QUANTUM KINEMATICS OF THE ELECTRIC CHARGE*

A. Staruszkiewicz

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

Dedicated to the memory of Professor Jan Rzewuski

(Received February 21, 1995)

The electric charge is a Lorentz invariant quantity. Its canonically conjugate partner, however, called phase, cannot be Lorentz invariant. We calculate explicitly the Lorentz transformation of the phase.

PACS numbers: 12.20. Ds

"This is the simplest example of a pathological representation of the Lorentz group. It may very well be that this pathological representation is essential for the physics of the future."

P.A.M. Dirac

1. Introduction

The quantum theory of the electric charge described in [1] has as dynamical variables the electric charge Q, the phase S_0 which is the canonically conjugate partner of the electric charge,

$$[Q, S_0] = ie \qquad (\hbar = 1 = c)\,,$$

e being the elementary charge, and the amplitudes of transversal zero-frequency waves c_{lm}; $l = 1, 2, \ldots$ and $m = -l, \ldots, l$ are quantum numbers known from the theory of angular momentum. The remaining canonical commutation relations are

$$[Q, c_{lm}] = 0\,, \qquad [S_0, c_{lm}] = 0\,, \qquad [c_{lm}, c^{+}_{l'm'}] = 4\pi e^2 \delta_{ll'} \delta_{mm'}\,.$$

* Research supported in part by KBN under research grant No 2 PO3B 090 08.

The theory is Lorentz invariant which means that there are six generators of the Lorentz group $M_{\mu\nu}$ with the usual Lie algebra

$$[M_{\mu\nu}, M_{\alpha\beta}] = i(g_{\mu\beta}M_{\nu\alpha} + g_{\nu\alpha}M_{\mu\beta} - g_{\mu\alpha}M_{\nu\beta} - g_{\nu\beta}M_{\mu\alpha}) .$$

Physically the generators $M_{\mu\nu}$ are constants of motion which must exist in a Lorentz invariant theory because of the first Noether theorem. M_{23}, M_{31}, and M_{12} are components of the angular momentum; M_{01}, M_{02} and M_{03} do not have a good name and are sometimes called "boosts".

(The habit of giving common names to scientific concepts is most unfortunate and reveals an alarming state of mind. Julian Schwinger evidently did not like the term "boost" [2]. Having this in mind I would propose to use the term "hyperbolic momentum". Thus the constants of motion associated with the full symmetry of space-time would have the names

— energy
— linear momentum
— angular momentum
— hyperbolic momentum

I hope that the logic and simplicity of this scheme will appeal to the reader.)

The electric charge Q is a Lorentz invariant quantity:

$$[M_{\mu\nu}, Q] = 0 .$$

On the other hand, the phase S_0 cannot be a Lorentz invariant quantity; this reflects the well known fact that a charged state cannot be Lorentz invariant. We shall calculate in this paper the commutators $[M_{\mu\nu}, S_0]$ and $[M_{\mu\nu}, c_{lm}]$, revealing in this way the full Lorentz symmetry of the quantum theory of the electric charge.

2. The spherical functions

The Lorentz transformation transforms a quantum state with a given quantum number l into a linear combination of such states with all kinematically possible values of l. This makes it necessary to fix in advance the relative phases of spherical functions for all values of l. We shall use in this paper the spherical functions given by Landau and Lifshitz in the *second* Polish edition of their book "Quantum Mechanics" [3] which is a translation of the *third* Russian edition. The *first* Polish edition, which is a translation of the *first* Russian edition, uses the spherical functions defined by Condon and Shortley [4]; they are convenient in the nonrelativistic quantum mechanics but less convenient in the relativistic one.

The spherical functions we shall use are given explicitly by the expression

$$Y_{lm}(\vartheta, \varphi) = (-1)^{(m+|m|)/2}\, i^l \left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\vartheta)e^{im\varphi}$$

and differ from those given by Condon and Shortley by the factor i^l. Using the formulae given by Condon and Shortley on page 53 one finds easily

$$i \sin\vartheta \frac{\partial}{\partial\vartheta} Y_{lm} = l\sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}Y_{l+1,m} + (l+1)\sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}Y_{l-1,m}\,,$$

$$i \cos\vartheta Y_{lm} = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}Y_{l+1,m} - \sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}Y_{l-1,m}\,.$$

3. The Lorentz transformation of the phase

The electric charge Q "lives" at the spatial infinity. The electromagnetic field at the spatial infinity is described completely by a single scalar function

$$S(x) = -ex^\mu A_\mu(x)\,,$$

where A_μ is the electromagnetic potential [5]. This function, called *phase* for reasons explained in Refs [1] and [5], has the following properties.

$S(x)$ is homogeneous of degree zero:

$$S(\lambda x) = S(x) \qquad \text{for all} \quad \lambda > 0\,.$$

$S(x)$ is gauge invariant.

$S(x)$ satisfies the equation

$$\Box S(x) = 0\,.$$

Because of the homogeneity property it is convenient to describe the phase $S(x)$ in the spherical coordinates

$$\begin{aligned} x^0 &= R\sinh\psi\,, \\ x^1 &= R\cosh\psi\sin\vartheta\cos\varphi\,, \\ x^2 &= R\cosh\psi\sin\vartheta\sin\varphi\,, \\ x^3 &= R\cosh\psi\cos\vartheta\,, \end{aligned}$$

$$0 < R < \infty\,, \quad -\infty < \psi < +\infty\,, \quad 0 \le \vartheta \le \pi\,, \quad 0 \le \varphi < 2\pi\,.$$

Note that R, ϑ and φ are space-like coordinates while ψ is a time-like coordinate. Moreover

$$R\frac{\partial S}{\partial R} \equiv x^\mu \frac{\partial S}{\partial x^\mu} = 0$$

from the Euler theorem on homogeneous functions. Thus $S(x)$ "lives" effectively on the three-dimensional hyperboloid $(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=-1$ which forms the spatial infinity of space-time.

The general solution of the wave equation satisfied by $S(x)$ can be written in the form [1]

$$S(x) = S(\psi,\vartheta,\varphi) = S_0 - eQ\tanh\psi + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{c_{lm}\, f_{lm}^{(+)}(\psi,\vartheta,\varphi) + \text{h.c.}\right\}.$$

Here S_0 is a constant operator, Q is the total charge, $f_{lm}^{(+)}$ was calculated in [1] as

$$\begin{aligned} f_{lm}^{(+)}(\psi,\vartheta,\varphi) =& Y_{lm}(\vartheta,\varphi)\left\{\frac{1}{2}\left[\frac{\Gamma(l/2)\Gamma(l/2+1)}{\Gamma(l/2+1/2)\Gamma(l/2+3/2)}\right]^{1/2} g_l(\psi)\right. \\ &\left. - i\left[\frac{\Gamma(l/2+1/2)\Gamma(l/2+3/2)}{\Gamma(l/2)\Gamma(l/2+1)}\right]^{1/2} f_l(\psi)\right\}, \\ f_l(\psi) =& \tanh\psi\cdot {}_2F_1\left(-\frac{l}{2},\frac{l+1}{2};\frac{3}{2};\tanh^2\psi\right), \\ g_l(\psi) =& {}_2F_1\left(-\frac{l+1}{2},\frac{l}{2};\frac{1}{2};\tanh^2\psi\right). \end{aligned}$$

$f_{lm}^{(+)}$ is a positive frequency solution of the d'Alembert equation in the usual sense of this term *i.e.* its Fourier transform vanishes on the lower half of the light-cone in the momentum space.

Let $M_{\mu\nu}$ denote the generators of the Lorentz group. $[M_{\mu\nu}, Q] = 0$ from the Lorentz invariance of the total charge Q. Hence

$$[M_{\mu\nu}, S(x)] = [M_{\mu\nu}, S_0] + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{[M_{\mu\nu}, c_{lm}]f_{lm}^{(+)} + [M_{\mu\nu}, c_{lm}^{+}]\overline{f_{lm}^{(+)}}\right\}$$

On the other hand

$$[M_{\mu\nu}, S(x)] = \tfrac{1}{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)S(x).$$

To compare these two equations one has to use the formulae given at the end of the second section and, additionally, the following formulae which can be obtained from the explicit expression for $f_{lm}^{(+)}$.

For $l > 1$

$$[x_0\partial_1 - x_1\partial_0 + i(x_0\partial_2 - x_2\partial_0)]f_{lm}^{(+)} = \sqrt{l(l+2)}\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}f_{l+1,m+1}^{(+)} + \sqrt{(l-1)(l+1)}\sqrt{\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}}f_{l-1,m+1}^{(+)}.$$

For $l = 1$

$$[x_0\partial_1 - x_1\partial_0 + i(x_0\partial_2 - x_2\partial_0)]f_{lm}^{(+)} = \sqrt{l(l+2)}\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}f_{l+1,m+1}^{(+)} + \sqrt{\frac{2}{\pi}}Y_{l-1,m+1}\sqrt{\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}}.$$

For $l > 1$

$$[x_0\partial_1 - x_1\partial_0 - i(x_0\partial_2 - x_2\partial_0)]f_{lm}^{(+)} = -\sqrt{l(l+2)}\sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}f_{l+1,m-1}^{(+)} - \sqrt{(l-1)(l+1)}\sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}f_{l-1,m-1}^{(+)}.$$

For $l = 1$

$$[x_0\partial_1 - x_1\partial_0 - i(x_0\partial_2 - x_2\partial_0)]f_{lm}^{(+)} = -\sqrt{l(l+2)}\sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}f_{l+1,m-1}^{(+)} - \sqrt{\frac{2}{\pi}}Y_{l-1,m-1}\sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}.$$

For $l > 1$

$$(x_0\partial_3 - x_3\partial_0)f_{lm}^{(+)} = -\sqrt{l(l+2)}\sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}f_{l+1,m}^{(+)} + \sqrt{(l-1)(l+1)}\sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}f_{l-1,m}^{(+)}.$$

For $l = 1$

$$(x_0\partial_3 - x_3\partial_0)f_{lm}^{(+)} = -\sqrt{l(l+2)}\sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}f_{l+1,m}^{(+)} + \sqrt{\frac{2}{\pi}}Y_{l-1,m}\sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}.$$

In the result one obtains

$$[M_{01} + iM_{02}, S_0] = \frac{1}{i\pi\sqrt{3}}(c_{1,-1} - c^{+}_{11}),$$
$$[M_{01} - iM_{02}, S_0] = \frac{1}{i\pi\sqrt{3}}(c^{+}_{1,-1} - c_{11}),$$
$$[M_{03}, S_0] = \frac{1}{i\pi\sqrt{6}}(c_{10} + c^{+}_{10}).$$

These equations express the fact, well known from the classical electrodynamics, that a moving charge can be regarded as a charge at rest accompanied by transversal zero-frequency waves.

For completeness we give also the Lorentz transformation of the amplitudes of transversal zero-frequency waves.

$$[M_{01} + iM_{02}, c^{+}_{lm}] = i\sqrt{l(l+2)}\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}c^{+}_{l+1,m+1}$$
$$+ i\sqrt{(l-1)(l+1)}\sqrt{\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}}c^{+}_{l-1,m+1} - \frac{4}{\sqrt{3}}eQ\delta^1_l\delta^{-1}_m,$$
$$[M_{01} - iM_{02}, c^{+}_{lm}] = -i\sqrt{l(l+2)}\sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}c^{+}_{l+1,m-1}$$
$$- i\sqrt{(l-1)(l+1)}\sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}c^{+}_{l-1,m-1} + \frac{4}{\sqrt{3}}eQ\delta^1_l\delta^{1}_m,$$
$$[M_{03}, c^{+}_{lm}] = -i\sqrt{l(l+2)}\sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}c^{+}_{l+1,m}$$
$$+ i\sqrt{(l-1)(l+1)}\sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}c^{+}_{l-1,m} - \sqrt{\frac{8}{3}}eQ\delta^1_l\delta^{0}_m.$$

It is instructive to compare these formulae with those given by Gelfand, Minlos and Shapiro [6]. Let us imagine that the vacuum state *i.e.* the state $|0\rangle$ such that

$$Q|0\rangle = 0, \qquad c_{lm}|0\rangle = 0, \qquad M_{\mu\nu}|0\rangle = 0$$

is acted upon by both sides of the above equations. One obtains then the formulae identical with those given by Gelfand, Minlos and Shapiro (Ref. [6], page 199, Eqs (13), (14), (15) and (16)) in the special case $l_0 = 1$, $l_1 = 0$, l_0 and l_1 being the parameters of an irreducible representation of the proper, orthochronous Lorentz group introduced by Gelfand, Minlos and Shapiro. One sees thus that for $Q = 0$ our states $c^{+}_{lm}|0\rangle$ are identical with the states of Gelfand, Minlos and Shapiro *i.e.* they span an irreducible representation of the proper, orthochronous Lorentz group corresponding to the special choice of parameters $l_0 = 1$ and $l_1 = 0$. In general, however, $Q \neq 0$ and our formulae describe the kinematical content of the case $Q \neq 0$. Dirac, who seems to have discovered much of the present theory [7], describes the case $Q \neq 0$ as "pathological" although there is nothing pathological in it: it is just the kinematics of the electric charge implicit in the Maxwell equations!

REFERENCES

[1] A. Staruszkiewicz, *Ann. Phys.* (N.Y.) **190**, 354 (1989).

[2] J. Schwinger, *Particles, Sources, and Fields*, Vol. 1, Addison–Wesley Publ. Comp., 1970, page 8.

[3] L. Landau, E. Lifshitz, *Quantum Mechanics*, 2nd Polish edition which is a translation of 3rd Russian edition, PWN, Warsaw 1979, page 98.

[4] E.U. Condon, G.H. Shortley, *The Theory of Atomic Spectra*, Cambridge University Press, 1964, page 50.

[5] A. Staruszkiewicz, *Quantum Mechanics of the Electric Charge*, a contribution to the Yakir Aharonov Festschrift *Quantum Coherence and Reality*, ed. by J.S. Anandan and J.L. Safko, World Scientific, Singapore 1994, page 90.

[6] I.M. Gelfand, R.A. Minlos, Z.Ya. Shapiro, Representations of the rotation group and the Lorentz group and their applications, Moscow 1958, page 199 (in Russian).

[7] P.A.M. Dirac, *Int. J. Theor. Phys.* **23**, 677 (1984).