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ABELIAN–HIGGS–NAVIER–STOKES HYDRODYNAMICS FOR NEMATIC FILMS WITH DEFECTS*

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A new theory of hydrodynamics of uniaxial nematic liquid crystal films in the presence of defects is developed. A gauge field incorporating screening is introduced, resulting in the static elastic free energy having the form of a two-dimensional Abelian–Higgs model. Hydrodynamic equations are derived via the standard methods of de Groot and Mazur. By working in the vicinity of the Bogomol'nyi equations consequences for defect centre motion are outlined.

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Liquid crystals are important for technical applications such as liquid crystal displays. However, the dynamics of defects studied in experiments is also used to model the evolution of defects like strings in cosmology [1]. Here we propose a theory to model the dynamics of nematic liquid crystals with defects.

1. Description of defects

Liquid crystals are states of matter which are in between those of a liquid and crystal. In terms of their mechanical properties they are akin to isotropic fluids, but in terms of their optical properties they behave more like crystals. There are various types of liquid crystals. We shall only consider uniaxial nematics. The nematic molecule is a rod-like molecule and the

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average direction of these rods is given by a vector $\boldsymbol{n}$ with fixed length, $\boldsymbol{n} = 1$. The distribution of the molecules is random, but there is orientation order. For this macroscopic continuum theory the molecules are like headless arrows, so $\boldsymbol{n}$ and $-\boldsymbol{n}$ have to be identified since they represent the same physical state [2,3]. Here we study nematic liquid crystals in two-dimensional systems. These are thin layers, or films, where the molecules align parallel with the surface layer, *i.e.* the director is confined to two dimensions.

The symmetry of possible rotations of the director is spontaneously broken, since the molecules tend to align themselves in parallel with one another. This is reflected in the form of the distortion free-energy density describing the statics of nematics. A ground state which minimizes this density is locally aligned (with possible exceptions at singular points or lines, also called defects) [2]. The most general form of the distortion free-energy density is

$$\phi_{\mathrm{d}} = \tfrac{1}{2}K_1[\nabla \cdot \boldsymbol{n}]^2 + \tfrac{1}{2}K_2[\boldsymbol{n} \cdot (\nabla \times \boldsymbol{n})]^2 + \tfrac{1}{2}K_3[\boldsymbol{n} \times (\nabla \times \boldsymbol{n})]^2 . \qquad (1)$$

The positive constants K_1, K_2, and K_3 are associated respectively with splay, twist, and bend deformation. For a classic nematic like *p*-azoxyanisol (PAA) at 120°C their values are $K_1 = 0.7 \times 10^6$ dyn, $K_2 = 0.43 \times 10^6$ dyn, $K_3 = 1.7 \times 10^6$ dyn [2]. As ϕ_d is even in $\boldsymbol{n}$, the states $\boldsymbol{n}$ and $-\boldsymbol{n}$ are indeed indistinguishable. We remark that in two dimensions the three kinds of deformation are not independent, since there we have

$$[\boldsymbol{n} \cdot (\nabla \times \boldsymbol{n})]^2 = [\nabla \cdot \boldsymbol{n}]^2 + [\boldsymbol{n} \times (\nabla \times \boldsymbol{n})]^2 .$$

For our purpose we use a simplified version of ϕ_{d} with enhanced symmetry, the one-constant approximation where one assumes $K_1 = K_2 = K_3 =: K$ [2]

$$\phi_1 = \tfrac{1}{2}K\, \partial_j n_l\, \partial_j n_l . \qquad (2)$$

(The convention of summing over two repeated indices is used throughout, unless mentioned otherwise.)

The ground state of the nematic is described by a direction or orientation. Consequently in the three dimensions the manifold of such states (known also as the vacuum manifold) has the topology of a sphere S_2. However, since $\boldsymbol{n}$ and $-\boldsymbol{n}$ are identified, this manifold is further restricted to the projective plane. In two dimensions the manifold is a circle with opposite points on a diameter identified, which is still homeomorphic to a circle, S_1 [4].

The possible topological defects are given by the homotopy groups of the vacuum manifold. For S_1 the only non-trivial homotopy group is $\pi_1(S_1)$ which determines the possible types of defect points (also known as disclinations). These points can be considered as strings, if we see our two-dimensional system as the cross-section of a three-dimensional one. Other

homotopy groups for systems with different vacuum manifolds determine walls, monopoles and textures; a brief introduction is given in [1]. $\pi_1(S_1)$ reflects the number of topologically distinct noncontractible loops on S_1. This homotopy group can be represented through integers, or more precisely through half-integers. Defect points are described mathematically like vortices, and in nematics they can have half-integer winding numbers because of the identification of $\boldsymbol{n}$ and $-\boldsymbol{n}$. Consequently the defect points in two-dimensional nematic liquid crystals are characterized by their winding numbers $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$. Schematic illustrations of director configurations and pictures from experiments, both for various types of defects, can be found *e.g.* in [3, 5, 2, 6].

To deal with the possible singularities of the director, it has been found useful to soften the constraint $\boldsymbol{n} = 1$. The length of the director is regulated instead by a potential term which is added to the energy density. We take the potential to be proportional to $(|\boldsymbol{n}|^2 - 1)^2$. This is minimized for $|\boldsymbol{n}| = 1$. So the ground states remain the same, and so does the vacuum manifold of the system.

In the past the introduction of gauge fields has been suggested and used to model systems with defects [7–11]. A gauge field can model screening when many vortices are present. Here we propose a new form of the distortion free-energy density which will be used later on to derive the equations of hydrodynamics, in the same way as ϕ_{d} was the basis for the Ericksen–Leslie equations which describe the hydrodynamics of nematics. It is recognized that the Ericksen–Leslie equations do not satisfactorily incorporate disclinations [2].

For two-dimensional liquid crystals the relevant gauge theory with the vacuum manifold S_1 is the Abelian–Higgs model. This is a relativistic model formulated for three space plus one time dimension. We take on just the static on space part. It gives us a generalized form of the one-constant approximation ϕ_1 in (2). When written in terms of the complex director $n = n_1 + in_2$ relating to the two-dimensional system, it has the form

$$\phi = \tfrac{1}{2} K D_j n \overline{D_j n} + \tfrac{1}{4}\lambda(|n|^2 - 1)^2 + \tfrac{1}{2}F^2 \,. \tag{3}$$

The ordinary derivatives in ϕ_1 are replaced with covariant ones, $D_j = \partial_j - igA_j$ and a term in the derivatives of the gauge field $\boldsymbol{A}$, the gauge field strength $F = \partial_1 A_2 - \partial_2 A_1$, is added. We also have the potential mentioned above. Kawasaki and Brand [11] claim to derive a free-energy density close to (3) from (1) by carefully separating the director into singular and non-singular pieces. These arguments are delicate, especially those, involving the number of the degrees of freedom of the system.

Whereas ϕ_1 is invariant under global rotations $n_j(\boldsymbol{r}) \mapsto R_{jk}\, n_k(\boldsymbol{r})$, with R_{jk} being a constant rotation matrix, ϕ is covariant, *i.e.* invariant under a

local rotation of the form

$$\begin{aligned} n(\boldsymbol{r}) &\mapsto e^{i\varphi(\boldsymbol{r})} n(\boldsymbol{r})\,, \\ A_j(\boldsymbol{r}) &\mapsto A_j(\boldsymbol{r}) + \tfrac{1}{g}\,\partial_j \varphi(\boldsymbol{r})\,. \end{aligned} \tag{4}$$

This is referred to as a gauge transformation determined by the space-dependent phase angle $\varphi(\boldsymbol{r})$.

The covariant energy density (3) implies that the interaction between disclinations is screened. Mathematically the decay of the force between two defects in a model without gauge fields is inversely proportional to the distance between defects [12]. In the Abelian–Higgs model the force falls off exponentially with the distance [13,14] so it has a weaker long range interaction. Even though very little is known about the dynamics of disclinations in nematics [6] it is still expected that a large number of them causes screening and that they influence dissipative flow properties. Our model proposes to take account of this by introducing the gauge field, which has also topological and geometrical significance.

2. Hydrodynamic equations for nematics with defects

We derive hydrodynamic equations to model nematics with defects based on the covariant energy density (3). We will follow the standard procedure of hydrodynamics based on the concept of the fluxes and forces of de Groot and Mazur [15]. This procedure was successful in deriving the usual dynamics of nematics. The resulting equations are referred to as the Ericksen–Leslie equations and the variables are the velocity field $\boldsymbol{v}$ and the director $\boldsymbol{n}$. On suppressing $\boldsymbol{n}$ they reduce to the Navier–Stokes equations. In our hydrodynamics for defects in addition to $\boldsymbol{v}$ and $\boldsymbol{n}$, the gauge field $\boldsymbol{A}$ is introduced. Consequently not only additional terms are introduced into the existing equations, but there is a new dissipative equation for $\boldsymbol{A}$.

In this section we will employ the notation $\boldsymbol{n}=(n_1, n_2)$ and $\boldsymbol{A}=(A_1, A_2)$ for the two-dimensional vector fields. The components are functions of the two-dimensional coordinate $\boldsymbol{r}$. Instead of the complex number i the covariant derivative contains the matrix $\boldsymbol{I}$ which is given by

$$\boldsymbol{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

($\boldsymbol{I}$ is the generator for the group of rotations in the plane). The covariant derivative is then $D_{j,kl} = \partial_j \delta_{kl} - g A_j I_{kl}$. Its action on $\boldsymbol{n}$ we shall also note in short as $(Dn)_{jk} = D_{j,kl} n_l$. This expression translates into components the complex one used in equation (3) by making the real and imaginary part

the first and second component in the second index in $(Dn)_{jk}$. Furthermore, the expression for the gauge field strength $F_{jk}=\partial_j A_k-\partial_k A_j$ is used instead of F. Our theory is dissipative and does not have the relativistic invariance of field theories. Consequently time and space components are not on an equal footing, and we are not forced to introduce an A_0 field. We will consider $A_0 = 0$ and time independent gauge transformations. The covariant form of the distortion free energy density for nematics from (3) can then be rewritten as

$$\phi = \tfrac{1}{2}K(Dn)_{jk}(Dn)_{jk} + \tfrac{1}{4}\lambda(n_j n_j - 1)^2 + \tfrac{1}{4}F_{jk}F_{jk}\,. \tag{5}$$

Physically the system is understood as a layer with a small but homogeneous thickness, and the vector fields are confined to the plane, as indicated above. The energy of the whole system is written as $\Phi = \int d^3x\,\phi$.

Instead of equation (4) the gauge transformation, under which the energy density is invariant, now looks like

$$\begin{aligned} n_j(\boldsymbol{r}) &\;\mapsto\; R_{jk}(\boldsymbol{r})n_k(\boldsymbol{r})\,, \\ A_j(\boldsymbol{r})\,I_{kl} &\;\mapsto\; A_j(\boldsymbol{r})\,I_{kl} + \tfrac{1}{g}\left(\partial_j R_{ki}(\boldsymbol{r})\right)R^{-1}_{il}(\boldsymbol{r})\,, \end{aligned} \tag{6}$$

where $R_{jk}(\boldsymbol{r})$ is a two-dimensional rotation matrix. As a consequence F_{jk} is again invariant and the covariant derivative of the director transforms like $(Dn)_{jk}(\boldsymbol{r}) \mapsto R_{kl}(\boldsymbol{r})\,(Dn)_{jl}(\boldsymbol{r})$.

The dynamics of nematics requires the following two variables appropriate for isotropic fluids in addition to the director $\boldsymbol{n}$(r) and the gauge field $\boldsymbol{A}$(r): the fluid velocity $\boldsymbol{v}(r) = (v_1(\boldsymbol{r}), v_2(\boldsymbol{r}))$ and the pressure $p(\boldsymbol{r})$ (discussed further below). The density ρ of the fluid is assumed to be constant. This implies the incompressibility condition

$$\partial_j v_j = 0\,. \tag{7}$$

The acceleration of the fluid is given by the derivative of the tensor of the total stress $\boldsymbol{\sigma}(\boldsymbol{r})$ (by Newton's law combined with the definition of the stress tensor in $df_i = d^2S_j\,\sigma_{ji}$, which the force df_i on a surface element d^2S_j).

$$\rho\,\partial_t v_j = \partial_l \sigma_{lj}\,. \tag{8}$$

The total stress is split into the elastic stress $\boldsymbol{\sigma}^{\mathrm{e}}$ (also called the Ericksen stress) and the viscous stress $\boldsymbol{\sigma}^{\mathrm{v}}$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{e}} + \boldsymbol{\sigma}^{\mathrm{v}}\,. \tag{9}$$

In addition to (7) and (8) the dynamics of nematics requires equations for the first-order time derivatives of the director and gauge field. These, together with the expressions for elastic and viscous stress as functions of the above mentioned variables, are derived in what follows.

2.1. The entropy source

From the expression of the entropy source for a flowing nematic liquid the hydrodynamic fluxes and forces are identified. For small fluxes the forces are linear in the fluxes. The aim is to find these linear equations or constitutive relations. Their proportionality factors are of the form of a viscosity coefficient multiplied by a monomial in the hydrodynamic variables. The viscosity coefficients need to be determined phenomenologically.

The dissipation (product of temperature and entropy source) is the decrease in stored free energy. Its contributions are the kinetic and distortion energy. Any external fields (electric, magnetic or gravity) are not taken into account.

$$T\dot{S} = -\frac{d}{dt}\int d^3x \left(\tfrac{1}{2}\rho\, v_j v_j + \phi\right) , \tag{10}$$

where S is the entropy.

To calculate the first term of the right hand side we use the fluid acceleration equation (8). On using integration by parts we obtain

$$-\frac{d}{dt}\int d^3x\, \rho\, v_j v_j = \int d^3x\, \sigma_{jk}\, \partial_j v_k + \text{surface terms}\,. \tag{11}$$

To calculate the second term of the right hand side in (10) we perform two types of variations of the distortion energy density. One is a variation of the energy density by changing the orientation of the director and the gauge field. The transformations are $\boldsymbol{n} \mapsto \boldsymbol{n}' + \delta\boldsymbol{n}$ and $\boldsymbol{A} \mapsto \boldsymbol{A}' + \delta\boldsymbol{A}$. Through partial integration we get for the energy

$$\delta\Phi = \int d^3x \left(-h_j^n\, \delta n_j - h_j^A\, \delta A_j\right) + \text{surface terms} \tag{12}$$

with

$$h_j^n \;=\; \partial_l \frac{\partial\phi}{\partial(\partial_l n_j)} - \frac{\partial\phi}{\partial n_j} = K(DDn)_{llj} - \lambda(n_l n_l - 1)n_j\,, \tag{13}$$

$$h_j^A \;=\; \partial_l \frac{\partial\phi}{\partial(\partial_l A_j)} - \frac{\partial\phi}{\partial A_j} = \partial_l F_{lj} - Kg(Dn)_{jk}\, I_{kl}\, n_l\,. \tag{14}$$

Usually, $\boldsymbol{h}^n$ is called the molecular field; here we mark it as related to the director $\boldsymbol{n}$. The additional expression which appears in our model has the analogous notation $\boldsymbol{h}^A$ and is called the molecular field related to the gauge field $\boldsymbol{A}$. The double application of the covariant derivative in (13) is explicitly given by $(DDn)_{lmj} = D_{l,jk}D_{m,ki}n_i$. We notice that the two molecular field are non-zero only out of equilibrium, as the Euler–Lagrange equations belonging to ϕ are $\boldsymbol{h}^n = 0$ and $\boldsymbol{h}^A = 0$.

The second variation of the energy density consists of displacing the liquid without changing the director orientation. The variation parameter is $\boldsymbol{u}(\boldsymbol{r})$ which gives the displacement of the coordinate $\boldsymbol{r}$ to $\boldsymbol{r'} = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r})$. The director transforms such that

$$\boldsymbol{n}(\boldsymbol{r}) \mapsto \boldsymbol{n'}(\boldsymbol{r'}) = \boldsymbol{n'}(\boldsymbol{r} + \boldsymbol{u}) = \boldsymbol{n}(\boldsymbol{r})\,. \tag{15}$$

The transformation of $\boldsymbol{r}$ to $\boldsymbol{r}$' causes the derivative operator to transform as $\partial_j \mapsto \partial'_j = (\delta_{jk} - \partial_j u_k(\boldsymbol{r}))\partial_k$. Gauge invariance is obtained by imposing the same transformation on the gauge field. All calculations are just up to linear order in $\boldsymbol{u}$. This gives the following results for the variation of the variables:

$$\begin{aligned}
\delta\,(\partial_j n_k) = \;& \partial'_j n'_k(\boldsymbol{r'}) - \partial_j n_k(\boldsymbol{r}) \;&&= -\partial_l n_k\,\partial_j u_l\,,\\
\delta A_j = \;& A'_j(\boldsymbol{r'}) - A_j(\boldsymbol{r}) \;&&= -A_l\,\partial_j u_l\,,\\
\delta\,(\partial_j A_k) = \;& \partial'_j A'_k(\boldsymbol{r'}) - \partial_j A_k(\boldsymbol{r}) \;&&= -\partial_l A_k\,\partial_j u_l - \partial_j A_l\,\partial_k u_l
\end{aligned} \tag{16}$$

and by using the anti-symmetry of F_{jk} in the term involving $\delta(\partial_j A_k)$, we get the variation of the energy. The result is written in terms of the elastic stress $\boldsymbol{\sigma}^{\mathrm{e}}$, which is the usual variable obtained from this kind of variation.

$$\delta\Phi = \int d^3x\,\sigma^{\mathrm{e}}_{jk}\,\partial_j u_k\,, \tag{17}$$

$$\sigma^{\mathrm{e}}_{jk} = -K(Dn)_{jl}(Dn)_{kl} - F_{jl}F_{kl} - p\,\delta_{jk}\,. \tag{18}$$

The function $p = p(\boldsymbol{r})$ is a Lagrange multiplier that enforces the incompressibility condition (7) on the displacement given by $\boldsymbol{u}(\boldsymbol{r})$. It is called pressure and has the negative sign reflecting that generally the stress tensor $\boldsymbol{\sigma}$ for an isotropic fluid without friction reduces to $\sigma_{jk} = -\delta_{jk}p$ where p is the ordinary pressure.

The total variation is obtained by adding (12) and (17). This gives the total time derivative

$$-\frac{d}{dt}\int d^3x\,\phi = \int d^3x\,\left(-\sigma^{\mathrm{e}}_{jk}\,\partial_j v_k + h^n_j\,\dot{n}_j + h^A_j\,\dot{A}_j\right) + \text{surface terms}\,, \tag{19}$$

where $\dot{\boldsymbol{u}}$ was replaced by $\boldsymbol{v}$, $\dot{\boldsymbol{n}}$ is understood as the material time derivative, the rate of change of the director as experienced by a moving molecule, and is written in terms of covariant quantities

$$\dot{n}_j = (Dn)_{0j} + v_l(Dn)_{lj}\,. \tag{20}$$

Analogously $\dot{\boldsymbol{A}}$ is defined by

$$\dot{A}_j = F_{0j} + v_l\,F_{lj}\,. \tag{21}$$

Under gauge transformations $\dot{\boldsymbol{n}}$ behaves like $\boldsymbol{n}$, and $\dot{\boldsymbol{A}}$ is invariant.

Finally we get the entropy source by putting (11) and (19) into equation (10)

$$T\dot{S} = \int d^3x \left[\sigma^{\mathrm{v}}_{jk}\,\partial_j v_k + h^n_j\,\dot{n}_j + h^A_j\,\dot{A}_j\right] + \text{surface terms}\,, \tag{22}$$

where equation (9) was used to replace the total and elastic stress by the viscous stress.

2.2. The balance of torques

From the entropy source (22) the hydrodynamic fluxes and forces could be identified already. However we continue to follow de Gennes' derivation and rewrite it further by splitting the viscous stress into a symmetric and an anti-symmetric part.

$$\sigma^{\mathrm{vs}}_{jk} = \tfrac{1}{2}\left(\sigma^{\mathrm{v}}_{jk} + \sigma^{\mathrm{v}}_{jk}\right)\,, \tag{23}$$

$$\Gamma_j = -\varepsilon_{jkl}\,\sigma^{\mathrm{v}}_{kl}\,, \tag{24}$$

where ε_{jkl} is Levi–Civita's anti-symmetric symbol. The same is done for the tensor formed by the derivative of the velocity

$$s_{jk} = \tfrac{1}{2}\left(\partial_j v_k + \partial_k v_j\right)\,, \tag{25}$$

$$\omega_j = \tfrac{1}{2}\varepsilon_{jkl}\,\partial_k v_l\,. \tag{26}$$

(For the cross products the variables like $\boldsymbol{v}$ and $\boldsymbol{\sigma}^{\mathrm{v}}$ are seen as three-dimensional where the components relating to the third dimension are zero. As a result $\boldsymbol{\Gamma}$ and $\boldsymbol{\omega}$ are perpendicular to the plane of our two-dimensional system.) This allows us to write

$$\sigma^{\mathrm{v}}_{jk}\,\partial_j v_k = \sigma^{\mathrm{vs}}_{jk}\,s_{jk} - \Gamma_j\,\omega_j\,. \tag{27}$$

We will now show that $\boldsymbol{\Gamma}$ may be identified as the torque density and can be written as $\boldsymbol{\Gamma} = \boldsymbol{n} \times \boldsymbol{h}^n$. To further this aim the first step is a variation of the energy density for a constant director rotation around the axis parallel to $\delta\boldsymbol{\omega}$ with a small rotation angle $|\delta\boldsymbol{\omega}|$

$$\boldsymbol{n} \mapsto \boldsymbol{n} + \delta\boldsymbol{\omega} \times \boldsymbol{n}\,. \tag{28}$$

This is a gauge rotation as given in equation (6), but with a constant rotation matrix, and so the corresponding variation of the gauge field is zero. By separating into volume and surface terms and using the molecular field $\boldsymbol{h}^n$

of (13), we get a change of the energy $\Phi_V = \int_V d^3x\,\phi$ of any part V of our system given by

$$\delta\Phi_V = \left[-\int\limits_V d^3x\, h_j^n\, \varepsilon_{jkl}\, n_l + \int\limits_{\partial V} d^2S_i\, \frac{\partial\phi}{\partial(\partial_i n_j)}\, \varepsilon_{jkl}\, n_l \right] \delta\omega_k \tag{29}$$

due to the variation. The second integral inside the bracket is the torque on the director on the surface. Since the energy density is invariant under the gauge rotation (28), we have $\delta\Phi_V = 0$, and the torque on the director can be replaced by the first integral inside the bracket. The second contribution to the torque comes from the stress on the surface through $\int \varepsilon_{jkl}\, r_k\, df_l = \int \varepsilon_{jkl}\, r_k \sigma_{il}\, d^2S_i$, and so the total torque on the volume V is

$$T_j = \int\limits_V d^3x\, \varepsilon_{jkl}\, n_k\, h_l^n + \int\limits_{\partial V} \varepsilon_{jkl}\, r_k \sigma_{il}\, d^2S_i\,. \tag{30}$$

On the other hand the torque is given by the rate of change of the angular momentum $\boldsymbol{L}$, *i.e.* $\dot{\boldsymbol{L}} = \boldsymbol{T}$. It is calculated from $\dot{\boldsymbol{L}} = \int_V d^3x\, \rho\, \boldsymbol{r} \times \dot{\boldsymbol{v}}$ by putting in the total stress from (8), integrating by parts and by using the equations (9) and (24) to rewrite the volume integral. We then get

$$\dot{L}_j = \int\limits_{\partial V} \varepsilon_{jkl}\, r_k \sigma_{il}\, d^2S_i + \int\limits_V d^3x\, \left(\Gamma_j - \varepsilon_{jkl}\, \sigma^{\mathrm{e}}_{kl} \right)\,. \tag{31}$$

The term in $\boldsymbol{\sigma}^{\mathrm{e}}$ is zero. This is based on the fact that the energy density is invariant, *i.e.* $\delta\Phi_V = 0$, under coordinate rotation $\boldsymbol{r} \mapsto r + \delta\boldsymbol{\omega} \times r$ (without changing the director) around the axis $\delta\boldsymbol{\omega}$ with a small rotation angle. This variation is the same as that carried out in equations (15) to (18) with $\boldsymbol{u}(r) = \boldsymbol{\omega} \times r$. (It is also valid for any part V of the system.) Hence $\delta\Phi_V$ can be obtained from (17) where $\partial_j u_k = \varepsilon_{jkl}\, \omega_l$.

Comparing equations (30) and (31) and using the fact that they are valid for any volume V within the system, we can conclude that

$$\boldsymbol{\Gamma} = \boldsymbol{n} \times \boldsymbol{h}^n\,. \tag{32}$$

By using equation (27) and (32) in (22), the new form of the entropy source is obtained as

$$T\dot{S} = \int d^3x \left[\sigma^{\mathrm{vs}}_{jk}\, s_{jk} + h_j^n\, N_j + h_j^A\, \dot{A}_j \right] + \text{surface terms}\,, \tag{33}$$

where we have introduced the vector $\boldsymbol{N}$, the rate of change of the director with respect to the background fluid

$$\boldsymbol{N} = \dot{\boldsymbol{n}} - \boldsymbol{\omega} \times \boldsymbol{n}\,. \tag{34}$$

Since $\boldsymbol{\omega}$ is perpendicular to the plane of the film containing $\boldsymbol{n}$, $\boldsymbol{\omega}\times\boldsymbol{n}$ behaves in the same way as $\boldsymbol{n}$ under gauge rotation (and so does $\dot{\boldsymbol{n}}$); so $\boldsymbol{N}$ rotates like $\boldsymbol{n}$ for a change of gauge.

The entropy source (33) displays three types of dissipation: dissipation by shear flow, by rotation of the director with respect to the background fluid, and by change of the gauge field.

2.3. The linear equations between fluxes and forces

In the entropy source (33) each contribution is identified as a product of a flux with the conjugate force. We choose them in the same way as de Gennes [2]: s_{ii}, s_{12}, $\boldsymbol{N}$, and $\dot{\boldsymbol{A}}$ as hydrodynamic fluxes conjugate to the forces $\sigma^{\mathrm{vs}}_{ii}$, $2\sigma^{\mathrm{vs}}_{12}$, $\boldsymbol{h}^n$, and $\boldsymbol{h}^A$ respectively (the factor 2 reflects that $\sigma^{\mathrm{vs}}_{12}\,s_{12}$ occurs twice). In the limit of weak fluxes, the forces are linear functions of the fluxes. We can express this in the form

$$\begin{pmatrix} \sigma^{\mathrm{vs}}_{ii} \\ 2\sigma^{\mathrm{vs}}_{12} \\ h^n_j \\ h^A_k \end{pmatrix} = \begin{pmatrix} \cdots \\ \text{constituent} \\ \text{matrix} \\ \cdots \end{pmatrix} \begin{pmatrix} s_{ll} \\ s_{12} \\ N_p \\ \dot{A}_q \end{pmatrix} . \tag{35}$$

A priori this matrix is a tensor constructed from the variables of the problem. This gives the matrix a large number of possible terms. However, certain conditions restrict the form of the matrix. We look at the behaviour of the fluxes and forces under time reversal. From equation (8) and (18) it can be seen that the components of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\mathrm{e}}$ are even under time reversal, and so $\boldsymbol{\sigma}^{\mathrm{v}}$ is even too. The same goes for the other hydrodynamic forces, whereas all the fluxes are odd. The Onsager reciprocal relations [15] then imply that the constituent matrix is symmetric. We demand that the matrix be compatible with the local symmetry of uniaxial nematics ($\mathrm{D}_{\infty h}$). This excludes derivatives, the gauge field and certain tensor types from the matrix. Furthermore the matrix elements need to be such that the equations are invariant under director inversion $\boldsymbol{n}\mapsto-\boldsymbol{n}$, which is essential for nematics. (This inversion leaves the gauge field unchanged, since $\boldsymbol{n}\mapsto-\boldsymbol{n}$ corresponds to a gauge transformation with constant phase angle $\varphi=\pi$.) The only scalar invariant is $|\boldsymbol{n}|^2$, but outside the core of a disclination it is of value one and near the defect centre it is small; like in the Ericksen–Leslie theory it will not be used in the constituent matrix.

With these conditions (35) translates into the following equations

$$\begin{aligned} \sigma^{\mathrm{vs}}_{ij} \;=\; & \alpha_1\, n_i n_j n_k n_l s_{kl} + \tfrac{1}{2}\gamma_2\,(n_i N_j + n_j N_i) \\ & + \alpha_4\, s_{ij} + \tfrac{1}{2}\,(\alpha_5+\alpha_6)\,(n_i s_{kj} + n_j s_{ki})\, n_k\,, \end{aligned} \tag{36}$$

$$h_i^n = \gamma_1 N_i + \gamma_2 n_j s_{ji} \,, \tag{37}$$
$$h_i^A = \beta_1 \dot{A}_1 + \beta_2 n_i n_j \dot{A}_j \,. \tag{38}$$

The viscosity coefficients α_i in $\boldsymbol{\sigma}^{\rm vs}$ are set up with the result for $\boldsymbol{\sigma}^{\rm v}$ in mind. From equations (23) and (24) we know that $\sigma_{ij}^{\rm v} = \sigma_{ij}^{\rm vs} - \frac{1}{2}\varepsilon_{ijk}\Gamma_k$. Hence with the help of equations (36), (32), and (37) we obtain the total viscous stress

$$\begin{aligned}\sigma_{ij}^{\rm v} = {} & \alpha_1\, n_i n_j n_k n_l s_{kl} + \alpha_2\, n_i N_j + \alpha_3\, n_j N_i \\ & + \alpha_4\, s_{ij} + \alpha_5\, n_i n_k s_{kj} + \alpha_6\, n_j n_k s_{ki} \,,\end{aligned} \tag{39}$$

where α_2 and α_3 are defined such that $\gamma_1 = \alpha_3 - \alpha_2$ and $\gamma_2 = \alpha_3 + \alpha_2$. (In addition we have the relation $\gamma_2 = \alpha_6 - \alpha_5$ [2].) The coefficients α_i and γ_i are viscosities known from the Ericksen–Leslie theory [2]. The coefficients β_i, related to the dissipation of the gauge field, are new and have been introduced by us.

In summary we have the hydrodynamics of nematic liquid crystals with defects given by the three dissipative equations (8), (37), (38), corresponding to the three hydrodynamic variables, the velocity field $\boldsymbol{v}$, the director field $\boldsymbol{n}$, and the gauge field $\boldsymbol{A}$. In addition we have the incompressibility condition (7) giving rise to the pressure variable p. To specify the three hydrodynamic equations in detail we need the total stress $\boldsymbol{\sigma}$ in (9), the elastic stress $\boldsymbol{\sigma}^{\rm e}$ in (18), the viscous stress $\boldsymbol{\sigma}^{\rm v}$ in (39), the rate of change of the director $\boldsymbol{N}$ in (34), the material time derivatives $\dot{\boldsymbol{n}}$ and $\dot{\boldsymbol{A}}$ in (20) and (21), and the molecular fields $\boldsymbol{h}^n$ and $\boldsymbol{h}^A$ in (13) and (14).

If the gauge field is zero, the original form of the Ericksen–Leslie equations is obtained for the director and velocity field from (8) and (37), together with the incompressibility condition. Since the original equations due to the dissipation were not invariant under a global rotation, the new equations are not invariant under gauge transformation. If we require gauge invariance we can keep only the diagonal terms in the constituent matrix in (35). Instead of (37) and (38) the dissipative equations for the director and gauge field become

$$h_i^n = \gamma_1 N_i \,, \tag{40}$$
$$h_i^A = \beta_1 \dot{A}_i \tag{41}$$

and the viscous stress from (39) reduces to

$$\sigma_{ij}^{\rm v} = \tfrac{1}{2}\gamma_1 \left(n_j N_i - n_i N_j\right) + \alpha_4\, s_{ij} \,. \tag{42}$$

(The γ_1 term is invariant under gauge rotation because we work in only two dimensions.)

In a first attempt to solve these equations we truncated the system by assuming zero velocity, *i.e.* the configuration of the system changes through reorientation of the molecules without displacing them. We then have to consider just the two dissipative equations (40) and (41), and this is a simplification used earlier [16].

3. Hydrodynamics of defects in the Abelian–Higgs model

In this section we summarize very briefly how the general equations for the hydrodynamics of nematics with defects derived in the previous section are used to find the dynamics of disclination points in a simplified system with zero velocity field. A detailed account of the related calculations and results can be found in [17].

First we consider the distortion free-energy density (3). For a certain value of the gauge coupling g, the Bogomol'nyi limit, a configuration of disclinations with winding numbers of the same sign is a static solution. In general, of course, several defects are not a static solution, and there is a force between them which makes them move. However, if the coupling deviates only slightly from the Bogomol'nyi limit we can still assume that the configuration takes on the form of a static solution at every instant. Such a configuration change over time is known as quasi-static motion and is induced by the force due to the deviation from the Bogomol'nyi limit. A solution with defects in the Bogomol'nyi limit can be parameterized by the defect positions [18]. The motion of the configuration is then given equivalently by the motion of the defect positions in the space of possible static solutions (also known as moduli space). The concept of quasi-static motion was developed by Manton [19] and has been applied by Samols and Dziarmaga [20–22].

Physically a defect is a macroscopic region with a certain topological charge. Mathematically this is modelled as a topological defect point with a core region and a well defined centre point (providing the defect position coordinate). If the distance between the defects is large compared to the core size, a static multi-defect solution in the Bogomol'nyi limit is found to a good approximation by superposition of solutions with single defects in different positions. A static solution for several defects with overlapping core regions is modelled by a single defect solution which is modified such that it allows for disintegration of this defect into smaller ones.

In both cases, for large distances or overlapping cores, the static solution in the Bogomol'nyi limit can be parameterized by the defect positions. Their dynamics is given by the field equations (40) and (41). These are reduced to equations just for the defect positions by projection on the moduli space. The moduli space is spanned by zero-modes of the linear operator given by

linearization of equations (40) and (41) in the field variables $\boldsymbol{n}$ and $\boldsymbol{A}$. It turns out that the zero-modes are derivatives of the defect solutions with respect to the defect coordinates. The result of the projection is a set of equations in the defect positions and their first time derivatives.

We derive and solve these equations for both cases mentioned above. For defects with overlapping cores they indicate that an unstable disclination with higher winding number can disintegrate into smaller ones which move away from one another radially with exponentially increasing speed (as long as their core regions overlap). Two disclination far apart from one another move on a straight line, where their distance increases logarithmically with time. Disclinations with smaller winding number have faster motion.

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