

Define creation and annihilation operators

$$\hat{\bar{\xi}}_1 = \frac{\gamma_1 - i\gamma_2}{2}, \qquad \hat{\bar{\xi}}_2 = \frac{\gamma_3 - i\gamma_4}{2}$$

$$\hat{\xi}_1 = \frac{\gamma_1 + i\gamma_2}{2}, \qquad \hat{\xi}_2 = \frac{\gamma_3 - i\gamma_4}{2} \tag{14}$$

(where γ_μ are the Dirac Euclidean matrices) which satisfy the anticommutation relations:

$$\{\hat{\bar{\xi}}_\alpha, \hat{\xi}_\beta\} = \delta_{\alpha\beta},$$

$$\{\hat{\bar{\xi}}_\alpha, \hat{\xi}_\beta\} = \{\hat{\bar{\xi}}_\alpha, \hat{\bar{\xi}}_\beta\} = 0, \qquad \alpha, \beta = 1, 2. \tag{15}$$

Note that in this representation

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = -(1 - 2\hat{\bar{\xi}}_1\hat{\xi}_1 - 2\hat{\bar{\xi}}_2\hat{\xi}_2 + 4\hat{\bar{\xi}}_1\hat{\bar{\xi}}_2\hat{\xi}_1\hat{\xi}_2) \tag{16}$$

plays the role of $(-)^F$ since it anticommutes with any odd number of fermionic operators. The vectors in Fock space are, as usual, represented by polynomials in the creation operators.

To these vectors, we associate the following representative:

$$V(\bar{\xi}) = V_1 + V_2\bar{\xi}_1 + V_3\bar{\xi}_2 + V_4\bar{\xi}_1\bar{\xi}_2, \tag{17}$$

where $\bar{\xi}_1$ and $\bar{\xi}_2$ are anticommuting generators of a Grassmann algebra. Defining as usual

$$\int d\bar{\xi}\xi = 1$$

and

$$\int d\bar{\xi} = 0$$

we have an inner product

$$(V, W) = \sum_{n=1}^{4} V_n^* W_n = \int \prod_{\alpha=1}^{2} d\bar{\xi}_\alpha d\xi_\alpha e^{-\sum_{\alpha=1}^{2} \bar{\xi}_\alpha \xi_\alpha} V^*(\xi) W(\bar{\xi}) \tag{18}$$

with

$$V^*(\xi) = V_1^* + V_2^*\xi_1 + V_3^*\xi_2 + V_4^*\xi_2\xi_1, \qquad (\xi_\alpha = \text{Grassmann}). \tag{19}$$

The elements $(1, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_1\bar{\xi}_2)$ form an orthonormal basis in the state space and to each matrix operator $\hat{M}$ is associated a kernel:

$$K(\bar{\xi}, \xi) = N_{00}+N_{01}\xi_1+N_{02}\xi_2+N_{03}\xi_1\xi_2+\bar{\xi}_1(N_{10}+N_{11}\xi_1+N_{12}\xi_2+N_{13}\xi_1\xi_2)$$
$$+\bar{\xi}_2(N_{20}+N_{21}\xi_1+N_{22}\xi_2+N_{23}\xi_2\xi_1)+\bar{\xi}_1\bar{\xi}_2(N_{30}+N_{31}\xi_1+N_{32}\xi_2+N_{33}\xi_1\xi_2), \quad (20)$$

where the $N_{\alpha\beta}$ are the matrix elements of $\hat{M}$ in this basis. To the action of an operator $\hat{M}$ on the space of spinors

$$V_\alpha = M_{\alpha\beta}W_\beta \quad (21)$$

corresponds in this formulation

$$V(\bar{\xi}) = \int d\bar{\eta}_\alpha d\eta_\alpha e^{-\sum_\alpha \bar{\eta}_\alpha \eta_\alpha} K(\bar{\xi}, \eta)W(\bar{\eta}). \quad (22)$$

Alternatively to the normal ordered form of an operator

$$\hat{M} = M_{00}+M_{01}\hat{\xi}_1+M_{10}\hat{\bar{\xi}}_1+M_{11}\hat{\bar{\xi}}_1\xi_1 \quad (23)$$

is associated an element of the Grassmann algebra called the normal symbol and defined as

$$N(\bar{\xi}, \xi) = M_{00}+M_{01}\xi_1+M_{10}\bar{\xi}_1+M_{11}\bar{\xi}_1\xi_1. \quad (24)$$

The relation between the kernel and the normal symbol being

$$K(\bar{\xi}, \xi) = e^{\sum_{\alpha=1}^{2} \bar{\xi}_\alpha \xi_\alpha} N(\bar{\xi}, \xi). \quad (25)$$

We will suppress summation symbols and indices when there is no ambiguity. Now consider the simplest case where $H = (p_\mu \gamma^\mu)^2$. We have:

$$\mathrm{Tr}\, e^{-\beta H} = \lim_{N\to\infty} (1-\varepsilon H)^N, \quad N = \frac{\beta}{\varepsilon}. \quad (26)$$

Every matrix multiplication in (26) gives a factor

$$\int d\bar{\xi}_i d\xi_i e^{-\bar{\xi}_i \xi_i}$$

[see (22)] while the ith factor in the product gives

$$e^{+\bar{\xi}_i \xi_{i-1}}(1-\varepsilon p^2).$$

The factors of $(1-\varepsilon p^2)$ will be treated as usual to produce the path integral over bosonic degrees of freedom. Then

$$\mathrm{Tr}\, e^{-\beta H} = \int d\bar{\xi}_N d\xi_0 e^{-\bar{\xi}_N \xi_0} e^{\bar{\xi}_N \xi_{N-1}} \prod_{i=1}^{N-1} d\bar{\xi}_i d\xi_i e^{-\sum_{i=1}^{N-1} \bar{\xi}_i(\xi_i - \xi_{i-1})} \int \mathrm{BOSONS} \quad (27)$$

where the first two exponentials come from the trace and the last factor in $(1-\varepsilon H)^N$ respectively. Anti-periodic boundary conditions are then required for the fermion field

$$\xi_0 = -\xi_N.$$

For the case of interest, i.e. $\mathrm{Tr}\,\gamma_5 e^{-\beta H}$ it is sufficient to change the boundary conditions of the fermions to periodic ones since as we have seen γ_5 corresponds to $(-)^F$. This can be seen explicitly by inserting the kernel $e^{-\bar{\xi}\xi}$ corresponding to γ_5. Notice that SUSY requires invariance under translation in time which impose PBCs. Taking the limit $\varepsilon \to 0$:

$$\mathrm{Tr}\,\gamma_5 e^{-\beta H} = \int \mathscr{D}\bar{\xi}_\alpha(t)\mathscr{D}\xi_\alpha(t)\mathscr{D}x^\mu(t)\exp\left[-\tfrac{1}{2}\int_0^\beta (\dot{x}^\mu)^2 + \bar{\xi}_\alpha\dot{\xi}_\alpha|_{\mathrm{PBC's}}\right]$$

$$\mu = 1, 4, \qquad \alpha = 1, 2 \tag{28}$$

or in a more convenient form

$$\mathrm{Tr}\,\gamma_5 e^{-\beta H} = \mathrm{Tr}\,(-)^F e^{-\beta H} = \int \mathscr{D}x^\mu(t)\mathscr{D}\psi^\mu(t)\exp\left[-\tfrac{1}{2}\int_0^\beta (\dot{x}^\mu)^2 + \psi^\mu\dot{\psi}^\mu|_{\mathrm{PBC's}}\right] \tag{29}$$

with

$$\bar{\xi}_1 = \frac{\psi^1 + i\psi^2}{\sqrt{2}}, \qquad \bar{\xi}_2 = \frac{\psi^3 + i\psi^4}{\sqrt{2}}$$

$$\xi_1 = \frac{\psi^1 - i\psi^2}{\sqrt{2}}, \qquad \xi_2 = \frac{\psi^3 - i\psi^4}{\sqrt{2}}. \tag{30}$$

The right-hand side of (29) is supersymmetric by construction since it was built with $H = Q^2$ and SUSY boundary conditions. The generalization to more complicated cases is relatively straightforward. For a Dirac operator on a d-dimensional Riemanian manifold, the action reads

$$S = \int_0^\beta dt\left[\tfrac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \tfrac{1}{2}\, g_{\mu\nu}(x)\psi^\mu(t)\,(D_t^g\psi)^\nu\right], \tag{31}$$

where

$$(D_t^g)^\mu{}_\nu = \partial_t\delta^\mu{}_\nu + \dot{x}^\varrho\Gamma^\mu_{\varrho\nu}, \qquad \mu, \nu = 1, d \tag{32}$$

$g_{\mu\nu}(x)$ being the metric tensor and $\Gamma^\mu_{\varrho\nu}$ the Christoffel connection. For fermion fields coupled to an external gauge field A^{ab}_μ, the Dirac operator is

$$\gamma^\mu(\partial_\mu + A_\mu) \equiv \hat{\mathbf{A}}$$

with

$$\hat{\mathbf{A}}^2 = (\partial_\mu + A_\mu)^2 + \sigma_{\mu\nu}F_{\mu\nu} \tag{33}$$

acting on spin ⊗ isospin space. Treating the isospin degrees of freedom in a way analogous to the spin degrees of freedom gives in the most general case:

$$S = \int_0^\beta dt\left[\tfrac{1}{2}\, g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{1}{2}\, g_{\mu\nu}\psi^\mu(D_t^g\psi)^\nu + \bar{\eta}^a D_t^A\eta^a - \tfrac{1}{2}\,\bar{\eta}^a F^{ab}_{\mu\nu}\psi^\mu\psi^\nu\eta^b + i\,\frac{\alpha}{\beta}\,\bar{\eta}^a\eta^a\right], \tag{34}$$

with

$$D_t^A = \partial_t + \dot{x}^\alpha A_\alpha(x) \tag{35}$$

and

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + [A_\mu, A_\nu]^{ab} \tag{36}$$

whose coupling to the ψ_μ is readily understood from (33). The "mass" term for the η field has been added for later convenience ($\int dt\bar{\eta}\eta$ is the isospin number). The general form of the analytical index expressed as a functional integral is

$$\mathrm{Tr}\,(-)^F e^{-\beta H} = \mathrm{Tr}\,\gamma_5(-)^N e^{-\beta H - i\alpha N} = \int_{\mathrm{PBCs}} \mathscr{D}x^\mu \mathscr{D}\psi^\mu \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a e^{-S}, \tag{37}$$

where H is given by the square of the Dirac operator ensuring the supersymmetry algebra. The explicit transformations of the fields are

$$\begin{aligned} \delta x^\mu &= \varepsilon\psi^\mu, \\ \delta\psi^\mu &= -\varepsilon\dot{x}^\mu, \\ \delta\eta^a &= -\varepsilon\psi^\mu A^{ab}_\mu \eta^b, \\ \delta\bar{\eta}^a &= -\varepsilon\bar{\eta}^b A^{ba}_\mu \psi^\mu, \end{aligned} \tag{38}$$

which yields using the Bianchi identities for $F_{\mu\nu}$ and $R^\mu_{\alpha\beta\varrho}$ (Rieman tensor):

$$\delta S = \tfrac{1}{2}\int dt\partial_t(\psi^\mu \dot{x}_\mu) = 0. \tag{39}$$

One sees clearly how the periodic boundary conditions ensure SUSY in (39). It is possible to give a superfield formulation of what precedes [4].

4. *Proof of the Atiyah-Singer index theorem*

In this Section we will compute $\mathrm{Tr}(-)^F e^{-\beta H}$ explicitly using its supersymmetric path integral representation and in this way obtain the general expression for the topological indices.

The outline of the proof is as follows:

1) The path integral being invariant under gauge and coordinate transformations it is possible to work without loss of generality in a special gauge and coordinate system. A simple choice is

$$A_\mu(x) = -\tfrac{1}{2}\,x^\nu F_{\mu\nu}, \tag{40a}$$

$$\psi_\mu \Gamma^\mu_{\nu\varrho}\psi^\nu = \tfrac{1}{2}\,R_{\alpha\varrho\mu\nu}\psi^\mu\psi^\nu x^\alpha. \tag{40b}$$

2) The result being independent of β, one can get rid of all the terms with an explicit β dependence.

If we use the standard heat kernel result for the free Lagrangian we know that

$$\int \mathscr{D}x^\mu(t) e^{-\frac{1}{2\beta}\int(\dot{x}^\mu)^2} \sim \int dx^1 \ldots dx^d \beta^{-\frac{1}{2}d}.$$

Then, scaling the ψ field zero mode $\psi \to \beta^{-1/2}\psi$ and $t \to \beta t$, together with $x \to \sqrt{\beta}x$ one obtains:

$$I(\alpha) = \mathrm{Tr}\,(-)^F e^{-\beta H - i\alpha N} = \mathrm{Tr}\,(-)^N \gamma_5 e^{-\beta H - i\alpha N}$$

$$= \int \mathscr{D}\psi^\mu \mathscr{D}x^\mu \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a \exp\Big[-\int_0^\beta dt\big[\tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{1}{2} g_{\mu\nu}\psi^\mu (D_t^g\psi)^\nu$$

$$+\bar{\eta}^a (D_t^A \eta)^a - \tfrac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}\psi^\mu\psi^\nu\eta^b + i\frac{\alpha}{\beta}\bar{\eta}^a\eta^a\big]\Big]$$

$$= \int_{\text{zero modes}} \prod_{\mu=1}^{d} dx^\mu d\psi^\mu \int_0 \mathscr{D}x^\mu \exp\Big[-\tfrac{1}{2}\int_0^1 dt(\dot{x}^\mu\dot{x}_\mu + x_\mu \mathscr{R}^\mu{}_\nu \dot{x}^\nu)\Big]$$

$$\times \int_0 \mathscr{D}\psi^\mu \exp\Big[-\tfrac{1}{2}\int_0^1 dt[\psi^\mu\dot{\psi}_\mu]\Big]$$

$$\times \int \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a \exp\Big[-\int_0^1 dt[\bar{\eta}^a\dot{\eta}^a + \bar{\eta}^a(-\tfrac{1}{2}\mathscr{F} + i\alpha)^{ab}\eta^b\Big], \tag{41}$$

where $\int_0$ = integral without the zero modes and

$$\mathscr{R}^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\alpha\beta}\psi^\alpha\psi^\beta, \quad \text{and} \quad \mathscr{F} = F_{\mu\nu}\psi^\mu\psi^\nu,$$

$$(\psi^\mu \text{ being the zero modes}). \tag{42}$$

The detailed computation of the path integrals obtained in (41) is given in Appendix A. Using (A.13) with $T = -\mathscr{F}/2 + i\alpha$ and (A.16), we readily derive the following general formula

$$I(\alpha) = \int dx^1 \ldots dx^d \int d\psi^1 \ldots d\psi^d \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} (-i)^{\frac{d}{2}} \det{}^{-1/2}\left(\frac{\sinh \mathscr{R}^\mu{}_\nu/2}{\mathscr{R}^\mu{}_\nu/2}\right) \det(1 - e^{\mathscr{F}/2 - i\alpha}) \tag{43}$$

where $\mathscr{R}_{\mu\nu}$ and $\mathscr{F}$ are given by (42). Notice that for any tensor $A_{\mu_1,\ldots,\mu_d}$

$$\int dx^1 \ldots dx^d \int d\psi^1 \ldots d\psi^d A_{\mu_1\ldots\mu_d}\psi^{\mu_1}\ldots\psi^{\mu_d} = (-)^{\frac{d}{2}} \int_{\text{space}} A_{\mu_1\ldots\mu_d} dx^{\mu_1}\ldots dx^{\mu_d}. \tag{44}$$

Equations (44) permits the rewriting of Eq. (43) in terms of differential forms

$$I(\alpha) = \sum_{k=0}^{n} (-)^k e^{-i\alpha k} I_k = \left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int_{\text{space}} \det{}^{-1/2}\left(\frac{\sinh \mathscr{R}^\lambda{}_\nu/2}{\mathscr{R}^\lambda{}_\nu/2}\right) \det(1 - e^{\mathscr{F}/2 - i\alpha}), \tag{45}$$

where now

$$\mathscr{R}^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\alpha\beta} dx^\alpha dx^\beta \quad \text{and} \quad \mathscr{F} = F_{\mu\nu} dx^\mu dx^\nu. \tag{46}$$

It follows immediately from Eq. (45) (which is our main result) that

$$I_1 = \int_{\text{space}} \left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \det{}^{-1/2}\left(\frac{\sinh \mathscr{R}^\mu{}_\nu/2}{\mathscr{R}^\mu{}_\nu/2}\right) \mathrm{Tr}\,(e^{\mathscr{F}/2}). \tag{47}$$

Equation (47) gives trivially

— the winding number [5] (2 dimension, Abelian gauge group)

$$I_1 = \int_{\text{space}} \left(\frac{i}{2\pi}\right) \tfrac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \frac{i}{4\pi} \int d^2x \varepsilon^{\mu\nu} F_{\mu\nu} \tag{48}$$

— the Pontryagin number [5] (4 dimension, non-Abelian gauge group)

$$I_1 = \int \left(\frac{i}{2\pi}\right)^2 \frac{1}{2!} \operatorname{Tr}\left(\tfrac{1}{2} F_{\mu\nu} \tfrac{1}{2} F_{\alpha\beta}\right) dx^\mu dx^\nu dx^\alpha dx^\beta = -\frac{1}{32\pi^2} \int d^4x \varepsilon^{\mu\nu\alpha\beta} \operatorname{Tr}(F_{\mu\nu} F_{\alpha\beta}). \tag{49}$$

Similarly, one can derive expressions for the Euler number and the Hirzebruch signature. We will devote the next section to the analysis of these two cases.

5. *Euler number and Hirzebruch signature*

We will first review the classical definition of the index of an elliptic complex in the special cases of the Euler number and Hirzebruch signature [6]. We will then sketch the steps which link them to the index of the Dirac operator. This amounts to a simple redefinition of the isospin space introduced in Section 2. Let Λ^p ($p = 0, 1, \ldots d$) be the space of p forms on the tangent space of a given d-dimensional manifold. One defines as usual the operator d_p (here the index p is kept for clarity)

$$d_p : \Lambda^p \to \Lambda^{p+1} \; (\text{ex} : \hat{\omega}^1 \in \Lambda^1 : d_1 \hat{\omega}^1 = \tfrac{1}{2} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu \wedge dx^\nu \tag{50}$$

and their adjoints

$$d_p^* : \Lambda^{p+1} \to \Lambda^p (\text{ex} : \hat{\omega}^1 \in \Lambda^1 ; d_0^* \hat{\omega}^1 = -\partial_\mu \omega^\mu). \tag{51}$$

This defines the de Rham elliptic complex

$$0 \to \Lambda^0 \xrightarrow{d_0} \Lambda^1 \xrightarrow{d_1} \ldots \to \Lambda^d \to 0. \tag{52}$$

The Laplacian operators are

$$\Delta p = d_p^* d_p + d_{p-1} d_{p-1}^* : \Lambda_p \to \Lambda_p. \tag{53}$$

By definition the Euler index is

$$\chi = \sum_{p=0}^{d-1} (-)^p \dim \ker \Delta_p \equiv \sum_{p=0}^{d-1} (-)^p \dim H_p, \tag{54}$$

where $\ker \Delta_p$ is the space of harmonic p forms ($d\hat{\omega} = d^* \hat{\omega} = 0$) and is identified with the cohomology class H_p (the space of closed p-forms $d\omega_p = 0$ modulo the exact ones $\hat{\omega}_p = d\hat{\omega}_{p-1}$.

It is now easy to recast the Euler index for a complex of elliptic operators in a form similar to the one defined in Section 2. One first splits the space of forms in two: the even and the odd forms. Then define the two step complex

$$D = d + d^* : \Lambda^{2p} \to \Lambda^{2p+1} \tag{55}$$

with

$$D(\hat{\omega}_0, \hat{\omega}_2, \ldots) = (d_0\hat{\omega}_0 + d_1^*\hat{\omega}_2,\, d_2\hat{\omega}_2 + d_3^*\hat{\omega}_4, \ldots)$$

$$(\hat{\omega} \in \Lambda^{2p}) \quad \text{since} \quad d^2 = 0,\, d^{*2} = 0$$

and

$$D^*: \Lambda^{2p+1}: \Lambda^{2p+1} \to \Lambda^{2p}$$

with

$$D^*(\hat{\omega}_1, \hat{\omega}_3, \ldots) = (d_0^*\hat{\omega}_1,\, d_1\hat{\omega}_1 + d_2^*\hat{\omega}_3, \ldots). \tag{56}$$

It follows that

$$D^*D = \oplus_i \Delta_{2i}: \Lambda^{2p} \to \Lambda^{2p}$$

$$DD^* = \oplus_i \Delta_{2i-1} \tag{57}$$

where the Δ_i are the Laplacians defined above. Then

$$\ker D = \oplus_i \ker \Delta_{2i}$$

$$\ker D^* = \oplus_i \ker \Delta_{2i-1}, \tag{58}$$

since if $\phi \in \ker \Delta_i$ then $d_i\phi = 0$ and $d_{i-1}^*\phi = 0$ and $\ker D \equiv \ker D^*D$. Then

$$\text{Ind}\, D = \dim\ker D - \dim\ker D^*$$

$$= \sum_i (-)^i \dim\ker \Delta_i = \chi. \tag{59}$$

However, to preserve the supersymmetric structure built in the previous sections, it is necessary to work with the Dirac operator **D** and not D. Let us see how this operator acts on the space of forms. For that, we have to choose the isospin space of Sections 1 and 2 in an appropriate way. Take for it (*isospin*) = (*spin*)* so acts on $S \otimes S^*$.

Since $S \otimes S^* \sim \gamma$ matrices, then $\omega \in S \otimes S^*$ is of the form:

$$\omega = \omega^{sc} + \omega_\mu\gamma^\mu + \frac{1}{2!}\omega_{\mu_1\mu_2}\gamma^{\mu_1}\gamma^{\mu_2} + \ldots + \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\gamma^{\mu_1}\ldots\gamma^{\mu_p}, \tag{60}$$

with $\omega_{\mu_1,\ldots,\mu_k}$ skew-symmetric. Tr ω corresponds to a form $\hat{\omega} \in \Lambda^*$ through the obvious substitutions $\gamma^\mu \to dx^\mu$

$$\hat{\omega} \to \omega^{sc} + \omega_\mu dx^\mu + \ldots + \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}dx^{\mu_1}\ldots{,}dx^{\mu_p}. \tag{61}$$

The identification of the Euler index will be done in two steps:

1) $\mathbf{D}\omega = (d - d^*)\omega$ and $\ker \mathbf{D}^+\mathbf{D}$ = *harmonic forms*

2) separate

$$S \otimes S^* \text{ in } C_+ = S \otimes S_+^* = \{\omega : \omega\gamma_5 = \omega\}, \tag{62}$$

$$C_- = S \otimes S_-^* = \{\omega : \omega\gamma_5 = -\omega\} \tag{63}$$

and show that on

$$C_+ : \gamma_5 \omega = (-)^k \omega,$$
$$C_- : \gamma_5 \omega = (-)^{k+1} \omega \tag{64}$$

if ω is of degree k.
3) It follows that

$$I_+ = \text{Ind}\,(\mathbf{D} \text{ on } C_+) = \sum_k (-)^k \dim H_k^- \tag{65}$$

and

$$I_- = \text{Ind}\,(\mathbf{D} \text{ on } C_-) = -\sum_k (-)^k \dim H_k^+, \tag{66}$$

where H^+ and H^- are the spaces of *harmonic forms* $\in C_+$ and C_- respectively. Then

$$I_+ - I_- = \sum_k (-)^k \dim H_k = \chi. \tag{67}$$

For the clarity of the exposition, the details of steps 1) to 3) are given in Appendix B. The proof that the index (67) is equal to the corresponding topological index is now merely a specialization of the general formula (47) to the case where $F_{\mu\nu} = -1/4\, R^d_{\beta\mu\nu}\gamma_\alpha\gamma_\beta$ (isospin $= s^*$). In order to compute $I_\pm = \text{Ind}\,(\mathbf{D} \text{ on } C_\pm)$, we need $\text{Tr}\,[e^{\mathscr{F}/2}(1+\gamma_5)/2]$. Using (A.1) with $\omega^\mu_\nu = 1/2\, R^\mu_\nu$:

$$\text{Tr}\,(e^{\frac{1}{2}\mathscr{F}}\gamma_5) = \frac{2^{-\frac{1}{2}d}}{i^{\frac{1}{2}d}\left(\dfrac{d}{2}\right)!} \det{}^{1/2}\left(\frac{\sinh \mathscr{R}/2}{\mathscr{R}/2}\right) \varepsilon^{\mu_1 \ldots \mu_d} (\tfrac{1}{2})^{\frac{d}{2}} \mathscr{R}_{\mu_1\mu_2} \ldots \mathscr{R}_{\mu_{d-1}\mu_d}. \tag{68}$$

Combining (47) and (68) gives

$$I_+ - I_- = \chi = \int_{\text{space}} \left(\frac{i}{2\pi}\right)^{\frac{1}{2}d} \left(\frac{1}{2i}\right)^{\frac{1}{2}d} (\tfrac{1}{2})^{\frac{1}{2}d} \left(\frac{1}{(\frac{1}{2}d)!}\right) \varepsilon^{\mu_1\ldots\mu_d} \mathscr{R}_{\mu_1\mu_2} \ldots \mathscr{R}_{\mu_{d-1}\mu_d}$$
$$= \int \left(\frac{1}{8\pi}\right)^{\frac{1}{2}d} \frac{1}{(\frac{1}{2}d)!} \varepsilon^{\mu_1\ldots\mu_d} \mathscr{R}_{\mu_1\mu_2} \ldots \mathscr{R}_{\mu_{d-1}\mu_d}. \tag{69}$$

For example, for a 2-sphere one has

$$R^\mu{}_{\nu\alpha\beta} = \tfrac{1}{2} R(\delta^\mu{}_\alpha g_{\nu\beta} - \delta^\mu{}_\beta g_{\nu\alpha}), \quad \varepsilon^{\mu\nu}\mathscr{R}_{\mu\nu} = 2R d^2x$$

(R is the scalar curvature) and finally

$$\chi = \int_{\text{space}} \frac{1}{4\pi} R d^2x = \int \frac{K}{2\pi}\, d^2x, \quad \left(K = \frac{R}{2} = \text{Gauss curvature}\right). \tag{70}$$

The Hirzebruch signature is given by

$$\text{sign} = I_+ + I_- = \text{Ind}\,(\mathbf{D} \text{ on } C_+) + \text{Ind}\,(\mathbf{D} \text{ on } C_-) \tag{71}$$

which, using (47) and (A.12) is

$$\text{sign} = I_+ + I_- = \int_{\text{space}} (2)^{\frac{1}{2}d} \left(\frac{i}{2\pi}\right)^{\frac{1}{2}d} \det{}^{-1/2}\left(\frac{\sinh \mathscr{R}^{\mu}{}_{\nu}/2}{\mathscr{R}^{\mu}{}_{\nu}/2}\right) \det{}^{1/2}\left(\cosh \frac{\mathscr{R}^{\mu}{}_{\nu}}{4}\right)$$

$$= \int \left(\frac{i}{\pi}\right)^{v} \det{}^{-1/2}\left(\frac{\tanh \mathscr{R}^{\mu}{}_{\nu}/2}{\mathscr{R}^{\mu}{}_{\nu}/2}\right), \quad (v = \tfrac{1}{2}d). \tag{72}$$

In the particular $d = 4$ case

$$\text{sign} = \int \frac{1}{192\pi^2} \operatorname{Tr}(\mathscr{R}^2). \tag{73}$$

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APPENDIX A

We give here the computation of the path integrals used to derive the results of Sections 4 and 5. They fall in four categories.

1) $$\operatorname{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}}) = \int_{\text{PBCs}} \mathscr{D}\psi^{\mu} \exp\left[-\tfrac{1}{2}\int_0^1 dt(\psi^{\mu}\dot{\psi}^{\mu} + \omega_{\mu\nu}\psi^{\mu}\psi^{\nu})\right]. \tag{A.1}$$

2) $$\operatorname{Tr}(e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}}) = \int_{\text{APBCs}} \mathscr{D}\psi^{\mu} \exp\left[-\tfrac{1}{2}\int_0^1 dt(\psi^{\mu}\dot{\psi}^{\mu} + \omega_{\mu\nu}\psi^{\mu}\psi^{\nu})\right]. \tag{A.2}$$

3) $$\operatorname{Tr}((-)^N e^{-\bar{\eta}^a T^{ab}\eta^b}) = \int_{\text{PBCs}} \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a \exp\left[-\int_0^1 dt(\bar{\eta}^a\dot{\eta}^a - \tfrac{1}{2}\bar{\eta}^a T^{ab}\eta^b)\right]. \tag{A.3}$$

4) $$\int_{\text{PBCs}} \mathscr{D}x^{\mu} \exp\left[-\tfrac{1}{2}\int_0^1 dt((\dot{x}^{\mu})^2 + x_{\mu}\mathscr{R}^{\mu}{}_{\nu}\dot{x}^{\nu})\right]. \tag{A.4}$$

The detailed computation of (A.1) is given below. (The other three cases being very similar.)

$$2^{-\frac{1}{2}d} \operatorname{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}}) = Z' \det{}^{1/2}_{0,\text{PBC}}(\partial_t + \omega^{\mu}{}_{\nu}) \int_{\substack{\text{zero}\\ \text{modes}}} d\psi^1 \ldots d\psi^d e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}}, \tag{A.5}$$

where Z' is a normalization factor to be determined and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $\dot{\omega}_{\mu\nu} = 0$, while ψ^μ corresponds to $\gamma^\mu/\sqrt{2}$. Define $g(z) = \Pi_{k\neq 0}(k+z)$ with $g(0) = 1$ and $g'(0) = b$; b is an unknown constant.

$$\frac{g'}{g} = \sum_{k\neq 0} \frac{1}{k+Z} + \dots \tag{A.6}$$

Except at $Z = 0$. g'/g has poles with residue 1 at every $Z = k$, $k \in Z$. Then

$$\frac{g'}{g} = \frac{\pi \cos \pi Z}{\sin \pi Z}, \tag{A.7}$$

$$g(Z) = \frac{\sin \pi Z}{\pi Z} e^{-bZ}; \qquad g(0) = 1, \qquad g'(0) = b. \tag{A.8}$$

It follows that

$$2^{-\frac{1}{2}d} \operatorname{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = Z \det{}^{1/2} \left(\frac{\sin \frac{\omega^\mu{}_\nu}{2i}}{\frac{\omega^\mu{}_\nu}{2i}} e^{b\frac{\omega^\mu{}_\nu}{2i}} \right) \int_{\text{zero modes}} d\psi^1 \dots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu},$$

with $\omega^{\mathrm{T}} = -\omega$, $\operatorname{Tr}\omega = 0$, and the determinant is the one of a finite $d\times d$ matrix.

$$2^{-\frac{1}{2}d} \operatorname{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = Z \det{}^{1/2} \left(\frac{\sin \frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}} \right) \int d\psi^1 \dots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu}. \tag{A.9}$$

We can now determine the normalization constant Z by inserting γ_5 into (A.1) and choosing $\omega = 0$

$$2^{-\frac{d}{2}} \operatorname{Tr}(\mathbf{1}) = Z \int d\psi^1 \dots d\psi^d \psi^1 \dots \psi^d (-i)^{\frac{d}{2}} 2^{\frac{d}{2}}, \tag{A.10}$$

using

$$\gamma_5 = (-i)^{\frac{d}{2}} \gamma_1 \dots \gamma_d, \quad \gamma_5^2 = 1, \quad \text{and} \quad \psi^\mu \to \frac{1}{\sqrt{2}} \gamma^\mu.$$

For d even we find

$$Z = (-i)^{\frac{d}{2}} 2^{-\frac{d}{2}}. \tag{A.11}$$

The final result is:

$$2^{-\frac{d}{2}}\,\mathrm{Tr}\,(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = \left(\frac{-i}{2}\right)^{\frac{d}{2}} \int d\psi^1 \ldots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu} \det{}^{1/2}\left(\frac{\sinh \frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}}\right)$$

$$= \frac{(-i)^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!}\, 2^{-d}\varepsilon^{\mu_1\ldots\mu_d}\omega_{\mu_1\mu_2}\ldots\omega_{\mu_{d-1}\mu_d} \det{}^{1/2}\left(\frac{\sinh \frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}}\right), \tag{A.12}$$

where we have used

$$\int d\psi^1 \ldots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu} = \frac{1}{\left(\frac{d}{2}\right)!}(-)^{\frac{d}{2}}\omega_{\mu_1\mu_2}\ldots\omega_{\mu_{d-1}\mu_d}\int d\psi^1 \ldots d\psi^d \psi^{\mu_1}\ldots\psi^{\mu_d}$$

$$= \frac{1}{\left(\frac{d}{2}\right)!}(-)^{\frac{d}{2}}\omega_{\mu_1\mu_2}\ldots\omega_{\mu_{d-1}\mu_d}\varepsilon^{\mu_1\ldots\mu_d}(-)^{\frac{d}{2}}. \tag{A.13}$$

The second case is treated in a similar way

$$2^{-\frac{d}{2}}\,\mathrm{Tr}\,(e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu})_{\mathrm{P1}} = \int_{\mathrm{APBCs}} \mathscr{D}\psi^\mu \exp\left[-\int_0^1 dt(\tfrac{1}{2}\,\psi^\mu\dot{\psi}^\mu + \tfrac{1}{2}\,\omega_{\mu\nu}\psi^\mu\psi^\nu\right). \tag{A.14}$$

We have to compute

$$g_{\mathrm{AP}}(Z) = \prod_{k\in \mathbf{Z}}(k+\tfrac{1}{2}+Z) = \cosh \pi Z, \quad g(0) = 1, \quad g'(0) = 0.$$

Then

$$2^{-\frac{d}{2}}\,\mathrm{Tr}\,(e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) \doteq \det{}^{1/2}\left(\cosh\frac{\omega^\mu{}_\nu}{2}\right).$$

In the isospin case we want

$$\mathrm{Tr}\,((-)^{N_\eta}e^{-\tilde{T}}) = \int_{\mathrm{PBCs}} \mathscr{D}\bar{\eta}^a\mathscr{D}\eta^a \exp\left[-\int_0^1 dt(\bar{\eta}^a\dot{\eta}^a + \bar{\eta}^a T^{ab}\eta^b)\right], \tag{A.15}$$

where $\tilde{T}$ is the matrix T^a_b promoted to Grassmann algebra in a way completely similar to what we did in the spinor case, and the trace is over the space of forms $\Lambda^*(\bar{\eta})$ (scalar, vector, tensor, ...) over this algebra. The computation is very analogous to case 1) since now

$$\operatorname{Tr}((-)^{N_n}e^{-\tilde{T}}) = Z\,\det{}_{0,\mathrm{PBC}}(\partial_t + T)\int_{\substack{\text{zero}\\ \text{modes}}} d\bar{\eta}_1 d\eta_1 \ldots d\bar{\eta}_n d\eta_n e^{-\bar{\eta}^a T^a{}_b \eta^b}$$

$$= Z\det\left(\frac{\sinh\frac{T}{2}}{\frac{T}{2}}\,e^{\frac{bT}{2i}}\right)\det(T) = Z\det\left(e^{\frac{T}{2}\left(1+\frac{b}{i}\right)} - e^{-\frac{T}{2}\left(1-\frac{b}{i}\right)}\right). \tag{A.16}$$

The last factor comes from the integration over the zero modes while the first one derives from the case 1) above. By taking $n = 1$ and T diagonal, b and Z are determined. In this case, we have two representations, one trivial with $N = 0$ and another corresponding to the 1-fermion state of the Hilbert space with $N_1 = 1$. We have

$$\operatorname{Tr}((-)^{N}e^{-\tilde{T}}) = (1 - e^{-T}),$$

which compared to (A.16) gives $b = -i$ and $Z = 1$. The final result

$$\operatorname{Tr}((-)^{N_n}e^{-\tilde{T}}) = \int_{\mathrm{PBCs}} \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a \exp\left[-\int_0^1 dt(\bar{\eta}^a\dot{\eta}^a + \bar{\eta}^a T^{ab}\eta^b)\right] = \det(1 - e^{-T}) \tag{A.17}$$

$$\int_{\mathrm{PBCs}} \mathscr{D}x^\mu \exp\left[-\int_0^1 dt\left(\tfrac{1}{2}(\dot{x}^\mu)^2 + \tfrac{1}{2}x_\mu \mathscr{R}^\mu{}_\nu \dot{x}^\nu\right)\right] = Z'\,\det{}^{-1/2}_{0,\mathrm{PBC}}[(-\partial_t^2 + \mathscr{R}\partial_t)^\mu{}_\nu]\int_{\substack{\text{zero}\\ \text{modes}}} dx^1 \ldots dx^d$$

$$= Z\,\det{}^{-1/2}_{0,\mathrm{PBC}}[(\partial_t - \mathscr{R})^\mu{}_\nu]\int dx^1 \ldots dx^d = (2\pi)^{-\frac{d}{2}}\det{}^{-1/2}\left(\frac{\sinh\frac{\mathscr{R}^\mu{}_\nu}{2}}{\frac{\mathscr{R}^\mu{}_\nu}{2}}\right)\int dx^1 \ldots dx^d,$$

$$\mathscr{R} \text{ skew symmetric.} \tag{A.18}$$

The determinant was already computed in case 1) and the normalization factor known from the free field case (taking $\mathscr{R} = 0$) is $z = (2n)^{-d/2}$.

APPENDIX B

Define the space of (Clifford) forms

$$C \equiv (S\otimes S^*):\left\{\omega:\omega = \omega^{\mathrm{sc}} + \omega_\mu\gamma^\mu + \frac{1}{2!}\omega_{\mu\nu}\gamma^\mu\gamma^\nu + \ldots + \omega_{\mu_1\ldots\mu_d}\gamma^{\mu_1}\ldots\gamma^{\mu_d};\right.$$

$$\left.\omega_{\mu_1\ldots\mu_d} = \text{skew}\right\} \tag{B.1}$$

on which one has the inner product

$$\langle \omega, \omega \rangle = \mathrm{Tr}\, \omega^+ \omega = |\omega^{sc}|^2 + \bar{\omega}_\mu \omega_\mu + \frac{1}{2!} \bar{\omega}_{\mu_1 \mu_2} \omega_{\mu_1 \mu_2} + \dots . \tag{B.2}$$

Let

$$(-i)^{\frac{d}{2}} \gamma_5 = \tilde{\gamma}_5 = \gamma_1 \dots \gamma_d = \frac{1}{d!} \varepsilon_{\mu_1 \dots \mu_d} \gamma^{\mu_1} \dots \gamma^{\mu_d}. \tag{B.3}$$

Then

$$\tilde{\gamma}_5^+ = (-)^{d(d-1)/2} \tilde{\gamma}_5, \quad \text{and} \quad \langle \tilde{\gamma}_5, \tilde{\gamma}_5 \rangle = 1. \tag{B.4}$$

One can check

$$* \omega = \tilde{\gamma}_5 \omega^+, \tag{B.5}$$

where * is the Hodge star operator [6],

$$\tilde{\gamma}_5 \omega_k = (-)^{k(d-k)} \omega_k \tilde{\gamma}_5, \tag{B.6}$$

where ω_k is a k-form, and from (B.5) and (B.6)

$$* * \omega = * (\tilde{\gamma}_5 \omega^+) = \tilde{\gamma}_5 (\omega \tilde{\gamma}_5^+) = (-)^{k(d-k)}. \tag{B.7}$$

For d *even*, one can consider the decomposition

$$C = C_+ + C_- \tag{B.8}$$

with

$$\begin{aligned} C_+ &\equiv (S \otimes S_+^*) : \{\omega \in C : \omega \gamma_5 = \omega\}, \\ C_- &\equiv (S \otimes S_-^*) : \{\omega \in C : \omega \gamma_5 = -\omega\}. \end{aligned} \tag{B.9}$$

Then from (B.6) and (B.3), we find

$$\begin{aligned} \gamma_5 \omega_k &= (-)^k \omega_k, \quad \text{if} \quad \omega_k \in C_+ \\ \gamma_5 \omega_k &= (-)^{k+1} \omega_k, \quad \text{if} \quad \omega_k \in C_-. \end{aligned} \tag{B.10}$$

To every $\omega \in C$ corresponds a differential form $\in \Lambda$

$$\hat{\omega} = \omega^{sc} + \omega_\mu dx^\mu + \frac{1}{2!} \omega_{\mu\nu} dx^\mu dx^\nu + \dots, \tag{B.11}$$

and to $\tilde{\gamma}_5$ corresponds

$$\hat{\tilde{\gamma}}_5 = \frac{1}{d!} \varepsilon_{\mu_1 \dots \mu_d} dx^{\mu_1} \dots dx^{\mu_d} = dx^1 \dots dx^d. \tag{B.12}$$

Observe that if ω^+ and ω^- belong to C^+ and C^- the decomposition in even and odd forms introduced in Section 4 corresponds to

$$\frac{1 \pm \gamma_5}{2} \omega^+ + \frac{1 \mp \gamma_5}{2} \omega^-.$$

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