THE ISING MODEL ON A RANDOM LATTICE WITH A COORDINATION NUMBER EQUAL 3*

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The micro- and grand-canonical partition functions for a system of spins on a dynamical two-dimensional random spherical surface with a coordination number 3 restricted to the set of lattices without the 'tadpole' and 'self-energy' insertions is calculated. The critical properties are shown to be the same as in the case of the unrestricted set of the ϕ^3 lattices.

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1. *Introduction*

The Ising model on a dynamical two-dimensional random lattice with a fixed coordination number was proposed by Kazakov [1]. The model with a coordination number k is equivalent [1] to the ϕ^k theory of two interacting $N\times N$ hermitean matrices for $N \to \infty$, which is exactly solvable [4]. The solutions for surfaces with a spherical topology and $k = 3,4$ [1, 2] show the same critical behaviour. In the thermodynamic limit the properties of a system with a fixed number of spins n (the micro-canonical ensemble) seem to be independent on the coordination number and the topology of the surface [3]. The Ising model on a dynamical random surface can be treated as a regularization of a two-dimensional fermionic string theory [5]. The sum over all admissible metrics is represented in discrete models by the sum over all lattices. The critical exponent γ_{str} of the string susceptibility was found in spherical and toroidal cases for a grand-canonical partition function of the Ising model [1, 3]. These results are in agreement with those obtained in the continuous case [6, 7]. This coincidence of results confirm usefullness of random lattice discretization methods as a regularization of some string theories.

This paper is devoted to the generalization of the results of the paper [2] to the model on ϕ^3 spherical lattices with removed 'tadpole' and 'self-energy' sub-diagrams. We expect

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that in the discrete models of random surfaces the critical propertiès are independent on the concrete realization of the sum over metrics. This means that the critical behaviour should not depend on the local structure of the lattice (a coordination number, existence of the 'tadpole' insertions etc.). The only important thing is the random character of the lattice, which implies that models considered in [1–3] and the one discussed in this paper belong to the same universality class. We check that indeed the critical properties remain the same as in the unrestricted cases [1–3] i.e. the critical indices for the micro-canonical partition function are: $\alpha = 1$, $\beta = 1/2$, $\gamma = 2$, $\delta = 5$ and the critical exponent $\gamma_{str} = -1/2$ for noncritical temperatures and $\gamma_{str} = -1/3$ at the critical one.

2. *Description of the model*

Let us consider the free energy of the ϕ^3 theory of two $N \times N$ hermitean matrices in zero dimensions:

$$\exp(N^2\bar{F}(a_\pm, c, g, N)) = \int D_N A D_N B \exp(\mathrm{Tr}(-1/2(a_+A^2 + a_-B^2) + cAB + 1/3gN^{-1/2}(A^3+B^3))), \tag{2.1}$$

where $a_\pm \geqslant 0$, $c \geqslant 0$, $a_+a_- - c^2 \geqslant 0$.

Formally, integral (2.1) exists only for imaginary g. In order to extend the domain of existence of this integral, one can add to the action the ϕ^4-term with the g' coupling. In the perturbative expansion (with respect to g and g') all diagrams with the g' vertex may be neglected if $g' \exp(1/g) \to 0$. In this limit only the ϕ^3 diagrams survive and this defines the method of regularization of the integral (2.1).

As was shown by Mehta [4] integral (2.1) can be rewritten in the form

$$\exp(N^2\bar{F}((a_\pm, c, g, N)) = \mathrm{const} \int \prod_{i=1}^{N} (dx_i dy_i w(x_i, y_i))\Delta(x)\Delta(y), \tag{2.2}$$

where $\Delta(x) = \det[x_i^{j-1}]$, $i, j = 1 \ldots N$, and const is g-independent,

$$\bar{w}(x, y) = \exp(-1/2(a_+x^2 + a_-y^2) + cxy + 1/3gN^{-1/2}(x^3+y^3)). \tag{2.3}$$

Changing variables in (2.2)

$$x \to (a_-/a_+)^{1/4}x, \quad y \to (a_+/a_-)^{1/4}y$$

we obtain

$$\bar{w}(x, y) \to w(x, y) = \exp(-1/2(ax^2 + ay^2) + cxy + 1/3gN^{-1/2}(e^{+H}x^3 + e^{-H}y^3)), \tag{2.4}$$

here

$$a = (a_+a_-)^{1/2}, \quad e^{4H/3} = a_-/a_+.$$

The free energy $F(a, c, g, H, N)$ given by the function (2.4) was considered in [2]. It was shown that in the large N limit the terms of the order $O((1/N^2)^h)$, $h = 0, 1 \dots$ of the $1/N^2$ expansion of $F(a, c, g, H, N)$ are equal up to the g-independent constant to the grand--canonical partition function of the Ising model on a two-dimensional random surface with a coordination number 3 and a topology with genus h

$$Z_h(G, \beta, H) = \sum_{n=1}^{\infty} G^{2n} Z_h^{(2n)}(\beta, H). \tag{2.5}$$

Here H is the homogenous external magnetic field, G is the efffective coupling constant (in the thermodynamic interpretation $\log G$ is the chemical potential of a spin), β is the inverse Ising temperature, $Z_h^{(2n)}(\beta, H)$ is the micro-canonical partition function for a system with $2n$ spins (a lattice with coordination number 3, i.e. with the number of nearest neighbours of each spin equal 3, admits only an even number of vertices) on the lattice with genus h (genus is related to the topology of the the minimal surface spanned by the lattice)

$$Z_h^{(2n)}(\beta, H) = \sum_{\{A^{(2n)}\}} \sum_{\{\sigma^i\}} \exp\big(\beta \sum_{\langle ij\rangle} A_{ij}^{(2n)} \sigma^i \sigma^j + H \sum_i \sigma^i\big), \tag{2.6}$$

here $\sigma^i = \pm 1$. $A^{(2n)}$ is an adjacency matrix

$$A_{ij}^{(2n)} = \begin{cases} 1 \text{ if } \sigma^i \text{ and } \sigma^j \text{ are the nearest neighbours,} \\ 0 \text{ otherwise.} \end{cases}$$

The first sum in (2.6) extends over all lattices with $2n$ vertices and genus h. G, β and a, c, g are related in the following way:

$$G^2 = g^2/c^3 \left(\frac{(a/c)^{1/2}}{(a/c)^2 - 1}\right)^3, \qquad \exp(2\beta) = a/c. \tag{2.7}$$

3. *Calculation of the free energy $F(a, c, g, H, N)$*

In this paper we study only the leading term $O(1)$ of the $1/N^2$ expansion of $F(a, c, g, H, N)$ (we denote it by $F(a, c, g, H)$). This corresponds to the grand-canonical partition function for the Ising model in the spherical case. Without loss of generality we can choose $c = 1$.

Following Metha [4]:

$$F(a, c = 1, g, H) = 1/2 \log f(x = 1) - \int_0^1 dx f'(x)/f(x) x$$
$$+ 1/2 \int_0^1 dx f'(x)/f(x) x^2 + 3/4 + 1/2 \log \pi/\sqrt{2}, \tag{3.1}$$

where $f = f(x)$ is given by the system of algebraic equations:

$$x = -2ge^{+H}(t_+ + s_+ r_+) + as_+ - f, \quad 0 = -ge^{+H}(2s_+ + r_+^2) + ar_+ - r_-,$$
$$0 = (-2ge^{+H} + a)f - s_-, \quad 0 = -ge^{+H} f^2 - t_- \tag{3.2}$$

and the counterparts of these equations obtained by the replacement: $H \leftrightarrow -H$, $r_+ \leftrightarrow r_-$, $s_+ \leftrightarrow s_-$, $t_+ \leftrightarrow t_-$.

Introducing variables [2]

$$\varrho_\pm = 2r_\pm g e^{\pm H} - a, \quad z = g^2 f \tag{3.3}$$

one obtains

$$g^2 x = 2z^2 + z(\varrho_+\varrho_- - 1), \quad z = e^{\pm 2H}(\varrho_\mp^2 + 2e^{\mp 2H}\varrho_\pm + 2ae^{\mp 2H} - a^2)/8\varrho_\pm. \tag{3.4}$$

It is convenient to define new variable ϱ by

$$z = (\varrho^2 + 2\varrho + 2a - a^2)/8\varrho = z(\varrho, a). \tag{3.5}$$

Comparing (3.4) and (3.5) one gets

$$\varrho_\pm = \varrho_\pm(\varrho, a, H)$$

and substituting z, $\varrho_\pm$ in (3.4) as functions of ϱ, a, H

$$\begin{aligned} g^2 x &= \Gamma(\varrho, a, H) \\ &= 2z^2(\varrho, a) + [\varrho_+(\varrho, a, H)\varrho_-(\varrho, a, H) - 1]z(\varrho, a). \end{aligned} \tag{3.6}$$

Integral (3.1) can be rewritten as an integral in ϱ. We have

$$dx f'(x)/f(x) = d\varrho \frac{\varrho^2 - a(2-a)}{\varrho(\varrho^2 + 2\varrho + a(2-a))} = d\varrho \partial_\varrho z/z \tag{3.7}$$

and

$$x = \Gamma(\varrho, a, H)/\Gamma(R, a, H),$$

where R is defined by $g^2 = \Gamma(R, a, H)$. ϱ is varying from $-a$ to R while x from 0 to 1. This, together with (3.7) gives

$$F(a, c = 1, g, H) = \Phi(R, a, H) + 3/4 + 1/2 \log \pi/\sqrt{2}, \tag{3.8}$$

where

$$\begin{aligned} \Phi(R, a, H) &= 1/2 \log z(R, a)/\Gamma(R, a, H) - L(R, a, H) + 1/2 K(R, a, H), \\ L(R, a, H) &= \Gamma^{-1}(R, a, H) \int_{-a}^{R} d\varrho \partial_\varrho z/z \Gamma(\varrho, a, H), \\ K(R, a, H) &= \Gamma^{-2}(R, a, H) \int_{-a}^{R} d\varrho \partial_\varrho z/z \Gamma^2(\varrho, a, H). \end{aligned} \tag{3.9}$$

For $H = 0$ and $g = 0$ we have $R = -a$ and $L = 1$, $K = 1/2$, $z/\Gamma = (a^2-1)^{-1}$ and finally

$$F(a, c = 1, g = 0, H = 0) = 1/2 \log \pi/(\sqrt{2}\,(a^2 - 1)).$$

It reproduces the Gaussian result. In the Appendix we give an explicit formula for Φ in the case $H = 0$ and $g \neq 0$.

4. Subtraction of tadpole and self-energy diagrams

To remove the tadpole and self-energy subdiagrams from the model we consider free energy $\tilde{F}(a_\pm, c, \lambda_\pm, g)$ generated by the weight $\tilde{w}(x, y)$ (2.3) with an extra linear term

$$\bar{w}(x, y) = \exp(-1/2(a_+c^2+a_-y^2)+cxy + 1/3\, gN^{-1/2}(x^3+y^3)+N^{1/2}(\lambda_+x+\lambda_-y)). \tag{4.1}$$

We introduce new dynamical variables, which are dual to $a_\pm, c, \lambda_\pm$

$$p_{a_\pm} = -\partial_{a_\pm}\tilde{F}, \quad p_c = \partial_c\tilde{F}, \quad p_{\lambda_\pm} = \partial_{\lambda_\pm}\tilde{F}. \tag{4.2}$$

p_c and $2p_{a_\pm}$ are propagators $\langle AB\rangle, \langle AA\rangle, \langle BB\rangle$ respectively.
To kill tadpoles we choose

$$p_{\lambda_\pm} = 0. \tag{4.3}$$

To subtract self-energy insertions we define Ising temperature and the magnetic field H as follows:

$$p_{a_+}/p_{a_-} = \exp(4H/3); \; 4p_{a_+}p_{a_-}/p_c^2 = \exp(4\beta) \tag{4.4}$$

and the "renormalized coupling" constant G:

$$G^2 = g^2(p_{a_+}p_{a_-}p_c^2)^{3/4}. \tag{4.5}$$

We can now treat the above renormalization procedure as a thermodynamic process which keeps values of $p_{a_\pm}, p_c, p_{\lambda_\pm}$ unchanged. The thermodynamic potential appropriate to describe such a process is the "free enthalpy" obtained from the free energy by the Legendre transform

$$\tilde{E}(p_{a_\pm}, p_c, p_{\lambda_\pm}, g) = \tilde{F}(a_\pm, c, \lambda_\pm, g) + a_+p_{a_+}+a_-p_{a_-}-cp_c-\lambda_+p_{\lambda_+}-\lambda_-p_{\lambda_-}, \tag{4.6}$$

where (4.2) is satisfied and simultaneously

$$\partial_{p_{a_\pm}}\tilde{E} = a_\pm, \quad \partial_{p_c}\tilde{E} = -c, \quad \partial_{p_{\lambda_\pm}}\tilde{E} = -\lambda_\pm. \tag{4.7}$$

To find the explicit form of $p_{a_\pm}, p_c$ we change variables in the free energy $\tilde{F}(a_\pm, c, \lambda_\pm, g)$:

$$x \to x+N^{1/2}\varepsilon_+, \quad y \to y+N^{1/2}\varepsilon_-.$$

We can rewrite $\tilde{F}(a_\pm, c, \lambda_\pm, g)$ in terms of the free energy without the linear term in the action (2.1):

$$\tilde{F}(a_\pm, c, \lambda_\pm, g) = \bar{F}(A_\pm, c, g)+\delta, \tag{4.8}$$

where

$$A_\pm = a_\pm - 2g\varepsilon_\pm,$$

$$g\varepsilon_\pm^2 - a_\pm\varepsilon_\pm + c\varepsilon_\mp + \lambda_\pm = 0,$$

$$\delta = 1/3g(\varepsilon_+^3+\varepsilon_-^3)+1/2A_+\varepsilon_+^2+1/2A_-\varepsilon_-^2-c\varepsilon_+\varepsilon_-.$$

The condition for removing the tadpoles (4.3) implies

$$\varepsilon_{\pm} = \frac{2g}{(A_+A_- - c^2)}(A_{\mp}\partial_{A_\pm}\bar{F} + c\partial_{A_\mp}\bar{F}) \tag{4.9}$$

and

$$p_{a_\pm} = -\partial_{A_\pm}\bar{F} - 1/2\varepsilon_{\pm}^2, \qquad p_c = \partial_c\bar{F} - \varepsilon_+\varepsilon_-. \tag{4.10}$$

Taking $A_\pm = A\exp(\pm 2B/3)$ we express (4.9)-(4.10) in terms of the free energy of the Ising system in the external magnetic field B given by $F(A, c, g, B)$ which is determined by the weight (2.4) and was studied in the previous Section:

$$\partial_{A_\pm}\bar{F} = 1/2\exp(\pm 2B/3)(\partial_A F \mp 3\partial_B F/2A),$$

$$\varepsilon_{\pm} = \frac{g\exp(\pm B/3)}{A^2 - c^2}(A\exp(\pm B)(\partial_A F \mp 3\partial_B F/2A)$$

$$+c\exp(\mp B)(\partial_A F \pm 3\partial_B F/2A)). \tag{4.11}$$

We are now in a position to express the right-hand sides of (4.11) as functions of variables R, A, B. As in previous Section we chose $c = 1$ to simplify calculations:

$$\partial_{A_\pm}\bar{F} = -\phi_\pm(R, A, B) = 1/2\exp(\pm 2B/3)((\partial_A\Phi - \partial_R\Phi\partial_A\Gamma/\partial_R\Gamma)$$

$$\mp 3(\partial_B\Phi - \partial_R\Phi\partial_B\Gamma/\partial_R\Gamma)/2A),$$

$$\varepsilon_\pm = \varepsilon_\pm(R, A, B) = \frac{\sqrt{\Gamma}\exp(\pm B/3)}{A^2 - 1}((A\exp(\pm B) + \exp(\mp B))(\partial_A\Phi - \partial_R\Phi\partial_A\Gamma/\partial_R\Gamma)$$

$$\pm 3(\exp(\mp B) - A\exp(\pm B))(\partial_B\Phi - \partial_R\Phi\partial_B\Gamma/\partial_R\Gamma)/2A), \tag{4.12}$$

where $\Phi = \Phi(R, A, B)$ and $\Gamma = \Gamma(R, A, B)$ are given by (3.9) and (3.6), respectively. We obtain

$$p_{a_\pm} = p_{a_\pm}(R, A, B) = \phi_\pm - 1/2\varepsilon_\pm^2,$$

$$p_c = p_c(R, A, B) = A(\exp(-2B/3)\phi_+ + \exp(+2B/3)\phi_-) - 3\Gamma\partial_R\Phi/\partial_R\Gamma - 1. \tag{4.13}$$

To derive the second equation we used

$$A_+\partial_{A_+}\bar{F} + A_-\partial_{A_-}\bar{F} + c\partial_c\bar{F} + 3/2g\partial_g\bar{F} + 1 = 0,$$

which is a simple consequence of the scaling relation

$$\bar{F}(sA_\pm, sc, s^{3/2}g) + \log s = \bar{F}(A_\pm, c, g).$$

All physical quantities: the free enthalpy $\tilde{E}$, the effective coupling constant G^2, the Ising temperature β and the magnetic field H can be parametrized by R, A, B:

$$\tilde{E} = \tilde{E}(R, A, B) = \Phi(R, A, B) + \delta(R, A, B) + a_-(R, A, B)p_{a_-}(R, A, B)$$

$$+a_+(R, A, B)p_{a_+}(R, A, B) - cp_c(R, A, B),$$

$$G^2 = G^2(R, A, B) = (p_{a+}p_{a-}p_c^2)^{3/4}\Gamma,$$
$$\beta = 1/4 \log(4p_{a+}p_{a-}/p_c^2)^{3/4} = \beta(R, A, B),$$
$$H = 3/4 \log(p_{a+}/p_{a-}) = H(R, A, B). \tag{4.14}$$

Equations (4.14) give an implicit dependence of the free enthalpy on the physical parameters β, G^2, H. Inverting three last equations of (4.14) we find

$$\tilde{E} = \tilde{E}(G^2, \beta, H) = \tilde{E}(R(G^2, \beta, H), A(G^2, \beta, H), B(G^2, \beta, H)).$$

For fixed β and H, the free enthalpy $\tilde{E}(R, A, H)$ has a finite radius of convergence in parameter G^2. We denote it by $G_c^2(\beta, H)$. It means (see 2.5) that

$$Z_h^{(2n)}(\beta, H) \xrightarrow[n\to\infty]{} n^{(\gamma_{str}-3)}(G_c^2(\beta, H))^{-n}, \tag{4.15}$$

where γ_{str} is a critical exponent of the string susceptibility [1–3]. Equation (4.15) determines a micro-canonical partition function of the Ising model in the thermodynamic limit. In this limit we have, due to (4.15), a following formula for the free energy per spin

$$e(\beta, H) = -1/2 \log G_c^2(\beta, H). \tag{4.16}$$

To find a radius of convergence $G_c^2(\beta, H)$ of $\tilde{E}(G^2, \beta, H)$, one has to know the singularity of $\tilde{E}(G^2, \beta, H)$ determined by the smallest value of $|G^2|$. The singularity in question may be hidden in the R, A, B parametrization. Let us consider the point R_0, A_0, B_0 at which $\tilde{E}(R, A, B)$ is regular. We can perform a Taylor expansion:

$$\Delta\tilde{E} = \tilde{E} - \tilde{E}_0 = \partial_{R_0}\tilde{E}\Delta R + \partial_{A_0}\tilde{E}\Delta A + \partial_{B_0}\tilde{E}\Delta B + \dots$$

and analogously

$$\Delta G^2 = G^2 - G_0^2 = \partial_{R_0}G^2\Delta R + \partial_{A_0}G^2\Delta A + \partial_{B_0}G^2\Delta B + \dots$$
$$\Delta\beta = \beta - \beta_0 = \partial_{R_0}\beta\Delta R + \partial_{A_0}\beta\Delta A + \partial_{B_0}\beta\Delta B + \dots$$
$$\Delta H = H - H_0 = \partial_{R_0}H\Delta R + \partial_{A_0}H\Delta A + \partial_{B_0}H\Delta B + \dots . \tag{4.17}$$

Let us consider the curve $H, \beta =$ constant (that means that $\Delta\beta = \Delta H = 0$). One can see that along this curve vanishing of Jacobian at R_0, A_0, B_0

$$\mathrm{Jac}\,(R_0, A_0, B_0) = \frac{\partial(G^2, \beta, H)}{\partial(R_0, A_0, B_0)} = 0 \tag{4.18}$$

implies

$$\Delta G^2{}_{|H,\beta=\mathrm{const}} \propto (\Delta R)^p \Leftrightarrow \Delta R \propto (\Delta G)^{1/p}, \tag{4.19}$$

where p is an integer number and $p \geqslant 2$. Substituting (4.19) into the Taylor series of $\tilde{E}$:

a singular part of $\Delta\tilde{E}_{|H,\beta=\mathrm{const}} \propto (\Delta G)^{n/p}, \quad n \geqslant 1$

one sees that the condition for vanishing of the Jacobian (4.18) is a signal of the appearance of the singularity of $\tilde{E}(G^2, \beta, H)$ if n/p is not an integer number. To show that in the investigated model such a situation takes place we first calculate the ∂_R-derivatives:

$$\partial_R(\partial_A\Phi - \partial_R\Phi\partial_A/\partial_R\Gamma) = (\partial_R\Gamma/\Gamma)(\partial_A L - \partial_A K) - (\partial_A\Gamma/\Gamma)(\partial_R L - \partial_R K)$$
$$= (\partial_R\Gamma/\Gamma)(\partial_A L - \partial_A K + (\partial_A\Gamma/\Gamma)(L - 2K)) \propto \partial_R\Gamma,$$

and

$$\partial_R(\partial_B\Phi - \partial_R\Phi\partial_B\Gamma/\partial_R\Gamma) = (\partial_R\Gamma/\Gamma)(\partial_B L - \partial_B K) - (\partial_B\Gamma/\Gamma)(\partial_R L - \partial_R K)$$
$$= (\partial_R\Gamma/\Gamma)(\partial_B L - \partial_B K + (\partial_B\Gamma/\Gamma)(L - 2K)) \propto \partial_R\Gamma. \tag{4.20}$$

It can be seen by using (4.20) in (4.12) that $\partial_R\phi_\pm$ and $\partial_R\varepsilon_\pm$ have zeros along $\partial_R\Gamma$:

$$\partial_R\phi_\pm \propto \partial_R\Gamma, \qquad \partial_R\varepsilon_\pm \propto \partial_R\Gamma$$

and as a consequence

$$\partial_R p_{a\pm} \propto \partial_R\Gamma \quad \text{and} \quad \partial_R p_c \propto \partial_R\Gamma.$$

It implies

$$\partial_R G^2 \propto \partial_R\Gamma, \qquad \partial_R\beta \propto \partial_R\Gamma, \qquad \partial_R H \propto \partial_R\Gamma \tag{4.21}$$

and finally

$$\mathrm{Jac}\,(R, A, B) \propto \partial_R\Gamma \Leftrightarrow \mathrm{Jac}\,(R, A, B) = \partial_R\Gamma X(R, A, B). \tag{4.22}$$

In the parameter space (R, A, B) the condition $G^2 = 0$ is satisfied in the plane $R = -A$. The physical region of parameters R, A, B is spread between this plane and the critical surface. One can check for small B (it corresponds to small H as well) that between the plane $R = -A$ and the surface given by $\partial_R\Gamma = 0$ neither $X(R, A, B)$ has zeros nor $\tilde{E}(R, A, B)$ becomes singular. It implies that the solution of the equation

$$\partial_R\Gamma(R, A, B) = 0 \tag{4.23}$$

closest to the plane $R = -A$, determines the smallest singularity of $\tilde{E}(G^2, \beta, H)$ and, as was mentioned, it determines the radius of convergence $G_c^2(\beta, H)$ and the micro-canonical partition function of the Ising model in the thermodynamic limit as well. The critical condition (4.23) is the same as in the model on the unrestricted set of lattices [2]. Repeating all arguments from the work [2] and using (4.21) one finds a critical point by

$$\partial_R\Gamma(R_c, A_c, B = 0) = 0 \quad \text{and} \quad \partial_{RR}\Gamma(R_c, A_c, B = 0) = 0,$$

and the fact that the whole set of critical exponents for the micro-canonical partition function remains the same as in the full Φ^3-model as was announced in the introduction. In the Appendix we find the critical temperature: $\exp(2\beta_c) = 108/23$.

In the end we show that γ_{str} also remains unchanged if we remove the tadpole and self-energy sub-diagrams. Taking derivative ∂_g of both sides of the Lagendgre transform (4.6) at constant values of $p_{a\pm}, p_c, p_{\lambda\pm}$ one obtains:

$$\partial_g \tilde{E}(p_{a_\pm}, p_c, p_{\lambda_\pm}, g) = \partial_g \tilde{F}(a_\pm, c, \lambda_\pm, g), \tag{4.24}$$

where $a_\pm, c, \lambda_\pm$ are chosen to keep propagators constant. As was shown in [1, 2] $\gamma_{str} = -1/2$ for noncritical temperatures and $\gamma_{str} = -1/3$ at the critical one. It corresponds to the following behaviour of the singular part of $\tilde{F}$:

$$\Delta\tilde{F}_{sing} \propto (\Delta g)^{5/2} \text{ besides the critical temperature,}$$

$$\Delta\tilde{F}_{sing} \propto (\Delta g)^{7/3} \text{ at the critical temperature,} \tag{4.25}$$

where $\Delta g = g - g_c(\beta, H)$.

The higher derivatives of (4.6) are

$$\begin{aligned}\partial^2_{gg}\tilde{E}(p_{a_\pm}, p_c, p_{\lambda_\pm}, g) &= \partial^2_{gg}\tilde{F}(a_\pm, c, \lambda_\pm, g) + \partial^2_{ga_+}\tilde{F}\partial^2_{gp_{a+}}\tilde{E} - \partial^2_{ga_-}\tilde{F}\partial^2_{gp_{a-}}\tilde{E} \\ &\quad - \partial^2_{gc}\tilde{F}\partial^2_{gp_c}\tilde{E} - \partial^2_{g\lambda_+}\tilde{F}\partial^2_{gp_{\lambda+}}\tilde{E} - \partial^2_{g\lambda_-}\tilde{F}\partial^2_{gp_{\lambda-}}\tilde{E}, \\ \partial^3_{ggg}\tilde{E}(p_{a_\pm}, p_c, p_{\lambda_\pm}, g) &= \partial^3_{ggg}\tilde{F}(a_\pm, c, \lambda_\pm, g) + \dots .\end{aligned} \tag{4.26}$$

Since for finite $p_{a\pm}, p_c, p_{\lambda\pm}$ the values of $a_\pm, c, \lambda_\pm$ are also finite, we have from (4.26) that the singular behaviour of $\partial^2_{gg}\tilde{E}(p_{a\pm}, p_c, p_{\lambda\pm}, g)$ is determined by the singular part of the derivative $\partial^2_{gg}\tilde{F}(a_\pm, c, \lambda_\pm, g)$. Analogously for higher derivatives. Thus we have:

$$\partial_g \tilde{E}_{|\Delta g\to 0} < \infty, \quad \partial_{gg}\tilde{E}_{|\Delta g \to 0} < \infty, \quad \partial^3_{ggg}\tilde{E}_{|\Delta g\to 0} \to \infty. \tag{4.27}$$

Above we used the fact that if $\partial_g\tilde{E}$ and $\partial_g\tilde{F}$ are finite then also their derivatives with respect to $a_\pm, c, \lambda_\pm$ are finite. The singularity in $\partial^3_{ggg}\tilde{E}$ comes from the $\partial^3_{ggg}\tilde{F}$, so it is determined by the same critical exponent. Thus we reproduced exactly the value of $\gamma_{str} = -1/2$, which jumps at the critical temperature to $\gamma_{str} = -1/3$. Such a critical behaviour of γ_{str} was expected earlier in (4.19) which shows the behaviour

$$\Delta G^2{}_{|H,\beta=\text{const}} \propto (\Delta R)^2 \Leftrightarrow \Delta R \propto (\Delta G)^{1/2} \quad \text{for} \quad \beta \neq \beta_c,$$

$$\Delta G^2{}_{|H,\beta=\text{const}} \propto (\Delta R)^3 \Leftrightarrow \Delta R \propto (\Delta G)^{1/3} \quad \text{for} \quad H = 0 \text{ and } \beta = \beta_c.$$

5. *Summary*

In this paper we have shown that the critical properties of the Ising model on the random Φ^3 lattice remain unchanged when we restrict ourselves to the lattices without the tadpole and self-energy diagrams. It suggests that there exists an universal class of random surfaces for which the detailed lattice features are unimportant — they may affect the critical temperature but they do not affect the singular parts of thermodynamic functions in the critical region.

APPENDIX 1

In this appendix we give an explicit form of $\Gamma(R, A, H = 0)$ and $\Phi(R, A, H = 0)$. From here on the dependence on H is omitted.

$$\Gamma(R, A) = \frac{1}{32R^2}(R^2+2R+A(2-A))(4R^3+R^2-2R+A(2-A)). \tag{A.1}$$

The critical curve is given by the equation

$$\partial_R \Gamma = 0$$

which has the following solution:

$$(3R^2+A(2-A))(2R^3+3R^2-A(2-A)) = 0. \tag{A.2}$$

We draw it in Fig. 1. The physical region of parameters (A, R) is dotted in Fig. 1. It is bounded by the curve (a): $R = -A$ which corresponds to $G^2 = 0$ and by the curves (b or c) being the smallest root of (A.2). The critical point is determined by the common point of both roots or equivalently by

$$\partial_{RR}\Gamma = 0.$$

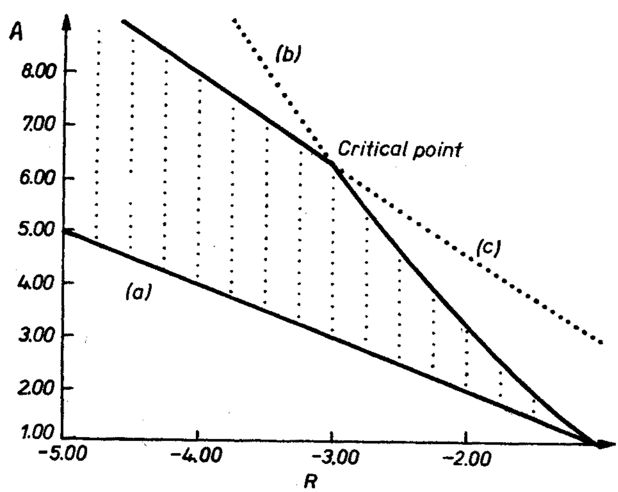
It is not difficult to check that critical values of A, R are

$$A_c = 2\sqrt{7}+1, \quad R_c = -3. \tag{A.3}$$

The exact forms of $L(R, A)$ and $K(R, A)$ are

$$L(R, A) = \frac{C(R, d)}{6(R^2+2R+A(2-A))(4R^3+R^2-2R+A(2-A))},$$

$$K(R, A) = \frac{D(R, d)}{12(R^2+2R+A(2-A))^2(4R^3+R^2-2R+A(2-A))^2},$$



where $d = A-1$ and

$$\begin{aligned} C(R, d) &= 8R^5+3R^4+(-36+24d^2)R^3+(-46+42d^2+32d^3)R^2 \\ &\qquad +(-12+12d^2)R+3(1-d^2)^2, \\ D(R, d) &= 32R^{10}+96R^9+3R^8+(-232+96d^2)R^7 \\ &\quad +(-108+180d^2-96d^4)R^6+(-360-168d^2-96d^4)R^5 \\ &\quad +(466-564d^2+210d^4+64d^6)R^4 \\ &\quad +(168-336d^2+264d^4-96d^6)R^3 \\ &\quad +(-12+12d^2+12d^4-12d^6)R^2 \\ &\qquad +(-8+24d^2-24d^4+8d^6)R+3(1-d^2)^4. \end{aligned} \tag{A.4}$$

Using these formulas and (4.4) we find the critical temperature β_c of the renormalized Φ^3 model corresponding to the values A_c, R_c:

$$\exp(2\beta_c) = 108/23. \tag{A.5}$$

APPENDIX 2

We show that in the low temperature limit $\beta \to \infty$ and for $H = 0$ this model reproduces the results of the one-matrix Φ^3 theory [8]. It is obvious that the low temperature limit ought to appear for $A \to \infty$ (4.11). Going along the critical curve $3R^2+A(2-A) = 0$ we introduce the rescaled variable $r = R/A$. r is of the order of unity along this curve. In this limit the effective coupling constant (4.5) must be redefined. It follows from the fact that the contributions of diagrams with antiparallel vertices are negligible because of the vanishing factor $\exp(-2\beta)$:

$$\sum (g^2(p_a p_c)^{3/2})^n Z^{(2n)}(p_a/p_c) \xrightarrow[\beta\to\infty]{} 2\sum (g^2 p_a^3)^n N^{(2n)}, \tag{A.6}$$

where $p_a = p_{a+} = p_{a-}$, $N^{(2n)}$ is a number of diagrams with $2n$-vertices. The factor 2 in front of the sum on the right-hand side of (A.6) follows from the existence of two spin directions. Thus the effective constant is

$$G_{\text{eff}}^2 = g^2 p_a^3.$$

In the r parametrization we have

$$G_{\text{eff}}^2 = \Gamma p_a^3 = -(r^2-1)(9r^2-1)^3/(8r^2)^4+O(1/A),$$

$$\exp(2\beta) = p_a/p_c = O(A) \xrightarrow[A\to\infty]{} \infty$$

or using parameter $\alpha = -(r^2-1)/(8r^2)$

$$G_{\text{eff}}^2 = \alpha(1-\alpha)^3, \quad a = a_+ = a_- = 1+\alpha(1-2\alpha) \tag{A.7}$$

and

$$\tilde{E} = -\tfrac{4}{3}\alpha^2 - \tfrac{2}{3}\alpha - \log|\alpha - 1| + \text{const.} \tag{A.8}$$

The last two expressions are in agreement with the results of [8].

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