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INFORMATION GEOMETRY, ONE, TWO, THREE (AND FOUR)*

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Although the notion of entropy lies at the core of statistical mechanics, it is not often used in statistical mechanical models to characterize phase transitions, a role more usually played by quantities such as various order parameters, specific heats or susceptibilities. The relative entropy induces a metric, the so-called information or Fisher–Rao metric, on the space of parameters and the geometrical invariants of this metric carry information about the phase structure of the model. In various models the scalar curvature, $\mathcal{R}$, of the information metric has been found to diverge at the phase transition point and a plausible scaling relation postulated. For spin models the necessity of calculating in non-zero field has limited analytic consideration to one-dimensional, mean-field and Bethe lattice Ising models. We extend the list somewhat in the current note by considering the *one*-dimensional Potts model, the *two*-dimensional Ising model coupled to two-dimensional quantum gravity and the *three*-dimensional spherical model. We note that similar ideas have been applied to elucidate possible critical behaviour in families of black hole solutions in *four* space-time dimensions.

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1. Introduction

The notion of entropy,

$$S = -\sum_{\alpha} p_\alpha \ln\left(p_\alpha\right), \tag{1}$$

lies at the heart of statistical mechanics where p_α is the probability of being in state α. It gives a measure of the number of microstates Ω accessible to a system $S = \ln \Omega$. It is also possible to define a relative entropy with respect to some reference set of configurations of probability r_α (such as those at the critical point or zero temperature),

$$G(p|r) = -\sum_{\alpha} p_\alpha \ln\left(\frac{p_\alpha}{r_\alpha}\right). \tag{2}$$

If θ represents parameters characterizing the class of models under consideration, the relative entropy, G, induces a metric for two "close" configurations θ and $\theta + \delta\theta$:

$$dl^2 = G\left(p(\theta)|p(\theta+\delta\theta)\right). \tag{3}$$

Since $G(p|p) = 0$ and $\partial G/\partial\theta = 0$ we find

$$dl^2 = \frac{\partial^2 G}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j\,. \tag{4}$$

If we assume a Gibbs-type distribution for the p_α,

$$p_\alpha = \frac{1}{Z}\exp(-\theta_i \Phi_i) \;\longrightarrow S = \langle \theta_i \Phi_i \rangle + \ln Z\,, \tag{5}$$

then the metric may be written as

$$dl^2 = \frac{\partial^2 \ln Z}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j\,. \tag{6}$$

This metric is well known in statistics as the Fisher–Rao metric, and characterizes the "closeness" of probability distributions [1]. For a one-parameter distribution the expectation value is the Fisher information, $I(\theta)$, given by

$$I(\theta) = \left\langle -\frac{\partial^2}{\partial\theta^2}\ln Z \right\rangle, \tag{7}$$

and is inversely related to the variance: $I(\theta)\mathrm{Var}(\theta) = 1$. In the multiparameter case this generalizes in an obvious manner to the covariance matrix.

Various authors [2–4] have observed that the geometric invariants of the metric, in particular the curvature, $\mathcal{R}$, might be used to characterize the phase structure of statistical mechanical models. The first observation to note is that a non-interacting model, the ideal gas, has a flat geometry since $\mathcal{R}_{\rm ideal} = 0$. The suggestion was then that the curvature was a measure of the strength of interaction, an observation confirmed by a calculation in the case of a van der Waals gas (which is interacting):

$$\mathcal{R}_{\rm vdW} = \frac{4}{3}\frac{\alpha\beta}{\bar{v}}\frac{F(\alpha,\beta)}{D(\alpha,\beta)^2}\,, \tag{8}$$

where β is the inverse temperature, α the pressure, and $\bar{v}(\alpha,\beta)$ and $F(\alpha,\beta)$ are expressions appearing in the equation of state. This shows that the curvature diverges along the spinodal line where $D(\alpha,\beta) = 0$.

These examples suggest that phase transitions are manifested as divergences in the scalar curvature associated with the Fisher–Rao metric, at least in two-parameter models. A natural question to ask is whether any of the standard scaling exponents associated with a transition can then be extracted from the behaviour of the curvature. This is best pursued by consideration of some specific examples. If we consider a spin model in field, the manifold of parameters, $\mathcal{M}$, is two-dimensional and parametrised by $(\theta^1,\theta^2) = (\beta,h)$. In this case, the components of the Fisher–Rao metric take the particularly simple form

$$G_{ij} = \partial_i\partial_j f\,, \tag{9}$$

where f is the reduced free energy per site and $\partial_i = \partial/\partial\theta^i$. With the metric given in Eq. (9), $\mathcal{R}$ may be calculated succinctly as

$$\mathcal{R} \;=\; -\frac{1}{2G^2}\begin{vmatrix} \partial_\beta^2 f & \partial_\beta\partial_h f & \partial_h^2 f \\ \partial_\beta^3 f & \partial_\beta^2\partial_h f & \partial_\beta\partial_h^2 f \\ \partial_\beta^2\partial_h f & \partial_\beta\partial_h^2 f & \partial_h^3 f \end{vmatrix}\,, \tag{10}$$

where $G = \det(G_{ij})$. Since the only scale present near criticality for a model displaying a higher-order transition is the correlation length, ξ, it has been hypothesised on dimensional grounds that $\mathcal{R} \sim \xi^d$, where d is the dimensionality of the system [2–4]. If we assume that hyperscaling holds, $\nu d = 2 - \alpha$, this leads to

$$\mathcal{R} \sim |\xi|^{(2-\alpha)/\nu}. \tag{11}$$

To test the behaviour of $\mathcal{R}$, one requires models which are solvable in field in order to obtain analytic expressions, and these are rather thin on the ground. Indeed, $\mathcal{R}$ has been calculated for the mean-field and Bethe-lattice Ising models [5] and the above scaling behaviour verified. It has also been calculated for the one-dimensional Ising model [3] where it takes the remarkably simple form

$$\mathcal{R}_{\rm Ising} = 1 + \frac{\cosh h}{\sqrt{\sinh^2 h + {\rm e}^{-4\beta}}} \,. \tag{12}$$

In this case $\mathcal{R}$ is positive definite and diverges only at the zero-temperature, zero-field "critical point" of the model. The correlation length is given by

$$\xi^{-1} = -\ln\left(\tanh(\beta)\right), \tag{13}$$

so that $\xi \sim \exp(2\beta)$ near criticality, and (11) holds there with $\alpha = 1, \nu = 1$ as expected[1].

Given this rather short list of examples, any further additions would be very worthwhile in order to see which features are generic and which are particular to the models concerned. We expand the list incrementally here, looking at the *one*-dimensional Potts model, the *two*-dimensional Ising model coupled to two-dimensional quantum gravity and the *three*-dimensional spherical model, which shares the critical exponents of the just mentioned coupled Ising model. We also discuss the application of similar ideas to critical behaviour in various families of black holes. The present paper is essentially an amalgam of results presented in [6–8], with some added discussion of related work in black hole physics.

2. One

The partition function for the 1D q-state Potts model is given by

$$Z_N(y,z) = \sum_{\{\sigma\}} \exp\left[\beta \sum_{j=1}^{N}\left(\delta(\sigma_j,\sigma_{j+1}) - \frac{1}{q}\right) + h\sum_{j=1}^{N}\left(\delta(\sigma_j,1) - \frac{1}{q}\right)\right], \tag{14}$$

where the spins, $\sigma_j \in \{1,\ldots,q\}$, are defined on the sites, $j \in \{1,\ldots,N\}$, of the lattice and where we have defined $y = \exp(\beta)$ and $z = \exp(h)$ for later calculational convenience. The model may be solved by transfer matrix methods [9], just as the 1D Ising model. For general q the full transfer matrix $T(y,z)$ may be written as $q-2$ diagonal elements, $(y-1)(yz)^{-1/q}$, and a 2×2 factor $t(y,z)$:

$$t(y,z) = \frac{1}{(yz)^{1/q}} \begin{pmatrix} yz & z^{1/2}\sqrt{q-1} \\ z^{1/2}\sqrt{q-1} & y+q-2 \end{pmatrix}. \tag{15}$$

[1] The Bethe lattice model also satisfies the postulated scaling, although there are some subtleties coming from the exponent α being zero [5].

The partition function is $Z_N(y,z) = \mathrm{Tr}\, T(y,z)^N$ and the eigenvalues of $T(y,z)$ are $\lambda_0, \lambda_1, \ldots, \lambda_{q-1}$, where

$$\left.\begin{array}{c}\lambda_0\\ \lambda_1\end{array}\right\} = \frac{1}{2}\left[y(1+z)+q-2 \pm \sqrt{(y(1-z)+q-2)^2+(q-1)4z}\right](yz)^{-1/q} \tag{16}$$

and $\lambda_2, \ldots, \lambda_{q-1} = (y-1)(yz)^{-1/q}$. The reduced free energy per site in the thermodynamic limit, $N \to \infty$, is thus given by $f = -\ln \lambda_0$.

It is straightforward to use this expression for the free energy in Eq. (10) to obtain the curvature, $\mathcal{R}$. In the current notation we re-derive the expression for the Ising model[2] as

$$\mathcal{R}_{\mathrm{Ising}} = 1 + \frac{y(1+z)}{\sqrt{y^2 - 2y^2 z + y^2 z^2 + 4z}}\,. \tag{17}$$

The expression for general q is similar in form to this, and is

$$\mathcal{R}_{\mathrm{Potts}} = A(q,y,z) + \frac{B(q,y,z)}{\sqrt{\eta(q,y,z)}}\,, \tag{18}$$

where the coefficients $A(q,y,z)$ and $B(q,y,z)$ are smooth functions of y and z and do not diverge for finite (physical) temperature or field. Furthermore

$$\eta(q,y,z) = [y(1-z)+q-2]^2 + (q-1)4z\,. \tag{19}$$

The expressions for $A(q,y,z)$ and $B(q,y,z)$ are very lengthy for general q (although still easily obtained) so we do not reproduce them here.

In zero-field ($z=1$) the expression for $\mathcal{R}$ is much more compact and is written for general q as

$$\mathcal{R} = \frac{(y+q-1)(4y^2 + (q-2)y - (q-2)(q-1))}{(q-1)(2y+q-2)^2}\,. \tag{20}$$

We see that as y ranges from 1 to ∞, $\mathcal{R}$ ranges from $(4-q)/(q-1)$ to ∞. In particular, the sign of the $y=1$ ($\beta=0$) limit of $\mathcal{R}$ changes at $q=4$, although the general morphology of $\mathcal{R}$ as a function of y and z remains the same for all $q>2$ as we shall see below.

The correlation length for the one-dimensional Potts model is defined in a similar manner to that of the Ising model,

$$\xi^{-1} = -\ln\left(\frac{\lambda_1}{\lambda_0}\right), \tag{21}$$

[2] There is a factor of two difference in the definitions of β, h between the Ising and Potts notations coming from the different spin definitions.

so $\xi \sim y$ for $z = 1, y \to \infty$. We thus retrieve the scaling of $\mathcal{R}$ for the one-dimensional Potts model expected from Eq. (11), namely $\mathcal{R} \sim y$ as $y \to \infty$. The exponents, as for the Ising model, are $\alpha = 1, \nu = 1$.

The general features of $\mathcal{R}$ at non-zero temperature and field are perhaps easiest seen in a contour plot as a function of y and z. In Fig. 1 we show the Ising ($q = 2$) case which has certain non-generic features. The $\pm h$ symmetry of the Ising model appears as a $z \to 1/z$ symmetry in the plot of $\mathcal{R}$. In addition, one can see that $\mathcal{R}$ is positive for all y and z. The maximum of $\mathcal{R}$ for a given y value lies along the zero field line at $z = 1$. In Fig. 2, $\mathcal{R}$ is plotted for the 3-state Potts model, We see that there is no longer a $z \to 1/z$ symmetry and that $\mathcal{R}$ is no longer positive definite.

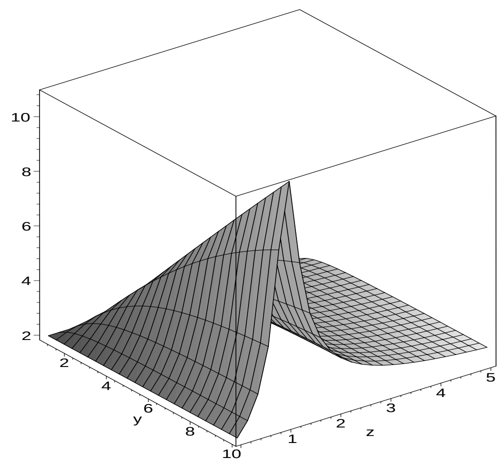

Fig. 1. A plot of $\mathcal{R}$ for the one-dimensional Ising model for $y = 1 \ldots 10$, $z = 0 \ldots 5$. The positivity of $\mathcal{R}$ and the expected $z \to 1/z$ symmetry are both apparent.

Some years ago Lee and Yang [10] addressed the question of how the singularities associated with field-driven phase transitions in Ising-like spin models on lattices arose in the thermodynamic limit. This was later extended by various authors to other models and to temperature-driven transitions [11,12]. Lee and Yang observed that the zeroes of the partition function for a spin model in a complex external field on a finite lattice would give rise to singularities in the free energy. In the thermodynamic limit these complex zeroes move in to pinch the real axis, signalling the onset of a physical phase transition. Typically, the loci of zeroes are lines in the complex field or temperature plane and when the endpoints of such lines occur at non-physical (*i.e.* complex) external field values they can be considered as ordinary critical points with an associated edge critical exponent, usually dubbed the Lee–Yang edge exponent [11].

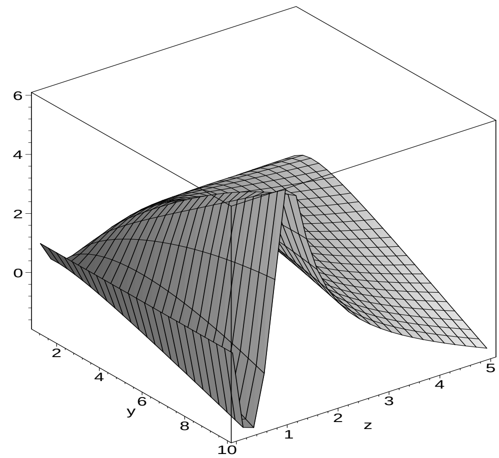

Fig. 2. A plot of $\mathcal{R}$ for the one-dimensional 3-state Potts model for $y = 1 \ldots 10$, $z = 0 \ldots 5$. $\mathcal{R}$ is no longer positive definite for physical values of y, z and there is no $z \to 1/z$ symmetry. In addition the maximum of $\mathcal{R}$ does not lie at $z = 1$.

The Lee–Yang zeroes for the one-dimensional Potts model on a periodic chain with N sites are given by the solutions [9, 13] of

$$Z_N = (\lambda_1)^N + (\lambda_0)^N = 0\,, \qquad \Leftrightarrow \qquad \lambda_1 = \exp\left(\frac{in\pi}{N}\right)\lambda_0\,, \tag{22}$$

where $\lambda_{0,1}$ are the eigenvalues given in Eq. (16) and n is odd. In the thermodynamic limit the locus of zeroes is determined by $|\lambda_0| = |\lambda_1|$ or

$$\eta(q, y, z) = [y(1-z) + q - 2]^2 + (q-1)4z = 0\,, \tag{23}$$

which can be satisfied for complex (in the $q = 2$ Ising case, purely imaginary) values of h.

From these considerations, it is clear that $\mathcal{R}$ will also diverge as the locus of zeroes is approached for both Ising and Potts models if we allow ourselves the liberty of an analytic continuation of the field to complex h values once $\mathcal{R}$ has been calculated, since $\mathcal{R} = A + B/\sqrt{\eta}$ and A, B are finite as $\eta \to 0$. The presence of the square root means that the divergence is characterised by an exponent $\sigma = -1/2$ which is the Lee–Yang edge exponent for the one-dimensional Potts (and Ising) model [11].

The status of these observations is a little unclear to us, since the calculation of $\mathcal{R}$ has assumed a real metric geometry throughout and such an arbitrary continuation in the final expression might be rather dangerous. It is nonetheless interesting that the Lee–Yang edge transition is still visible as a divergence in $\mathcal{R}$.

3. Two

The solution of the Ising model on an ensemble of Φ^4 (4-regular) or Φ^3 (3-regular) planar random graphs was first presented by Boulatov and Kazakov [14], who were motivated by string-theoretic considerations, since the continuum limit of the theory represents matter coupled to 2D quantum gravity. They considered the partition function for the Ising model on a single n vertex planar graph with connectivity matrix G^n_{ij}:

$$Z_{\mathrm{single}}(G^n,\beta,h)=\sum_{\{\sigma\}}\exp\left(\beta\sum_{\langle i,j\rangle}G^n_{ij}\sigma_i\sigma_j+h\sum_i\sigma_i\right), \tag{24}$$

then summed it over all n vertex graphs $\{G^n\}$ resulting in

$$Z_n=\sum_{\{G^n\}}Z_{\mathrm{single}}(G^n,\beta,h)\,, \tag{25}$$

before finally forming the grand-canonical sum over graphs with different numbers n of vertices

$$\mathcal{Z}=\sum_{n=1}^{\infty}\left(\frac{-4gc}{(1-c^2)^2}\right)^n Z_n\,, \tag{26}$$

where $c=\exp(-2\beta)$. This last expression could be calculated exactly as matrix integral over $N\times N$ Hermitian matrices,

$$\begin{aligned}\mathcal{Z}\;=\;&-\ln\int\mathcal{D}\phi_1\;\mathcal{D}\phi_2\;\exp\left(-\mathrm{Tr}\left[\frac{1}{2}(\phi_1^2+\phi_2^2)\right.\right.\\&\left.\left.-\;c\phi_1\phi_2-\frac{g}{4}\left(\mathrm{e}^h\phi_1^4+\mathrm{e}^{-h}\phi_2^4\right)\right]\right),\end{aligned} \tag{27}$$

where the $N\to\infty$ limit is to be taken to pick out the planar diagrams and the potential appropriate for Φ^4 (4-regular) random graphs has been shown.

When the matrix integral is carried out the solution is given in parametric form by

$$\mathcal{Z}=\frac{1}{2}\ln\left(\frac{z}{g}\right)-\frac{1}{g}\int_0^z\frac{dt}{t}g(t)+\frac{1}{2g^2}\int_0^z\frac{dt}{t}g(t)^2\,, \tag{28}$$

where the function $g(z)$ is

$$g(z)=\frac{1}{9}c^2z^3+\frac{z}{3}\left[\frac{1}{(1-z)^2}-c^2+\frac{zB}{(1-z^2)^2}\right] \tag{29}$$

and $B=2[\cosh(h)-1]$.

In the thermodynamic limit the free energy per site is given by

$$f = \ln\left(\frac{-4cg(z_0)}{(1-c^2)^2}\right), \tag{30}$$

where $z_0 = z_0(\beta, h)$ is the appropriate low- or high-temperature solution of $g'(z) = 0$. When $h = 0$ this may be solved in closed form, and although the solution is not available explicitly for non-zero h it can still be developed as a power series in h around the zero-field solutions in order to obtain expressions for quantities such as the energy, specific heat, magnetization and susceptibility. It was found that the critical exponents are given by $\alpha = -1$, $\beta = 1/2$, $\gamma = 2$, so the transition was *third* order with, intriguingly, the same exponents as the 3D spherical model on a regular lattice [15].

If we carry out a perturbative expansion for the high-temperature solution, which is symmetric in h and hence a series in even powers, we find

$$\begin{aligned} z_0 &= 1 - \frac{1}{u} - \frac{(u-1)(2u^2-2u+1)}{(2u-1)^4}h^2 \\ &+ \frac{(u-1)(2u^2-2u+1)(4u^5-10u^4+10u^3-5u^2+5u+1)}{24(2u-1)^9}h^4 + \dots\,, \end{aligned} \tag{31}$$

where the coefficients in the series are most naturally expressed in terms of $u = \exp(-\beta) = \sqrt{c}$, as above.

The expected scaling form of the various components of $\mathcal{R}$ for a generic spin model in field is discussed at some length in [16], and we now recapitulate these results briefly for comparison with the specific results for the Ising model on planar random graphs. The starting point is the scaling form of the free energy per spin near the critical point,

$$f(\varepsilon, h) = \lambda^{-1} f(\varepsilon\lambda^{a_\varepsilon}, h\lambda^{a_h}), \tag{32}$$

where $\varepsilon \equiv \beta_c - \beta$ and a_ε, a_h are the scaling dimensions for the energy and spin operators. For $\varepsilon > 0$, *i.e.*, in the unbroken high-temperature phase, we can use standard scaling assumptions to write this as

$$f(\varepsilon, h) = \varepsilon^{1/a_\varepsilon}\psi_+(h\varepsilon^{-a_h/a_\varepsilon}), \tag{33}$$

where ψ_+ is a scaling function and we also define $A = 1/a_\varepsilon$ and $C = -a_h/a_\varepsilon$ for later convenience. In terms of the standard critical exponents $A = 2-\alpha$ and $A + C = \beta$.

This generic scaling form can now be substituted into Eq. (10) to find the scaling behaviour of the scalar curvature (10) near criticality (*i.e.* $h = 0$, $\varepsilon \to 0$),

$$\mathcal{R} = \frac{(A+2C)[(A+2C)-(A-2)]}{2A(A-1)\psi_+(0)} \varepsilon^{-A} \tag{34}$$

or, translating back to the standard critical exponents,

$$\mathcal{R} = \frac{\gamma(\gamma-\alpha)}{2(2-\alpha)(1-\alpha)\psi_+(0)} \varepsilon^{\alpha-2} \,. \tag{35}$$

The discussion in [16] was intended to be as general as possible, one should note that for Ising-like models with a $\pm h$ symmetry all odd derivatives of the scaling function w.r.t. h will vanish so $\partial_h^3 f = 0$ rather than $\varepsilon^{A+3C}\psi_+'''(0)$. This does not affect the stated scaling relations.

However, one feature of these scaling relations does have an impact on the observed scaling for the Ising model. Generically one expects that $\partial_\beta^2 f = A(A-1)\varepsilon^{A-2}\psi_+(0)$, which contributes to both the metric and the determinant involved in calculating $\mathcal{R}$. If $A > 2$, *i.e.* $\alpha < 0$, this naively suggests a vanishing $\partial_\beta^2 f$ at criticality, which will in general *not* be the case. There would instead be a contribution from a regular term, which would give a constant at the critical point. Having such a constant term, which we take to be $A(A-1)\phi(0)$ for notational convenience, modifies the scaling form of $\mathcal{R}$ in the case $\alpha < 0$, $A > 2$ to

$$\mathcal{R} \;= \frac{(A+2C)^2}{2A(A-1)\phi(0)} \varepsilon^{-2} \tag{36}$$

or

$$\mathcal{R} = \frac{\gamma^2}{2(2-\alpha)(1-\alpha)\phi(0)} \varepsilon^{-2} \,, \tag{37}$$

so the critical exponent α no longer appears in the scaling exponent.

By virtue of the Boulatov and Kazakov solution of the Ising model on planar random graphs [14] we can explicitly confirm these observations. Since $\alpha = -1, \beta = 1/2, \gamma = 2$, we have $A = 3$, $C = -5/2$ and the modified discussion of scaling should apply. We can now take the series expansion for z_0 from Eq. (31), insert this into $g(z)$ and use the resulting expression for f in Eq. (30) to calculate the various terms that appear in the scalar curvature $\mathcal{R}$ as power series in h^2. We find that the leading terms at $h = 0$, with $\varepsilon_u \equiv u - u_{\mathrm{cr}} = \varepsilon/2 + \ldots$ and $u_{\mathrm{cr}} = 1/2$, and using β, h as co-ordinates are

$$\mathcal{R} \sim \frac{225}{704}\varepsilon_u^{-2} + \ldots = \frac{225}{176}\varepsilon^{-2} + \ldots \,. \tag{38}$$

A glance back at Eq. (37) shows that the modified scaling for $A > 2$ that these incorporate is, indeed, followed.

4. Three

Berlin and Kac [17] introduced the spherical model (and the Gaussian model) in an attempt to understand how generic some of the features of Onsager's solution [18] of the 2D Ising model are for ferromagnetic spin models, particularly for other dimensions. In the spherical model, the ± 1 condition on the value of the Ising spins is relaxed, whilst retaining a global constraint on the *total* spin magnitude. With s_i denoting the value of a spin at a site i of a hypercubic lattice, the partition function is [14]

$$\mathcal{Z} = \int ds_1 \dots ds_N \exp\left(\beta \sum_{\langle ij \rangle} s_i s_j + h \sum_i s_i\right) \delta\left(\sum_i s_i^2 - N\right), \tag{39}$$

where N is the total number of sites. This can be evaluated by exponentiating the constraint and using steepest descent, resulting in the following expression for the reduced free energy per site in the thermodynamic limit, $N \to \infty$:

$$f = \frac{1}{2} \ln\left(\frac{\pi}{\beta}\right) + \beta z - \frac{1}{2} g(z) + \frac{h^2}{4\beta(z-d)}, \tag{40}$$

where

$$g(z) = \frac{1}{(2\pi)^d} \int_0^{2\pi} d\omega_1 \dots d\omega_d \ln\left(z - \sum_{k=1}^{d} \cos(\omega_k)\right). \tag{41}$$

The saddle-point value of z which appears in the expression for the free energy in Eq. (40) is determined from

$$g'(z) = 2\beta - \frac{h^2}{2\beta(z-d)^2}. \tag{42}$$

The solution reveals no transition for $d = 1$ and 2, and a transition with the exponents $\alpha = -1,\ \beta = 1/2, \gamma = 2$ for $d = 3$.

It is useful to note that, with $h = 0$, Eq. (42) gives

$$\frac{dz}{d\beta} = \frac{2}{g''(z)}, \tag{43}$$

and hence

$$\frac{d^2 z}{d\beta^2} = -\frac{4g^{(3)}(z)}{[g''(z)]^3}. \tag{44}$$

The critical point is given by $z = d = 3$ and $h = 0$ [14], and the behaviour of $g(z)$ in this region is determined by differentiating Eq. (41) twice and then expanding for the small ω_k values which give the dominant contribution. In three dimensions one finds

$$g''(z) \sim -\frac{1}{2\sqrt{2}\pi}(z-3)^{-1/2}. \tag{45}$$

A further differentiation gives

$$g^{(3)}(z) \sim \frac{1}{4\sqrt{2}\pi}(z-3)^{-3/2}, \tag{46}$$

and an integration yields

$$g'(z) = \frac{1}{\sqrt{2}\pi}(z-3)^{1/2} + g'(3)\,, \tag{47}$$

where $g'(3) = (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})[K(2\sqrt{3}+\sqrt{6}-2\sqrt{2}-3)]^2 \approx 0.505\,462\,019\ldots$ is the massless 3D lattice propagator at the origin, which can be expressed in terms of one of the three classical Watson integrals [19] and hence is given by the standard elliptic integral $K(k^2)$. This latter expression can be combined with Eq. (42) with $h=0$ to give

$$(z-3) \sim 8\pi^2(\beta_c-\beta)^2 \sim \varepsilon^2, \tag{48}$$

in which $\beta_c = g'(3)/2 \approx 0.252\,731\,009\ldots$. Eqs. (45) and (46) may then be substituted in Eqs. (43) and (44) to give the scaling of $dz/d\beta$ and $d^2z/d\beta^2$,

$$\begin{aligned}
\lim_{z\to 3}\frac{dz}{d\beta} &= \lim_{z\to 3}\{-4\sqrt{2}\pi(z-3)^{1/2}\} = 0\,, \\
\lim_{z\to 3}\frac{d^2z}{d\beta^2} &= 16\pi^2\,,
\end{aligned} \tag{49}$$

which we shall employ below in the calculation of the scalar curvature.

We now move on to examine the scaling of the various terms contributing to $\mathcal{R}$ in Eq. (10) for the spherical model. As we have remarked, the $h\to -h$ symmetry in the free energy per site, f, of the spherical model means that any terms with an odd number of h derivatives will automatically be zero when $h=0$, hence $f_{\beta h} = f_{\beta\beta h} = f_{hhh} = 0$. This leaves the non-zero terms in the scaling region

$$\begin{aligned}
f_{\beta\beta} &\sim \frac{1}{2\beta_c^2}\,, \\
f_{hh} &\sim \frac{1}{16\pi^2\beta_c(\beta_c-\beta)^2} \sim \varepsilon^{-2}, \\
f_{\beta\beta\beta} &\sim 16\pi^2 - \frac{1}{\beta_c^3}\,, \\
f_{hh\beta} &\sim \frac{1}{8\pi^2\beta_c(\beta_c-\beta)^3} - \frac{1}{16\pi^2\beta_c^2(\beta_c-\beta)^2} \sim \varepsilon^{-3}.
\end{aligned} \tag{50}$$

We see that the expected general scaling of each term (for $\alpha < 0$) does indeed apply and that overall we have, as in Eq. (37),

$$\mathcal{R} \sim \varepsilon^{-2}. \tag{51}$$

We thus see that calculating the scaling of $\mathcal{R}$ for the 3D spherical model for which $\alpha = -1$ gives results in accordance with expectations from general scaling arguments which take into account the negative α, similarly to the Ising model on planar random graphs.

5. Four

The thermodynamics of black holes has been a subject of abiding interest since the pioneering work of Hawking [20] and similar ideas to those discussed in the previous sections for statistical mechanical models have also been applied to investigations of the critical behaviour of various families of black hole solutions in general relativity. Critical behaviour has arisen in several contexts in the study of black holes, ranging from the Hawking–Page [21] phase transition in hot Anti-de-Sitter space and the pioneering work by Davies [22] on the thermodynamics of Kerr–Newman black holes, to the idea that the extremal limit of various black hole families might be regarded as a *bona-fide* critical point [23–26].

It is the latter that is perhaps the closest to the work described here. In these studies the metrics used are the Ruppeiner metric

$$g_{ij} = -\partial_i \partial_j S(E, N^a)\,, \tag{52}$$

and the Weinhold [27] metric

$$g_{ij} = \partial_i \partial_j E(S, N^a)\,, \tag{53}$$

where S is the entropy and E the energy, with the N^a being other extensive variables such as scalar charges. Both metrics are related to the Fisher–Rao metric we have used by Legendre transforms of the appropriate variables, and have also been employed in a statistical mechanical context.

For example [26], a Reissner–Nordström black hole has a Weinhold metric of the form

$$dl_{\mathrm{W}}^2 = \frac{1}{8S^{(3/2)}} \left(\left(1 - \frac{3Q^2}{S} \right) dS^2 - 8Q\,dQ\,dS + 8S\,dQ^2 \right), \tag{54}$$

where Q is the charge, and we can see that one of the metric components vanishes at $S = 3Q^2$. This led Davies [22] to suggest that there was a phase transition at this point, but transforming to the Ruppeiner metric and choosing new co-ordinates gives a *flat* metric with $\mathcal{R} = 0$ everywhere.

On the other hand Kerr black holes which possess a spin J display a curved Ruppeiner geometry with a scalar curvature

$$\mathcal{R} = \frac{1}{4M^2} \frac{\sqrt{1 - \frac{J^2}{M^4}} - 2}{\sqrt{1 - \frac{J^2}{M^4}}} \tag{55}$$

that diverges at the extremal limit

$$\frac{J}{M^2} = \pm 1 \,. \tag{56}$$

It would thus appear that the general framework of looking for the signal of phase transitions in the geometric invariants of the information metric (or related metrics) can also be employed in these circumstances. Indeed, the AdS/CFT correspondence identifies Hawking–Page type transitions with deconfinement in the dual gauge theories [24,28] so it might be profitable to explore the information geometry view of such transitions further.

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