THE MOYAL DEFORMATION OF THE SECOND HEAVENLY EQUATION

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Some technique of reduction of the sdiff(Σ^2) SDYM equations to the second heavenly equation is proposed. Then it is shown that the same technique leads from the Moyal SDYM equations to the Moyal deformation of the second heavenly equation. The iterative solution of this latter equation is discussed.

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1. Introduction

In a sense the self-dual Yang–Mills (SDYM) equations can be considered to be a *master system* for the integrable systems in mathematical physics. According to Ward's conjecture [1–4] most of integrable systems should be a reduction of the SDYM equations. Then a big effort has been made to justify this conjecture [1], [2], [5–16]. Especially a great deal of interest has been devoted to the reduction of the SDYM equations to the self-dual Einstein equations [9–16]. Thus it has been shown, amongst other things, that the symmetry reduction of the sdiff(Σ^2) SDYM equations leads to the first or the second heavenly equation [9], [11], [13–16]; (sdiff(Σ^2) denotes the Lie algebra of the area preserving group of diffeomorphisms of the 2-surface Σ^2).

It is well known that the first or, equivalently, the second heavenly equation describe the general metric of the self-dual vacuum space-time [17] and that these equations are intègrable by the twistor methods [18–20]. In 1976 one of us (J.F.P.) and Robinson [21] brought ten vacuum Einstein equations in complex space-time to one second order nonlinear partial differential equation for one function under the only assumption that the anti-self-dual part of the Weyl tensor was algebraically special. (*Mutatis mutandi* this can be also done in the case of the real space-time of the signature $(++--)$). The equation obtained is called the *hyperheavenly equation* and in a sense it can be considered as a deformation of the second heavenly equation. Many interesting solutions of the hyperheavenly equation are known [22–24] but we are far from understanding the whole mystery of the hyperheavenly equation. In particular we do not know whether this equation is integrable.

Therefore it seems to be highly interesting to consider integrable deformations of the heavenly equation. Now as the Moyal algebra appears to be natural deformation of the sdiff(Σ^2) algebra it is almost obvious that one is interested in the Moyal deformation of the heavenly equation. In the case of the first heavenly equation this has be done by Strachan [31] and Takasaki [32]. However, from the "hyperheavenly" point of view one should rather analyse the Moyal deformation of the second heavenly equation. This has been done by Takasaki in his distinguished paper on the KP hierarchy [33]. Here we are going to give a slightly different approach.

Section 2 of the present paper is devoted to the connection between the sdiff(Σ^2) SDYM equations and the second heavenly equation. It is shown that some symmetry reduction of the sdiff(Σ^2) SDYM equations leads to the second heavenly equation. The same is done in the case of the *evolution second heavenly equation* introduced in Ref. [34].

In Section 3 the main points of the Weyl–Wigner–Moyal formalism are presented. This formalism, as it is well known, has been developed in order to represent the quantum mechanics in the form of the classical statistical mechanics [25–27], [35–38]. In our paper we are only interested in the parts of the Weyl–Wigner–Moyal formalism where the correspondence between operators and functions on the phase space is considered and where the Moyal algebra is analysed.

Then using the results of Sections 2 and 3 we find in Section 4 the Moyal deformation of the second heavenly equation , which is known to be integrable [33]. Following Strachan [31] we present the iterative method of finding the solution of this equation.

There exist many papers devoted to the heavenly equation and, especially, to its integrability. With the present work we would like to open a discussion on the integrability of the hyperheavenly equation, as in our opinion this equation is much more important for general relativity than the heavenly equation.

2. SDYM equations and the second heavenly equation

Here we propose a reduction od the sdiff(Σ^2) SDYM equations to the second heavenly equation ; (sdiff(Σ^2) denotes the Lie algebra of the area preserving group of diffeomorphisms of the real 2-surface Σ^2). Similar reductions are well known in the literature [9], [11], [13–16]. However, our approach can be easily modified so as to lead to the Moyal deformation of the second heavenly equation. We deal with the sdiff(Σ^2) SDYM equations in the flat 4-dimensional real manifold R^4 of the metric

$$ds^2 = 2(dx \odot d\tilde{x} + dy \odot d\tilde{y}) \,, \tag{2.1}$$

where $x, y, \tilde{x}, \tilde{y}$ are coordinates on R^4 and $\odot$ stands for the symmetrized tensor product, *i.e.*, $dx \odot d\tilde{x} \stackrel{\mathrm{def}}{=} \frac{1}{2}(dx \otimes d\tilde{x} + d\tilde{x} \otimes dx)$, *etc.* In what follows we assume that Σ^2 is such that [28–31], [39,40]

$$\mathrm{sdiff}(\Sigma^2) \cong \text{the Poisson algebra on} \Sigma^2 \,. \tag{2.2}$$

(*Remark*: In the case when Σ^2 is a 2-torus T^2 one has the isomorphism $\mathrm{sdiff}(\mathrm{T}^2) \cong \mathrm{su}(\infty)$ [28–31], [39]). Consequently the sdiff(Σ^2) Yang-Mills potentials on R^4 take the form of the hamiltonian vector fields [15]

$$A_i = \frac{\partial \Phi_i}{\partial q}\frac{\partial}{\partial p} - \frac{\partial \Phi_i}{\partial p}\frac{\partial}{\partial q} \,, i \in \{x, y, \tilde{x}, \tilde{y}\} \,, \tag{2.3}$$

where q, p are (local) coordinates in Σ^2 and $\Phi_i = \Phi_i(x, y, \tilde{x}, \tilde{y}, q, p), i \in \{x, y, \tilde{x}, \tilde{y}\}$, are some functions on $\mathrm{R}^4 \times \Sigma^2$.

It is known that the SDYM equations can be considered to constitute the compatibility condition of the following Lax pair [7], [15], [41]

$$\begin{aligned} (\partial_x + \lambda \partial_{\tilde{y}})\Psi &= -(A_x + \lambda A_{\tilde{y}})\Psi \\ (\partial_y - \lambda \partial_{\tilde{x}})\Psi &= -(A_y - \lambda A_{\tilde{x}})\Psi \,, \end{aligned} \tag{2.4}$$

where λ is the spectral parameter. Then the compatibility condition of the system (2.4) yields the SDYM equations in the form

$$[\partial_x + \lambda \partial_{\tilde{y}}, A_y - \lambda A_{\tilde{x}}] - [\partial_y - \lambda \partial_{\tilde{x}}, A_x + \lambda A_{\tilde{y}}] = [A_y - \lambda A_{\tilde{x}}, A_x + \lambda A_{\tilde{y}}] \,. \tag{2.5}$$

Substituting (2.3) into (2.5) and equating independent powers of λ one finds the following set of differential equations

$$\partial_x \Phi_y - \partial_y \Phi_x + \{\Phi_x, \Phi_y\}_P + X = 0 \,, \tag{2.6a}$$

$$\partial_{\tilde{x}} \Phi_{\tilde{y}} - \partial_{\tilde{y}} \Phi_{\tilde{x}} + \{\Phi_{\tilde{x}}, \Phi_{\tilde{y}}\}_P + Y = 0 \,, \tag{2.6b}$$

$$\begin{aligned} &\partial_x \Phi_{\tilde{x}} - \partial_{\tilde{x}} \Phi_x + \partial_y \Phi_{\tilde{y}} - \partial_{\tilde{y}} \Phi_y \\ &\quad + \{\Phi_x, \Phi_{\tilde{x}}\}_P + \{\Phi_y, \Phi_{\tilde{y}}\}_P + Z = 0 \,, \end{aligned} \tag{2.6c}$$

where $\{\bullet, \bullet\}_P$ denotes the Poissón bracket on Σ^2, *i.e.*,

$$\{f, g\}_P \stackrel{\text{def}}{=} \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{2.7}$$

for any functions $f = f(x, y, \tilde{x}, \tilde{y}, q, p)$ and $g = g(x, y, \tilde{x}, \tilde{y}, q, p)$. Moreover, X, Y and Z are arbitrary functions of $(x, y, \tilde{x}, \tilde{y})$ only. From (2.3) it follows that without any loss of generality one can put

$$X = Y = Z = 0\,. \tag{2.8}$$

Henceforth, we assume that the condition (2.8) holds. Consider now a special solution of the sdiff(Σ^2) SDYM equations when

$$A_x = 0 = A_y\,. \tag{2.9}$$

Then without any loss of generality we put

$$\Phi_x = 0 = \Phi_y\,. \tag{2.10}$$

Consequently, Eq. (2.6c) reads (remember that (2.8) holds)

$$\partial_x \Phi_{\tilde{x}} + \partial_y \Phi_{\tilde{y}} = 0\,. \tag{2.11}$$

From Eq. (2.11) one infers that there exists a function $\theta = \theta(x, y, \tilde{x}, \tilde{y}, q, p)$ such that

$$\Phi_{\tilde{x}} = -\partial_y \theta \quad \text{and} \quad \Phi_{\tilde{y}} = \partial_x \theta\,. \tag{2.12}$$

Inserting (2.12) into (2.6b) we get

$$\partial_x \partial_{\tilde{x}} \theta + \partial_y \partial_{\tilde{y}} \theta + \{\partial_x \theta, \partial_y \theta\}_P = 0\,. \tag{2.13}$$

Now we impose the following symmetries on $A_{\tilde{x}}$ and $A_{\tilde{y}}$

$$\begin{aligned} [\partial_x - \partial_q, A_{\tilde{x}}] &= 0 = [\partial_x - \partial_q, A_{\tilde{y}}] \\ [\partial_y - \partial_p, A_{\tilde{x}}] &= 0 = [\partial_y - \partial_p, A_{\tilde{y}}]\,. \end{aligned} \tag{2.14}$$

Thus there exists a function $\Theta = \Theta(x + q, y + p, \tilde{x}, \tilde{y})$ such that

$$\theta(x, y, \tilde{x}, \tilde{y}, q, p) = \Theta(x + q, y + p, \tilde{x}, \tilde{y})\,. \tag{2.15}$$

Substituting (2.15) into (2.13) one obtains the well known second heavenly equation [17] for one "key function" $\Theta = \Theta(x + q, y + p, \tilde{x}, \tilde{y})$

$$\partial_x \partial_{\tilde{x}} \Theta + \partial_y \partial_{\tilde{y}} \Theta + (\partial_x^2 \Theta)(\partial_y^2 \Theta) - (\partial_x \partial_y \Theta)^2 = 0\,. \tag{2.16}$$

Concluding: with (2.9) and (2.14) assumed one can reduce the sdiff(Σ^2) SDYM equations to the second heavenly equation (2.16).

Finally, observe that assuming the following symmetries

$$[\partial_x + \partial_{\tilde{x}}, A_{\tilde{x}}] = 0 = [\partial_x + \partial_{\tilde{x}}, A_{\tilde{y}}]$$
$$[\partial_y + \partial_q, A_{\tilde{x}}] = 0 = [\partial_y + \partial_q, A_{\tilde{y}}]\,, \tag{2.17}$$

we can find a function $H = H(x - \tilde{x}, y - q, \tilde{y}, p)$ such that

$$\theta(x, y, \tilde{x}, \tilde{y}, q, p) = H(x - \tilde{x}, y - q, \tilde{y}, p)\,. \tag{2.18}$$

Substituting (2.18) into (2.13) one gets

$$\partial_x^2 H - \partial_y \partial_{\tilde{y}} H + (\partial_x \partial_y H)(\partial_y \partial_p H) - (\partial_x \partial_p H)(\partial_y^2 H) = 0\,. \tag{2.19}$$

This is exactly the evolution form of the second heavenly equation found in Ref. [34].

3. The Weyl–Wigner–Moyal formalism

In this section we intend to consider some points of the Weyl–Wigner–Moyal formulation of the quantum mechanics in the phase space (for details see Refs. [25–27], [35–38]. Let $\mathcal{H}$ be the space of quantum states of a spinless particle in R^1. Given operator $\hat{f}$ in $\mathcal{H}$ one defines a function $f = f(q,p)$ on the phase space $\mathrm{R}^1 \times \mathrm{R}^1 = \mathrm{R}^2$ according to the formula

$$\mathrm{W}: \hat{f} \mapsto f = f(q,p) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{+\infty} \left\langle q - \frac{\xi}{2} | \hat{f} | q + \frac{\xi}{2} \right\rangle \exp(\frac{i}{\hbar}\xi p) d\xi\,. \tag{3.1}$$

Thus one arrives at the one to one correspondence (the *Weyl correspondence*) between some class $\mathcal{A}_1$ of linear operators in $\mathcal{H}$ and some class $\mathcal{C}_1 \subset C^\infty(\mathrm{R}^2; \mathrm{C})$ of functions on the phase space R^2. From (3.1), we obtain the mapping $\mathrm{W}^{-1} : \mathcal{C}_1 \ni f \mapsto \hat{f} \in \mathcal{A}_1$ (the *Weyl application*) to be

$$\mathrm{W}^{-1}(f) = \hat{f} = \frac{1}{(2\pi)^2} \int_{\mathrm{R}^2} \tilde{f}(\mu, \nu) \exp[i(\mu \hat{q} + \nu \hat{p})] d\mu d\nu\,, \tag{3.2}$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively, and $\tilde{f}$ stands for the Fourier transform of f

$$\tilde{f} = \tilde{f}(\mu, \nu) = \int_{\mathrm{R}^2} f(q,p) \exp[-i(\mu q + \nu p)] dq dp\,. \tag{3.3}$$

Let $\hat{f}, \hat{g} \in \mathcal{A}_1$ and let f and g be the corresponding functions *i.e.*, $f = \mathrm{W}(\hat{f})$ and $g = \mathrm{W}(\hat{g})$. Let also $\hat{f} \circ \hat{g} \in \mathcal{A}_1$. Then one states the natural question: What a function does correspond to $\hat{f} \circ \hat{g}$? The answer to this question is well known and it leads us to the *Moyal ∗-product.* Namely, we have

$$\mathrm{W}(\hat{f} \circ \hat{g}) = f * g\,, \tag{3.4}$$

where

$$\begin{aligned} f * g =& \frac{1}{(\pi\hbar)^2} \int_{\mathcal{R}^4} dq' dp' dq'' dp'' f(q', p') g(q'', p'') \\ & \times \exp\left\{ \frac{2i}{\hbar} [(q - q')(p - p'') - (q - q'')(p - p')] \right\} \\ =& \frac{1}{(\pi\hbar)^2} \int_{\mathcal{R}^4} dq' dp' dq'' dp'' f(q + q', p + p') g(q + q'', p + p'') \\ & \times \exp\left[\frac{2i}{\hbar} (q' p'' - q'' p') \right]. \end{aligned} \tag{3.5}$$

By simple but rather long manipulations the formula (3.5) can be brought to a more transparent differential form

$$f * g = f \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}} \right) g\,, \tag{3.6}$$

where $\overleftrightarrow{\mathcal{P}}$ is the *Poisson operator*

$$\overleftrightarrow{\mathcal{P}} \stackrel{\mathrm{def}}{=} \frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \tag{3.7}$$

acting according to the rule

$$f \overleftrightarrow{\mathcal{P}} g \stackrel{\mathrm{def}}{=} \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \stackrel{\mathrm{def}}{=} \{f, g\}_P\,. \tag{3.8}$$

Now as for any functions $f, g, h \in \mathcal{C}_1$ such that $(f * g) * h$ and $f * (g * h)$ are well defined the following relation holds

$$(f * g) * h = f * (g * h) \tag{3.9}$$

and, moreover, as by (3.6), (3.7) and (3.8)

$$\lim_{\hbar \to 0} f * g = fg \tag{3.10}$$

one can consider the Moyal $*$-product to be a deformation of the usual product of functions. Concluding, we arrive at the associative operator algebra $(\mathcal{A}, +, \circ)$, $\mathcal{A} \subset \mathcal{A}_1$, and associative function algebra $(\mathcal{C}, +, *)$, $\mathcal{C} \subset \mathcal{C}_1 \subset C^\infty(\mathrm{R}^2; \mathrm{C})$, such that the Weyl correspondence $\mathrm{W} : \mathcal{A} \to \mathcal{C}$ defines an algebra isomorphism. Let $\hat{f}, \hat{g} \in \mathcal{A}$ and let $[\hat{f}, \hat{g}] \stackrel{\mathrm{def}}{=} \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f}$. Then by (3.4) and (3.6) one gets

$$\mathrm{W}\Big(\frac{1}{i\hbar}[\hat{f}, \hat{g}]\Big) = \frac{1}{i\hbar}(f * g - g * f) = \frac{2}{\hbar} f \sin\Big(\frac{\hbar}{2} \overleftrightarrow{\mathcal{P}}\Big) g\,, \tag{3.11}$$

where $f = \mathrm{W}(\hat{f})$ and $g = \mathrm{W}(\hat{g})$. Define the *Moyal bracket* $\{\cdot\,,\cdot\}_M$ to be the following mapping

$$\{\cdot\,,\cdot\}_M : \mathcal{C} \times \mathcal{C} \to \mathcal{C}; \{f, g\}_M \stackrel{\mathrm{def}}{=} \frac{2}{\hbar} f \sin(\frac{\hbar}{2} \overleftrightarrow{\mathcal{P}}) g\,. \tag{3.12}$$

It is an easy matter to show that $(\mathcal{C}, +, \{\cdot\,,\cdot\}_M)$ is a Lie algebra which we denote by $\mathcal{M}$ and call the *Moyal algebra* [26–33]. It is evident that the Weyl correspondence defines an isomorphism between the Lie algebra $\mathcal{A}_L \stackrel{\mathrm{def}}{=} (\mathcal{A}, +, \frac{1}{i\hbar}[\cdot, \cdot])$ and the Moyal algebra $\mathcal{M}$ (see (3.11)).

From the definition(3.12) of the Moyal bracket one infers that

$$\lim_{\hbar \to 0} \{\cdot, \cdot\}_M = \{\cdot, \cdot\}_P\,. \tag{3.13}$$

Thus the Moyal algebra is a deformation of the Poisson algebra. Using the well known Taylor expansion of the sine we obtain

$$\{f, g\}_M = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \Big(\frac{\hbar}{2}\Big)^{2k} (f \overleftrightarrow{\mathcal{P}}^{2k+1} g)\,. \tag{3.14}$$

There exists an alternative approach to the operator representation of the Moyal algebra [28, 29, 31, 37]. Namely, substituting the Bopp operators

$$\hat{q} \mapsto \hat{Q} \stackrel{\mathrm{def}}{=} q + \frac{i\hbar}{2} \frac{\partial}{\partial p}\,, \quad \hat{p} \mapsto \hat{P} \stackrel{\mathrm{def}}{=} p - \frac{i\hbar}{2} \frac{\partial}{\partial q} \tag{3.15}$$

into (3.2) one gets the isomorphism between The Moyal algebra $\mathcal{M}$ and some Lie algebra $\mathcal{B}_L = (\mathcal{B}, +, \frac{1}{i\hbar}[\cdot, \cdot])$ of operators acting in $C^\infty(\mathrm{R}^2; \mathrm{C})$. More precisely, we define the mapping (the *Bopp application*) $\mathrm{B}^{-1} : \mathcal{C}_1 \ni f \mapsto \hat{f}^{(B)} \in \mathcal{B}_1$, where $\mathcal{B}_1$ is some class of operators acting in $C^\infty(\mathrm{R}^2; \mathrm{C})$ by

$$\mathrm{B}^{-1}(f) = \hat{f}^{(B)} = \frac{1}{(2\pi)^2} \int_{\mathrm{R}^2} \tilde{f}(\mu, \nu) \exp[i(\mu \hat{Q} + \nu \hat{P})] d\mu d\nu\,. \tag{3.16}$$

Using the Baker–Campbell–Hausdorff formula we obtain

$$\exp[i(\mu\hat{Q}+\nu\hat{P})] = \exp[i(\mu q+\nu p)]\exp[-\frac{\hbar}{2}(\mu\frac{\partial}{\partial p}-\nu\frac{\partial}{\partial q})]$$
$$= \exp[i(\mu q+\nu p)]\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}})\,. \tag{3.17}$$

Consequently, the Bopp application (3.16) can be rewritten in a compact form

$$\mathrm{B}^{-1}(f) = \hat{f}^{(B)} = f\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}})\,. \tag{3.18}$$

Therefore,

$$\hat{f}^{(B)}(g) = f * g\,. \tag{3.19}$$

From the very construction of the Bopp application it follows that for $f, g, f*g \in \mathcal{C}_1$

$$\mathrm{B}^{-1}(f*g) = \hat{f}^{(B)}\circ\hat{g}^{(B)}\,. \tag{3.20}$$

Hence by (3.18) one has

$$(f\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}})g)\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}) = (f\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}))\circ(g\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}))\,. \tag{3.21}$$

Finally, we arrive at the Lie algebra isomorphism $\mathcal{M}\cong\mathcal{B}_L$, where $\mathcal{B}_L$ denotes the Lie algebra $(\mathrm{B}^{-1}(\mathcal{C}), +, \frac{1}{i\hbar}[\cdot\,,\cdot])$.

Note that from (3.18) we find, assuming $\frac{\partial f}{\partial\hbar}=0$,

$$\frac{2}{i}\frac{\partial}{\partial\hbar}\hat{f}^{(B)} = f\overleftrightarrow{\mathcal{P}}\exp(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}) = f\overleftrightarrow{\mathcal{P}} + O(\hbar)\,, \tag{3.22}$$

i.e. $\frac{2}{i}\frac{\partial}{\partial\hbar}\hat{f}^{(B)}$ can be considered to be a deformation of the hamiltonian vector field.

Define

$$e_{(\mu,\nu)} = e_{(\mu,\nu)}(q,p) \overset{\mathrm{def}}{=} -\exp[i(\mu q+\nu p)],\quad \mu,\nu\in R^1\,. \tag{3.23}$$

Then by (3.12) one gets

$$\{e_{(\mu_1,\nu_1)}, e_{(\mu_2,\nu_2)}\}_M = \frac{2}{\hbar}\sin[\frac{\hbar}{2}(\mu_1\nu_2-\mu_2\nu_1)]e_{(\mu_1+\mu_2,\nu_1+\nu_2)}\,. \tag{3.24}$$

Hence for any $f, g\in\mathcal{C}$ we have

$$\{f,g\}_M = \frac{1}{(2\pi)^4}\int_{\mathrm{R}^4}\frac{2}{\hbar}\sin[\frac{\hbar}{2}(\mu_1\nu_2-\mu_2\nu_1)]\tilde{f}(\mu_1,\nu_1)$$
$$\times\,\tilde{g}(\mu_2,\nu_2)\mathrm{e}_{(\mu_1+\mu_2,\nu_1+\nu_2)}d\mu_1 d\nu_1 d\mu_2 d\nu_2\,. \tag{3.25}$$

Now it is an easy matter to carry over the above considerations to the case when the phase space appears to be a 2-torus T^2 endowed with the symplectic structure $dq \wedge dp$. Here, for any smooth function $f = f(q,p)$ on T^2 one has

$$W_T^{-1}(f) = \hat{f} = \frac{1}{(2\pi)^2} \sum_{m,n=-\infty}^{+\infty} \tilde{f}(m,n) \exp[i(m\hat{q} + n\hat{p})] \,, \tag{3.26}$$

where

$$\tilde{f}(m,n) = \int_{T^2} f(q,p) \exp[-i(mq+np)] dqdp \,, \quad m,n \in \mathcal{Z} \,. \tag{3.27}$$

(The index T means that we deal with a torus). Then analogously as before we arrive at the Moyal algebra $\mathcal{M}_T$, the Lie algebra of operators $\mathcal{A}_L^T$ ($\mathcal{M}_T \cong \mathcal{A}_L^T$) and the Bopp application B_T^{-1} with the corresponding operator Lie algebra $\mathcal{B}_L^T \cong \mathcal{M}_T$ (compare with Refs. [28–31]).

Define

$$e_{(m,n)} = e_{(m,n)}(q,p) \stackrel{\text{def}}{=} -\exp[i(mq+np)], \;\; m,n \in \mathcal{Z} \,. \tag{3.28}$$

Then (3.12) yields

$$\{e_{(m_1,n_1)}, e_{(m_2,n_2)}\}_M = \frac{2}{\hbar} \sin[\frac{\hbar}{2}(m_1 n_2 - m_2 n_1)] e_{(m_1+m_2, n_1+n_2)} \,. \tag{3.29}$$

Therefore, for any smooth functions f and g on T^2 one has

$$\begin{aligned} \{f,g\}_M = & \frac{1}{(2\pi)^4} \sum_{m_1,m_2,n_1,n_2 \in \mathcal{Z}} \frac{2}{\hbar} \sin[\frac{\hbar}{2}(m_1 n_2 - m_2 n_1)] \\ & \times \tilde{f}(m_1,n_1) \tilde{g}(m_2,n_2) e_{(m_1+m_2, n_1+n_2)} \,. \end{aligned} \tag{3.30}$$

Analogously we can deal with the Moyal algebra for other 2-surfaces [30].

4. Moyal deformation of the second heavenly equation

Now we are prepared to consider the Moyal deformation of the second heavenly equation.

Let $|\Psi> = |\Psi(x,y,\tilde{x},\tilde{y})>$ be a $\mathcal{H}$-valued function on R^4 and let $\hat{a}_i = \hat{a}_i(x,y,\tilde{x},\tilde{y})$, $i \in \{x,y,\tilde{x},\tilde{y}\}$, be $\mathcal{A}_L$-valued functions on R^4. Then the compatibility condition of the following Lax pair (compare with (2.4))

$$\begin{aligned} i\hbar(\partial_x + \lambda \partial_{\tilde{y}})|\Psi\rangle &= -(\hat{a}_x + \lambda \hat{a}_{\tilde{y}})|\Psi\rangle \,, \\ i\hbar(\partial_y - \lambda \partial_{\tilde{x}})|\Psi\rangle &= -(\hat{a}_y - \lambda \hat{a}_{\tilde{x}})|\Psi\rangle \,, \end{aligned} \tag{4.1}$$

where λ is the spectral parameter, yields the $\mathcal{A}_L$ SDYM equations in the form

$$[\partial_x + \lambda\partial_{\tilde{y}}, \hat{a}_y - \lambda\hat{a}_{\tilde{x}}] - [\partial_y - \lambda\partial_{\tilde{x}}, \hat{a}_x + \lambda\hat{a}_{\tilde{y}}] = \tfrac{1}{i\hbar}[\hat{a}_y - \lambda\hat{a}_{\tilde{x}}, \hat{a}_x + \lambda\hat{a}_{\tilde{y}}]\,. \quad (4.2)$$

Analogously as in Section 2, we are looking for the solution of (4.2) when

$$\hat{a}_x = 0 = \hat{a}_y\,. \quad (4.3)$$

Eqs (4.2), with (4.3) assumed, read

$$\partial_x \hat{a}_{\tilde{x}} + \partial_y \hat{a}_{\tilde{y}} = 0\,, \quad (4.4a)$$

$$\partial_{\tilde{x}} \hat{a}_{\tilde{y}} - \partial_{\tilde{y}} \hat{a}_{\tilde{x}} + \tfrac{1}{i\hbar}[\hat{a}_{\tilde{x}}, \hat{a}_{\tilde{y}}] = 0\,. \quad (4.4b)$$

Using the Weyl correspondence (3.1) one gets

$$\partial_x a_{\tilde{x}} + \partial_y a_{\tilde{y}} = 0\,, \quad (4.5a)$$

$$\partial_{\tilde{x}} a_{\tilde{y}} - \partial_{\tilde{y}} a_{\tilde{x}} + \{a_{\tilde{x}}, a_{\tilde{y}}\}_M = 0\,, \quad (4.5b)$$

where $a_i = a_i(x, y, \tilde{x}, \tilde{y}, q, p)$, $i \in \{x, y, \tilde{x}, \tilde{y}\}$, are the $\mathcal{M}$-valued functions on R^4

$$a_i \stackrel{\text{def}}{=} W(\hat{a}_i)\,. \quad (4.6)$$

From (4.5a) it follows that there exists a function $\theta = \theta(x, y, \tilde{x}, \tilde{y}, q, p)$ such that

$$a_{\tilde{x}} = -\partial_y\theta \quad \text{and} \quad a_{\tilde{y}} = \partial_x\theta\,. \quad (4.7)$$

Substituting (4.7) into (4.5b) we obtain the Moyal deformation of Eq. (2.13) in the form

$$\partial_x\partial_{\tilde{x}}\theta + \partial_y\partial_{\tilde{y}}\theta + \{\partial_x\theta, \partial_y\theta\}_M = 0\,. \quad (4.8)$$

Now, as before, we impose on a_i the symmetries

$$(\partial_x - \partial q)a_{\tilde{x}} = 0 = (\partial_x - \partial q)a_{\tilde{y}}$$
$$(\partial_y - \partial p)a_{\tilde{x}} = 0 = (\partial_y - \partial p)a_{\tilde{y}}\,. \quad (4.9)$$

From (4.7) and (4.9) it follows that there exists a function $\Theta = \Theta(x + q, y + p, \tilde{x}, \tilde{y})$ such that

$$\theta(x, y, \tilde{x}, \tilde{y}, q, p) = \Theta(x + q, y + p, \tilde{x}, \tilde{y})\,. \quad (4.10)$$

Finally inserting (4.10) into (4.8) one obtains the Moyal deformation of the second heavenly equation

$$\partial_x\partial_{\tilde{x}}\Theta + \partial_y\partial_{\tilde{y}}\Theta + \{\partial_x\Theta, \partial_y\Theta\}_M = 0\,. \quad (4.11)$$

$$\Theta = \Theta(x + q, y + p, \tilde{x}, \tilde{y}) .$$

It is of some interest to express the conditions (4.9) in terms of $\hat{a}_j$, $j = \tilde{x}, \tilde{y}$. First one quickly finds that, with (4.9) assumed, the Fourier transforms of $a_{\tilde{x}}$ and $a_{\tilde{y}}$ read

$$\begin{aligned} \tilde{a}_j &= \int_{\mathbf{R}^2} a_j(x + q, y + p, \tilde{x}, \tilde{y}) \exp[-i(\mu q + \nu p)] dq dp , \\ &= \exp[i(\mu x + \nu y)] \int_{\mathbf{R}^2} a_j(q', p', \tilde{x}, \tilde{y}) \exp[-i(\mu q' + \nu p')] dq' dp' , \\ j &= \tilde{x}, \tilde{y} . \end{aligned} \tag{4.12}$$

Therefore, substituting (4.12) into the formula (3.2) defining the Weyl application W^{-1}, we find that $\hat{a}_j = W^{-1}(a_j)$, $j = \tilde{x}, \tilde{y}$, have the following form

$$\hat{a}_j = \hat{a}_j(x + \hat{q}, y + \hat{p}, \tilde{x}, \tilde{y} \qquad j = \tilde{x}, \tilde{y} . \tag{4.13}$$

It is quite evident that the formulas (4.1) to (4.4b) and (4.13) can be easily written down in the Bopp operators language. Especially for (4.13) one gets

$$\hat{a}_j^{(B)} = \hat{a}_j^{(B)}(x + q + \frac{i\hbar}{2}\frac{\partial}{\partial p}, y + p - \frac{i\hbar}{2}\frac{\partial}{\partial q}, \tilde{x}, \tilde{y}) \quad j = \tilde{x}, \tilde{y} . \tag{4.14}$$

$$\hat{a}_j^{(B)} = B^{-1}(a_j) .$$

In order to find the Moyal deformation of the evolution heavenly equation (2.19) we assume the following symmetries

$$\begin{aligned} (\partial_x + \partial\tilde{x}) a_{\tilde{x}} &= 0 = (\partial_x + \partial\tilde{x}) a_{\tilde{y}} \\ (\partial_y + \partial q) a_{\tilde{x}} &= 0 = (\partial_y + \partial q) a_{\tilde{y}} . \end{aligned} \tag{4.15}$$

Then (4.7) and (4.15) imply the existence of a function $H = H(x - \tilde{x}, y - q, \tilde{y}, p)$ such that

$$\theta(x, y, \tilde{x}, \tilde{y}, q, p) = H(x - \tilde{x}, y - q, \tilde{y}, p) . \tag{4.16}$$

Substituting (4.16) into (4.8) we arrive at the Moyal deformation of Eq. (2.19).

$$\partial_x^2 H - \partial_y \partial_{\tilde{y}} H + \{\partial_y H, \partial_x H\}_M = 0 . \tag{4.17}$$

$$H = H(x - \tilde{x}, y - q, \tilde{y}, p) .$$

It is an easy matter to show that the conditions (4.15) as expressed in terms of $W^{-1}(a_j) = \hat{a}_j$, $j = \tilde{x}, \tilde{y}$, give now

$$\hat{a}_j = \hat{a}_j(x - \tilde{x}, y - \hat{q}, \tilde{y}, \hat{p}), \quad j = \tilde{x}, \tilde{y}, \tag{4.18}$$

or in the Bopp operators language

$$B^{-1}(a_j) = \hat{a}_j^{(B)} = \hat{a}_j^{(B)}(x - \tilde{x}, y - q - \frac{i\hbar}{2}\frac{\partial}{\partial p}, \tilde{y}, p - \frac{i\hbar}{2}\frac{\partial}{\partial q}), \quad j = \tilde{x}, \tilde{y}. \tag{4.19}$$

5. Integrability

It is known [33] that the Moyal deformation of the second heavenly equation (4.11) is integrable. Here, following Strachan [31], we present the iterative method for constructing the solution of this equation. From (3.14) one quickly finds

$$\{f, g\}_M = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{\hbar}{2}\right)^{2s} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} \times (\partial_q^{2s+1-j} \partial_p^j f)(\partial_q^j \partial_p^{2s+1-j} g). \tag{5.1}$$

(Compare with (2) of Ref. [31] or (1.1) of Ref. [30]. It seems that in those formulas the factor (-1) has been missed).

Now as $\Theta = \Theta(x + q, y + p, \tilde{x}, \tilde{y})$ (see(4.11)) the formula (5.1) yields

$$\{\partial_x \Theta, \partial_y \Theta\}_M = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{\hbar}{2}\right)^{2s} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} \times (\partial_x^{2s+2-j} \partial_y^j \Theta)(\partial_x^j \partial_y^{2s+2-j} \Theta). \tag{5.2}$$

Consider Θ to be the following power series in $\frac{\hbar}{2}$

$$\Theta = \sum_{n=0}^{\infty} \Theta_n \left(\frac{\hbar}{2}\right)^n. \tag{5.3}$$

$$\Theta_n = \Theta_n(x + q, y + p, \tilde{x}, \tilde{y}), \quad n = 0, 1, \ldots.$$

Substituting (5.3) into (4.11) and equating the coefficients of the same powers of $\frac{\hbar}{2}$ we get the system of differential equations

$$\partial_x \partial_{\tilde{x}} \Theta_r + \partial_y \partial_{\tilde{y}} \Theta_r + \sum_{s=0}^{[\frac{r}{2}]} \sum_{m=0}^{r-2s} \sum_{j=0}^{2s+1} \frac{(-1)^{s+j}}{(2s+1)!} \binom{2s+1}{j} \times (\partial_x^{2s+2-j} \partial_y^j \Theta_m)(\partial_x^j \partial_y^{2s+2-j} \Theta_{r-m-2s}) = 0, \quad r = 0, 1, \ldots. \tag{5.4}$$

Hence for $r = 0$ one finds

$$\partial_x \partial_{\tilde{x}} \Theta_0 + \partial_y \partial_{\tilde{y}} \Theta_0 + (\partial_x^2 \Theta_0)(\partial_y^2 \Theta_0) - (\partial_x \partial_y \Theta_0)^2 = 0\,, \tag{5.5}$$

i.e. the second heavenly equation.

Then for $r = 1$

$$\begin{aligned} &\partial_x \partial_{\tilde{x}} \Theta_1 + \partial_y \partial_{\tilde{y}} \Theta_1 + (\partial_y^2 \Theta_0)(\partial_x^2 \Theta_1) \\ &\qquad + (\partial_x^2 \Theta_0)(\partial_y^2 \Theta_1) - 2(\partial_x \partial_y \Theta_0)(\partial_x \partial_y \Theta_1) = 0\,. \end{aligned} \tag{5.6}$$

Therefore, given Θ_0 Eq. (5.6) is a linear partial differential equation on Θ_1.

Generally, for any $r \geq 1$ the formula (5.4) gives

$$\begin{aligned} &\partial_x \partial_{\tilde{x}} \Theta_r + \partial_y \partial_{\tilde{y}} \Theta_r + (\partial_y^2 \Theta_0)(\partial_x^2 \Theta_r) \\ &\qquad + (\partial_x^2 \Theta_0)(\partial_y^2 \Theta_r) - 2(\partial_x \partial_y \Theta_0)(\partial_x \partial_y \Theta_r) = V_r(\Theta_0, \ldots, \Theta_{r-1})\,, \end{aligned} \tag{5.7}$$

where $V_r = V_r(\Theta_0, \ldots, \Theta_{r-1})$ are some functions of their arguments. (From (5.6) and (5.7) it follows that $V_1 = 0$). Concluding, we observe that the system (5.4) consists of the second heavenly equation (5.5) and the set of linear partial differential equations of the form (5.7).

Similar considerations can be done for the Moyal deformation of the evolution second heavenly equation (4.17).

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REFERENCES

[1] R.S. Ward, *Nucl. Phys.* **B236**, 381 (1984).
[2] R.S. Ward, *Phil. Trans. R. Soc. Lond.* **A315**, 451 (1985).
[3] R.S. Ward, in *Springer Lecture Notes in Physics*, vol.246, *Field Theory, Quantum Gravity and Strings*, Proc. 1984/5, edited by H. J. de Vega and N. Sánchez, Springer-Verlag, Berlin 1986.
[4] R.S. Ward, *J. Geom. Phys.* **8**, 317 (1992).
[5] L.J. Mason, G.A.J. Sparling, *Phys. Lett.* **A 137**, 29 (1989).
[6] S. Chakravarty, M.J. Ablowitz, P.A. Clarkson, *Phys. Rev. Lett.* **65**, 1085 (1989).
[7] S. Chakravarty, M.J. Ablowitz, in *Painlevé Transcendents*, edited by D. Levi and P. Winternitz, Plenum Press, New York 1992.
[8] L.J. Mason, G.A.J. Sparling, *J. Geom. Phys.* **8**, 243 (1992).
[9] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, LMS Lecture Notes Series vol.149, Cambridge University, Cambridge 1991.
[10] L.J. Mason, E.T. Newman, *Commun. Math. Phys.* **121**, 659 (1989).

[11] I. Bakas, in *Non-linear Fields: Classical, Random Semiclassical*, edited by P. Garbaczewski and Z. Popowicz, World Scientific, Singapore 1991.
[12] S. Chakravarty, L. Mason, E.T. Newman, *J. Math. Phys.* **32**, 1458 (1991).
[13] Q. Han Park, *Int. J. Mod. Phys.* **A7**, 1415 (1992).
[14] C. Castro, *J. Math. Phys.* **34**, 681 (1993).
[15] J.F. Plebański, M. Przanowski, H. Garcia-Compeán, *Acta Phys. Pol.* **B25**, 1079 (1994).
[16] L.J. Mason, *Twistor Newsletter* **30**, 14 (1990).
[17] J.F. Plebański, *J. Math. Phys.* **16**, 2395 (1975).
[18] R. Penrose, *Gen. Rel. Grav.* **7**, 31 (1976).
[19] R. Penrose, R.S. Ward, in *General Relativity and Gravitation* Einstein Memorial Volume vol. 2, edited by A. Held, Plenum, New York 1980.
[20] C.P. Boyer, J.F. Plebański, *J. Math. Phys.* **26**, 229 (1985).
[21] J.F. Plebański, I. Robinson, *Phys. Rev. Lett.* **37**, 493 (1976).
[22] A. Garcia D., J.F. Plebański, *Nuovo Cimento* **B40**, 227 (1977).
[23] C.P. Boyer, J.D. Finley III, J.F. Plebański, in *General Relativity and Gravitation.* Einstein Memorial Volume vol.2, edited by A. Held, Plenum, New York 1980.
[24] J.F. Plebański, G.F. Torres del Castillo, *J. Math. Phys.* **23**, 1349 (1982).
[25] J.E. Moyal, *Proc. Cambridge Phil. Soc.* **45**, 99 (1949).
[26] J.F. Plebański, Institute of Physics of Nicolaus Copernicus University, Toruń, preprint No 69 (1969).
[27] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Ann. Phys.* (NY) **111**, 61 (1978); *Ann. Phys.* (NY) **111**, 111 (1978).
[28] D.B. Fairlie, C.K. Zachos, *Phys. Lett.* **B224**, 101 (1989).
[29] D.B. Fairlie, P. Fletcher, C.K. Zachos, *J. Math. Phys.* **31**, 1088 (1990).
[30] P. Fletcher, *Phys. Lett.* **B248**, 323 (1990).
[31] I.A.B. Strachan, *Phys. Lett.* **B283**, 63 (1992).
[32] K. Takasaki, *J. Geom. Phys.* **14**, 111 (1994).
[33] K. Takasaki, *J. Geom. Phys.* **14**, 332 (1994).
[34] J. D. Finley III, J. F. Plebański, M. Przanowski, H. Garcia-Compeán, *Phys. Lett.* **A181**, 435 (1993).
[35] H. Weyl, *Z. Phys.* **46**, 1 (1927).
[36] E.P. Wigner, *Phys. Rev.* **40**, 749 (1932).
[37] M. Hillery, R.F. O'Connell, M.O. Scully, E.P. Wigner, *Phys. Rep.* **106**, 121 (1984).
[38] W.I. Tatarskij, *Usp. Fiz. Nauk* **139**, 587 (1983).
[39] J. Hoppe, *Phys. Lett.* **B215**, 706 (1988).
[40] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, Berlin 1978.
[41] R.S. Ward, *Phys. Lett.* **A61**, 81 (1977).