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THE BRST ANALYSIS OF THE STÜCKELBERG FORMALISM

R. Kögerler

Fakultät für Physik, Universität Bielefeld
D-33615 Bielefeld, Universitätsstrasse 25, Germany

AND

J. Sładkowski*

Institute of Physics, University of Silesia,
Uniwersytecka 4, 40-007 Katowice, Poland

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The Stückelberg formalism can be regarded as a field-enlarging transformation that introduces an additional gauge symmetry to the considered model. We define and calculate the appropriate BRST charge. The physical state condition, demanding that a physical state is to be annihilated by the BRST charge, is shown to be equivalent to the Stückelberg physical state condition. Several applications of the approach to the formalism are presented. The comparison with the BFV procedure is given.

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1. Introduction

The choice of variables used to describe a quantum field theory should not have any physical significance. This field redefinition invariance is a quite nontrivial issue in quantum field theory. Complications may arise already at the level of free theories. An additional well known complication arises when one considers renormalizability of a gauge theory: it can only be shown after introducing extra degrees of freedom (the unitary gauge is formally nonrenormalizable). Recently, it has been proposed to apply the BRST symmetry idea to the field redefinition problem [1, 2]. We would

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* Alexander von Humboldt Fellow

like to show how these tools work in the Stückelberg formalism case [3–5]. The application of the general formalism presented in [1, 2] to a concrete and popular physical model allows for a deeper insight into the Stückelberg formalism. Moreover, it suggests its generalization to the general case of a vector field with the non-Yang-Mills type of self-interaction. We shall also discuss the relation of the field-enlarging transformation to the Batalin-Fradkin-Vilkovisky (BFV) formalism [6, 7] exemplifying it for the anomalous U(1) chiral gauge theory [8–14].

2. Abelian case

Let us consider an Abelian "massive gauge field" A_μ with the following Lagrange density:

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}A_\mu A^\mu \,, \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

We would like to restore the gauge symmetry broken by the mass term by introducing an additional scalar field. To this end, let us perform the following field-enlarging transformation [1, 2, 10, 11]:

$$A_\mu = A'_\mu + \frac{1}{m}\partial_\mu\phi \equiv g_\mu\left(A',\phi\right) \,. \tag{2}$$

The substitution of (2) into (1) gives (we will write A_μ instead of A'_μ)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}A_\mu A^\mu + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + mA_\mu\partial^\mu\phi \,. \tag{3}$$

Now,

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta A_\mu}\delta A_\mu + \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi = \frac{\delta\mathcal{L}}{\delta g_\mu}\left(\frac{\delta g_\mu}{\delta A_\nu}\delta A_\nu + \frac{\delta g_\mu}{\delta\phi}\delta\phi\right) \,, \tag{4a}$$

and the Lagrangian density (3) is invariant with respect to the following gauge transformations [1, 2]:

$$\delta\phi\left(x\right) = \alpha\left(x\right) \,, \tag{4b}$$

$$\delta A_\mu\left(x\right) = -\int d^4z d^4y \left[\frac{\delta g_\mu\left(A,\phi\right)}{\delta A_\nu}\right]^{-1}\left(x,y\right)\frac{\delta g_\nu\left(A,\phi\right)}{\delta\phi}\left(y,z\right)\alpha\left(z\right) \,, \tag{4c}$$

where α is an arbitrary function. The explicit form of the functional g_μ given by (2) leads to:

$$\delta A_\mu\left(x\right) = -\int d^4z d^4y \left[g^{\mu\nu}\delta\left(x-y\right)\right]^{-1}\frac{1}{m}\partial^\nu\delta\left(y-z\right)\alpha\left(z\right) = -\frac{1}{m}\partial_\mu\alpha\left(x\right) \,. \tag{4d}$$

In order to quantize this model we have to remove the gauge freedom. Let us consider the following gauge fixing term

$$\mathcal{L}_{\mathrm{gf}} = -\lambda \left(\partial_\mu A^\mu - \frac{m}{2\lambda}\phi \right)^2 . \tag{5}$$

The gauge-fixed Lagrangian density takes the form (the ghost term is omitted)

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{m^2}{2} A_\mu A^\mu - \lambda \left(\partial_\mu A^\mu \right)^2 + \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \tfrac{m^2}{4\lambda} \phi^2 . \tag{6}$$

This is the standard Stückelberg form of the Lagrangian for a massive Abelian gauge theory! The BRST charge of this model for the symmetry (4) is given by

$$Q_{\mathrm{BRST}} = \int d^3x \left(B \partial_0 c - \partial_0 B c \right) = i \sum_k \left(c_k^\dagger B_k - B_k^\dagger c_k \right) , \tag{7}$$

where B is the auxiliary field that linearizes the gauge-fixing term, c denotes the ghost field and the subscript k labels states in the momentum space representation [13]. The property that Abelian ghosts decouple imply that the state vector space V can be decomposed into a direct product $V = V' \otimes V_{\mathrm{FP}}$, where the V_{FP} contains only ghost fields and all other fields belong to V'. The BRST-physical-state condition, $Q_{\mathrm{BRST}}|\mathrm{phys}\rangle = 0$, takes in our case the form

$$B_k |\mathrm{phys}\rangle = 0 , \qquad \text{for all } k . \tag{8}$$

When one combines this with the B-field equation of motion, one gets:

$$\left(\partial_\mu A^\mu - \frac{m}{2\lambda}\phi \right) |\mathrm{phys}\rangle = 0 , \tag{9}$$

which is precisely the Stückelberg physical state condition. Let us notice that, although the gauge fixing term breaks the gauge symmetry (4), the Lagrange density (6) is still invariant with respect to (4) if

$$\Box \alpha - \frac{m}{2\lambda} \alpha = 0 . \tag{10}$$

This explains the nature of the "extra" symmetry of the Stückelberg model: the gauge fixing condition allows for such a restricted invariance. Of course, other gauge fixing conditions are also possible. They will give us other possible forms of a massive Abelian gauge field model. It is obvious that the condition $\phi = 0$ leads to (1) (unitary gauge).

3. Non-Abelian case

The non-Abelian "massive gauge field" has the following Lagrange density:

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + m^2 Tr\left(A_\mu A^\mu\right)\,, \tag{11}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\,.$$

To generalize the construction to the non-Abelian case, let us perform the field-enlarging transformation [1, 2, 4, 5]

$$A_\mu = g_\mu\left(A', U\right) = U^\dagger A'_\mu U - \frac{i}{g} U^\dagger \partial_\mu U\,, \tag{12}$$

where the scalar field U takes values in the adjoined (unitary) representation of the gauge group. This results in (as before we drop the prime sign over the gauge field)

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + m^2\mathrm{Tr}\left(A_\mu A^\mu\right) - 2\frac{im^2}{g}\mathrm{Tr}\left(\partial_\mu U U^\dagger A^\mu\right) - \frac{m^2}{g^2}\mathrm{Tr}\left(U^\dagger\partial_\mu U U^\dagger \partial^\mu U\right)\,. \tag{13}$$

It is convenient to rewrite the U field as

$$U(x) = \exp\left(\frac{ig}{m}\phi^a(x)T^a\right)\,,$$

where T^a denotes the Lie algebra generators of the gauge group. Eq. (13) then can be rewritten as

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + m^2\mathrm{Tr}\left(A_\mu A^\mu\right) + 2m\mathrm{Tr}\left(\partial_\mu\phi A^\mu\right) + \mathrm{Tr}\left(\partial_\mu\phi\partial^\mu\phi\right)\,. \tag{14}$$

As in the Abelian case, this Lagrange density is invariant with respect to the following gauge transformations (see Eq. (4))

$$\delta\phi^a(x) = \alpha^a(x)\,, \tag{15a}$$

$$\delta A^a_\mu(x) = -\left(D_\mu\alpha\right)^a(x)\,, \tag{15b}$$

where D_μ denotes the covariant derivative. To quantize the model we have to choose a gauge condition. The gauge-fixing condition

$$\mathcal{L}_{\mathrm{gf}} = -\lambda\mathrm{Tr}\left(\partial_\mu A^\mu - \frac{m}{\lambda}\phi\right)^2 \tag{16}$$

leads to the Lagrange density (we omit ghost fields)

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + m^2\mathrm{Tr}\,(A_\mu A^\mu) + \lambda\mathrm{Tr}\,(\partial_\mu A^\mu)^2 + Tr\,(\partial_\mu\phi\partial^\mu\phi) - \frac{m^2}{\lambda}Tr\phi^2\,, \tag{17}$$

which is the standard Stückelberg's one. Other gauge conditions provide us with more sophisticated forms of the massive non-Abelian gauge field Lagrangians. The BRST-physical-state condition, due to the presence of a more complicated ghost sector, has not such an obvious interpretation as in the Abelian case, but it still contains the condition that removes the scalar component of A_μ. Indeed, we can write [15, 16]

$$Q_{\mathrm{BRST}} = c^a\left(\partial_\mu A^\mu - \frac{m}{\lambda}\phi\right)_a - \tfrac{i}{2}f^c_{ab}c^ac^b\pi_c\,, \tag{18}$$

where c^a and π_a are the ghosts and their canonical conjugate fields. Then we have

$$Q_{\mathrm{BRST}}|\psi\rangle = c^aG_a|\psi^{(0)}\rangle + \tfrac{1}{2}c^ac^b\left[G_a|\psi^{(1)}_b\rangle - G_b|\psi^{(1)}_a\rangle - if^c_{ab}|\psi^{(1)}_c\rangle\right] + \cdots\,, \tag{19}$$

where $G = \left(\partial_\mu A^\mu - \frac{m}{\lambda}\phi\right)$ and $\psi^{(i)}$ denotes the ghost-number i component of the state

$$|\psi\rangle = \sum_{k=0}^{k=n}\frac{1}{k!}c^{a_1}\dots c^{a_k}|\psi^{(k)}_{a_1\dots a_k}\rangle\,.$$

Unfortunately, we cannot ensure that there are no BRST–physical states which comprise components with nonzero ghost numbers: at least academic examples of such theories can be given [16]. However, Yang-Mills theories seem to be safe from such complications and the BRST–physical-state condition is sufficient to guarantee that physical states have ghost number zero [16, 17]. This means that the BRST-physical-state condition implies the Stückelberg one:

$$Q_{\mathrm{BRST}}|\mathrm{phys}\rangle = 0 \;\Rightarrow\; \left(\partial_\mu A^{a\mu} - \frac{m}{\lambda}\phi^a\right)\psi^{(0)} = 0\,. \tag{20}$$

4. Applications

In this Section we would like to describe two possible applications of the described approach to the Stückelberg formalism. First, we shall generalize the approach to the case of a vector field with a non-Yang–Mills types of couplings, that are often introduced while discussing possible deviation from the orthodox standard model of the electroweak unification. Then we shall consider the anomalous U(1) chiral gauge theory and show how the Stückelberg formalism is related to the Batalin-Fradkin-Vilkovisky procedure.

4.1. Vector field with non-Yang–Mills types of couplings

Very often, one has to use an effective Lagrangian as a low energy approximation to a not yet known ultimate theory. For example, such considerations are important for analysing the possible existence of anomalous, that is not present in the standard model, weak vector bosons couplings for a triplet and a singlet vector field [18–23]. Of course, one have to find a clever way to reduce the enormous number of possible additional interaction terms. Usually, one takes symmetry as a guiding rule. One can impose only the conditions of Lorentz and $U(1)_{em}$ invariance [18, 19] on such an effective Lagrangian. It is also possible to require invariance with respect to the $SU_L(2)\otimes U_Y(1)$ but with the $SU_L(2)$ gauge symmetry nonlinearly realized [20–23]. We would like to show by using the Stückelberg formalism that these models are related. Let us suppose that the Lagrangian density, motivated by the observed electroweak particle spectrum, has the general form

$$\mathcal{L}\left(F^k_{\mu\nu}, A_\mu, W^\pm_\mu, Z_\mu, \psi_i\right) \tag{21}$$

that is constrained only by the requirement of invariance with respect to the Lorentz and the $U_{em}(1)$-gauge symmetries. $W^\pm_\mu$ and Z_μ denote field mediating weak interactions. A_μ is the photon. $F^k_{\mu\nu}$ denotes the different vector field kinetic terms and ψ_i all matter fields. The field-enlarging transformation (12) takes for this case the form

$$\frac{1}{2}\begin{pmatrix} g_Z Z_\mu & \sqrt{2}g_W W^+_\mu \\ \sqrt{2}g_W W^-_\mu & g_Z Z_\mu \end{pmatrix} = W_\mu \to W'_\mu = U^\dagger W_\mu U - iU^\dagger\partial_\mu U\,, \tag{22}$$

$$\psi \to R(U)\psi\,, \tag{23}$$

where

$$U(x) = \exp\left(\frac{ig}{m}\phi^a(x)T^a\right)$$

and R denotes the appropriate matter field representation. The condition $U^\dagger U = 1$ introduces a non-linearly realized $SU_L(2)$ gauge symmetry to the model. This condition removes also the scalar particle from the physical spectrum. Effectively, the transformation (22) can be realized by the substitution

$$g_W W^\pm_\mu \to \mathrm{tr}\left[\tau^\pm W'\right]\,, \tag{24a}$$

$$g_Z Z \to \mathrm{tr}\left[\tau^3 W'\right] , \tag{24b}$$

where $\tau^{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2)$ and τ_i denote the SU(2) generators (the Pauli matrices multiplied by $\frac{1}{2}$). Note, that if we do not perform (23) then the matter fields are gauge invariant. So, in fact, we have two types of gauge symmetry at our disposal (these symmetries are not equivalent in the chiral case, see the next subsection and Ref. 12). As before, various gauge fixing conditions lead to different representations of the model. This generalizes the considerations presented in [18]. Note, that the possible cut-off dependence of the results of calculations of physical quantities in effective models makes the above considerations a quite non-trivial issue. Examples od such calculations can be found in Ref. [18].

4.2. Anomalous chiral U(1) gauge theory

Let us consider the following Lagrange density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}A_\mu A^\mu + i\bar{\psi}_L\gamma^\mu\left(\partial_\mu + igA_\mu\right)\psi_L , \tag{25}$$

where $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$ is the left-handed Weyl field. The transformation (2) leads to the following Stückelberg Lagrangian (as usual, we omit the ghost sector)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}\left(A_\mu + \partial_\mu\theta\right)^2 + i\bar{\psi}_L\gamma^\mu\left(\partial_\mu + igA_\mu + ig\partial_\mu\theta\right)\psi_L . \tag{26}$$

The Fujikawa method [25] can be used to derive the equality for path integrands

$$g\bar{\psi}_L\gamma^\mu\partial_\mu\theta\psi_L = \frac{g^3}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\theta F_{\mu\nu}F_{\rho\sigma} . \tag{27}$$

So finally, we have (the path integral is understood)

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}\left(A_\mu + \partial_\mu\theta\right)^2 + i\bar{\psi}_L\gamma^\mu\left(\partial_\mu + igA_\mu\right)\psi_L \\ &- \frac{g^3}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\theta F_{\mu\nu}F_{\rho\sigma} .\end{aligned} \tag{28}$$

Here, the last term is the result of the anomalous transformation of the fermionic determinant. It depends on the spacetime dimension [11–12, 22–23]. Now, it is obvious that the Stückelberg formalism has to be put in force as a field-enlarging transformation. The addition of the scalar degrees of freedom alone would neglect the last term in (28) and the symmetry would not be restored. The Lagrangian density given by (28) can be obtained by the BFV quantization procedure [5, 7] (plus the gauge-fixing and ghosts sectors). The main idea of this formalism is to convert second class constraints to the first class ones by introducing new canonical variables (BFV-fields) [6–8]. Then one chooses the gauge by adding a gauge-fixing fermion field and, possibly, integrates out some fields. This leads to a correct quantum action. The new (effective) constraints for (25) have the form [8]

$$\phi = \pi_0 + m^2\theta\,, \qquad \phi' = \partial_i\pi^i + m^2 A^0 - j^0 + \pi_\theta\,, \tag{29}$$

where (θ, π_θ) is the canonical conjugate pair of BFV fields, π_μ denote the canonical conjugate momenta for A_μ and $j^\mu = g\bar{\psi}\gamma^\mu\psi$ is the current density. We have

$$[\phi(x), \phi(y)] = [\phi(x), \phi'(y)] = [\phi'(x), \phi'(y)] = 0\,. \tag{30}$$

It is well known that in the Hamiltonian formalism the first class constraints reflect the presence of gauge symmetry. To get the orthodox form of the Stückelberg Lagrangian in the BFV formalism one has to choose a special gauge condition [8] (the BFV procedure provides us with a gauge-fixed Lagrangian). In our approach, when the additional symmetry is explicitly introduced, there is full analogy between the Stückelberg scalar field and the BFV field: we still have to choose the gauge in (28). Different gauge conditions result in (equivalent) representations: no special gauge is required. The explicit form of the additional symmetry allows immediately to answer the question [8] why the simultaneous appearance of both the kinetic term of the scalar field θ and the Wess-Zumino term in the BFV formalism requires the presence of the gauge field mass term. The answer is: the mass term is necessary because it compensates the transformation of the scalar field kinetic term. Otherwise the symmetry would be broken.

5. Concluding remarks

The Stückelberg formalism can be regarded as a field-enlarging transformation that introduces an additional gauge symmetry to the model. Such a transformation does not influence the S-matrix because it is a point transformation. The well known theorems concerning point transformations imply this [26]. If one fully analyses the BRST structure of the model one gets that the Stückelberg-physical-state condition is implied by the requirement that the BRST charge annihilates physical states. It is also possible to visualize direct analogies with the Batalin–Fradkin–Vilkovisky quantization procedure. The Stückelberg approach allows to keep track of additional symmetries. This is not always possible in the abstract formulation. The origin of the "antifields" (a canonically conjugate field introduced for each field with the opposite Grassmann parity) can be understood in an analogous way [27, 28]. The formalism can be also used to analyse the bosonisation phenomenon [11, 29] and quantization of anomalous chiral theories [12, 13]. Wide application of the formalism in the effective Lagrangian models, along the lines discussed here can be anticipated [16].

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