HEAVY ELECTRON QUANTUM CRITICALITY*

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Although the concept of a quantum phase transition has been known since the nineteen seventies, their importance as a source of radical transformation in metallic properties has only recently been appreciated. A quantum critical point forms an essential singularity in the phase diagram of correlated matter. We discus new insights into the nature of this phenomenon recently gained from experiments in heavy electron materials.

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1. The challenge of quantum criticality

Over the past few years, condensed matter physicists have become fascinated by the phenomenon of quantum criticality. Classical phase transitions at finite temperature involve the development of an order parameter ψ. A material that is tuned close to a classical phase transition senses the imminent change of state as the order parameter develops thermal fluctuations over larger and larger regions of the sample, ultimately forming a scale-invariant state of fluctuating order called a "critical state". The understanding of the universal nature of the correlations that develop at a classical critical point is a triumph of twentieth century physics [1].

The analogous idea of quantum criticality was introduced by John Hertz during the hey-days of interest in critical phenomena, but was regarded as an intellectual curiosity [2]. Discoveries over the past decade and a half have radically changed this perspective, revealing the ability of quantum

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phase transitions to qualitatively transform the properties of a material at finite temperatures. For example, high temperature superconductivity is thought to be born from a new metallic state that develops at a certain critical doping in copper–perovskite materials [3]. Near a quantum phase transition, a material enters a weird state of "quantum criticality": a new state of matter where the wavefunction becomes a fluctuating entangled mixture of the ordered, and disordered state. The physics that governs this new quantum state of matter represents a major unsolved challenge to our understanding of correlated matter.

A quantum critical point (QCP) is a singularity in the phase diagram: a point $x = x_c$ at zero-temperature where the characteristic energy scale $k_B T_0(x)$ of excitations above the ground-state goes to zero (Fig. 1.) [4–6,8,9].

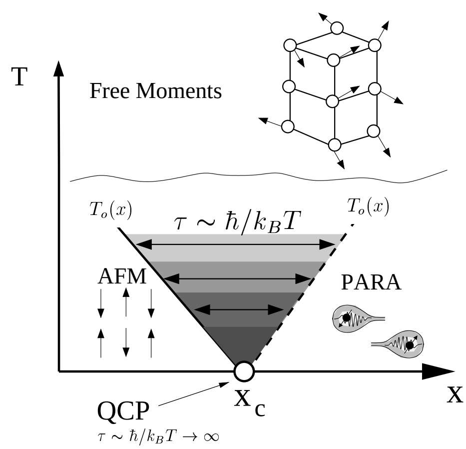

Fig. 1. Quantum criticality in heavy electron systems. High temperature: local moments. For $x < x_c$ spins become ordered for $T < T_0(x)$ forming an antiferromagnetic Fermi liquid; for $x > x_c$, composite bound-states form between spins and electrons at $T < T_0(x)$ producing a heavy Fermi liquid. "Non-Fermi liquid behavior", in which the characteristic energy scale is temperature itself, develops in the wedge shaped region between these two phases.

The QCP affects the broad wedge of phase diagram where $T > T_0(x)$. In this region of the material phase diagram, the critical quantum fluctuations are cut-off by thermal fluctuations after a correlation time given by the Heisenberg uncertainly principle[1]

$$\tau \sim \frac{\hbar}{k_B T}.$$

[1] Scaling of $\tau \sim \hbar/k_B T$ is an example of "naïve" scaling and is only expected to occur in quantum critical systems that lie below their upper critical dimension.

As a material is cooled towards a quantum critical point, the physics probes the critical quantum fluctuations on longer and longer timescales. Although the "quantum critical" region of the phase diagram where $T > T_0(x)$ is not a strict phase, the absence of any scale to the excitations other than temperature itself qualitatively transforms the properties of the material in a fashion that we would normally associate with a new phase of matter.

Heavy electron materials, offer a unique opportunity to study quantum criticality in a metal where the symmetry and character of the ground-state on either side of the QCP is unambiguous. These materials contain a dense array of local moments derived from rare earth or actinide atoms, embedded in a conducting host. At high temperature they display a Curie–Weiss temperature dependence of the magnetic susceptibility $\chi(T) \sim 1/T$ that is the hallmark of local moment metals. Depending on the exact conditions of the material, these local moments can order, forming an antiferromagnetically ordered metal, or they form composite bound-states with the surrounding electrons, giving rise to a highly renormalized Landau–Fermi liquid [10].

There is a growing list of heavy electron materials that can be tuned into the quantum critical point, by alloying, such as $CeCu_{6-x}Au_x$ [11], through the direct application of pressure, as in the case of $CeIn_3$ [12] and $CePd_2Si_2$ [13] or via the application of a magnetic field, as in the case of $YbRh_2Si_2$ [14, 15]. The recently discovered "1–1–5" materials [16–18] also appear to lie remarkably close to quantum criticality, with examples of chemically, pressure ($CeRhIn_5$ [16]) and field-tuned quantum criticality ($CeCoIn_5$ [17, 18]).

2. Key properties

In the ground-state near a quantum critical point, heavy electron materials display a linear specific heat $C_V = \gamma T$, and a quadratic temperature dependence of the resistivity $\rho = \rho_0 + AT^2$. Both of these properties are characteristic of Landau–Fermi liquid. As the QCP is approached, both γ and A appear to diverge, indicating a divergence in the effective mass at the QCP.

Some of the key properties at the QCP are:

- A divergent specific heat coefficient $\gamma(T) = C_V/T$ [19–21], which often displays a logarithmic temperature dependence [22]

$$\gamma(T) = \gamma_0 \log\left[\frac{T_0}{T}\right] . \tag{1}$$

- A quasi-linear temperature dependence of the resistivity [13, 14, 23]

$$\rho \propto T^{1+\varepsilon} , \tag{2}$$

with ε in the range of 0–0.6. Many compounds, such as $YbRh_2Si_2$ [24] and $CeCu_{6-x}Au_x$ [21] and $CeCoIn_5$ [16, 17] exhibit a perfectly linear resistivity, reminiscent of the cuprate perovskites.

- Anomalous exponents in the spin susceptibility, $\chi^{-1}(T) - \chi_0^{-1} \sim T^a$, with $a < 1$ for $CeCu_{5.9}Au_{0.1}$, $YbRh_2(Si_{1-x}Ge_x)_2$ ($x = 0.05$) and $CeNi_2Ge_2$ [13]. In $CeCu_{6-x}Au_x$, [21] neutron scattering measurements [21] reveal ω/T [25] scaling in the dynamic spin susceptibility

$$\chi^{-1}(\boldsymbol{q}, \omega) = f(\boldsymbol{q}) + (i\omega + T)^a \,, \tag{3}$$

where $f(\boldsymbol{q}) \to 0$ at the ordering wave vector(s).

The appearance of temperature as the only energy scale in the critical spin fluctuations with a non-trivial exponent $a < 1$, is an example of "naïve scaling", where the boundary condition (in this case, the periodicity of the fields over the imaginary time $\tau \in (0, \hbar/k_B T)$) determines the correlation time. This is a hallmark of a system where the critical modes lie beneath their upper critical dimension [26]. The q-independence of damping in the critical spin fluctuations suggests a local element to the underlying physics, and has stimulated efforts to develop a "locally quantum-critical" theory of the heavy electron QCP [27].

Recently, it has become possible to examine the evolution of the Fermi liquid properties at asymptotically low temperatures in the approach to a quantum critical point. Particularly interesting insights have been obtained from the material $YbRh_2Si_2$. This material has a 70 mK Neel temperature. By doping this material with Germanium, to form $YbRh_2$ $(Si_{1-x}Ge_x)_2$, ($x \sim 0.05$), the Néel temperature is driven to zero. In this quantum critical state, a tiny magnetic field is sufficient to drive the material into a Fermi liquid state. These studies indicate the presence of a single field-tuneable energy scale in both the specific heat C_V/T and the resistivity $\rho(T)$. The resistivity shows a field dependent cross-over between quadratic and linear temperature dependence, whilst the specific heat shows a field-dependent cross-over between a low-temperature upturn of the form $C_V/T \sim 1/T^{1/3}$ at $T >> b$ and $C_V/T \sim 1/b^{1/3}$, where $b = B - B_c$, in the field-tuned Fermi liquid. These results can be parameterized in the following form

$$\begin{aligned} \frac{d\rho}{dT} &\sim f\left(\frac{T}{T_0(b)}\right) , \\ \frac{C_V}{T} &\sim \frac{1}{T^{1/3}}\, \Phi\left(\frac{T}{T_0(b)}\right) , \end{aligned} \tag{4}$$

where $T_0(b) \propto b$, and $f(x) \sim \min(x, 1)$, $\Phi(x) \sim (\min(x, 1))^{1/3}$ (see Fig. 2). The existence of a single scale both the thermodynamics and the transport

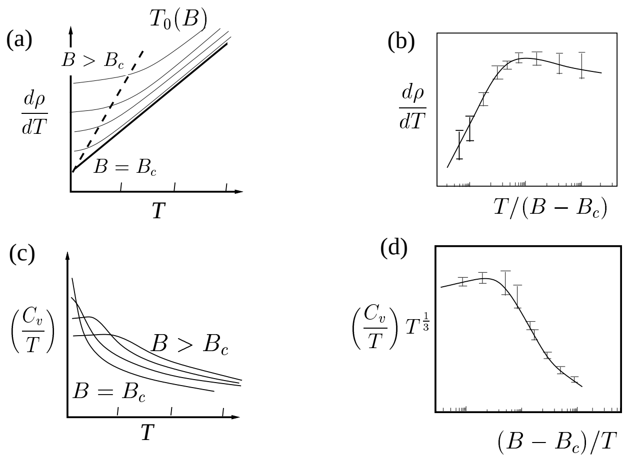

Fig. 2. Cartoon illustrating how the evolution of the resistivity and specific heat in $YbRh_2(Si_{1-x}Ge_x)_2$ ($x \sim 0.01$) is determined by a single scale $T_0(B) \sim (B - B_c)$, after [28]. (a) Linear resistance at criticality develops into quadratic dependence away from the QCP, (b) scaling of $d\rho/dT$ (error bars indicate spread of data), (c) field dependence of specific heat coefficient and (d) scaling of the specific heat coefficient C_V/T.

properties is striking evidence for the idea that the Fermi temperature goes to zero at a heavy fermion QCP. These results place very severe constraints on our understanding of the physics, as we now discuss.

3. Difficulties with the standard model

The standard model of heavy fermion quantum criticality, is provided by the Moriya–Hertz–Millis quantum spin density wave (QSDW) theory [2,6–8]. In this approach, critical behavior results from Bragg diffraction of electrons off quantum fluctuations in the spin density, described by an interaction of the form $H_I = g\sum_{\vec{q}} \vec{M}_{\vec{q}} \cdot \psi^{\dagger}_{\vec{k}-\vec{q}}\vec{\sigma}\psi_{\vec{k}}$. When the fermions are integrated out of the physics, the effective action for the slow quantum spin density modes is assumed to be local, and given by

$$\frac{F}{k_{\rm B}T} = \sum_{Q\equiv(\vec{q},i\nu_n)} |M(Q)|^2\chi^{-1}(Q) + \frac{U}{4}\int_0^{\frac{\hbar}{k_{\rm B}T}} d\tau \int d^dx M(x,\tau)^4 \,. \tag{5}$$

The inverse susceptibility

$$\chi^{-1}(Q) = \left(\left(\vec{q}-\vec{Q}_0\right)^2 + \xi^{-2} + \frac{|\nu_n|}{\Gamma_{\boldsymbol{Q}}}\right)\chi_0^{-1} \tag{6}$$

has an Orenstein–Zernicke form, where ξ is the correlation length, $\vec{Q}_0$ is the ordering wave-vector and the damping term, linear in frequency ν_n derives from coupling to the particle-hole excitations of the Fermi sea.

Critical fluctuations in this model strongly scatter electrons on "hot lines" around the Fermi surface which are separated by momentum $\vec{Q}_0$ — see Fig. 3(a) *(ii)*. On the hot lines, the electron scattering rate $\Gamma_{\rm sc} \propto \max(\omega, T)$ is linear in energy and temperature, and the quasiparticles masses are driven to infinity. This "marginal" Fermi liquid behavior [29] is confined to a narrow region of width $\delta k \sim \sqrt{T}$ around the hot lines, and even at criticality, the remainder of the Fermi surface would form a tranquil Landau–Fermi liquid.

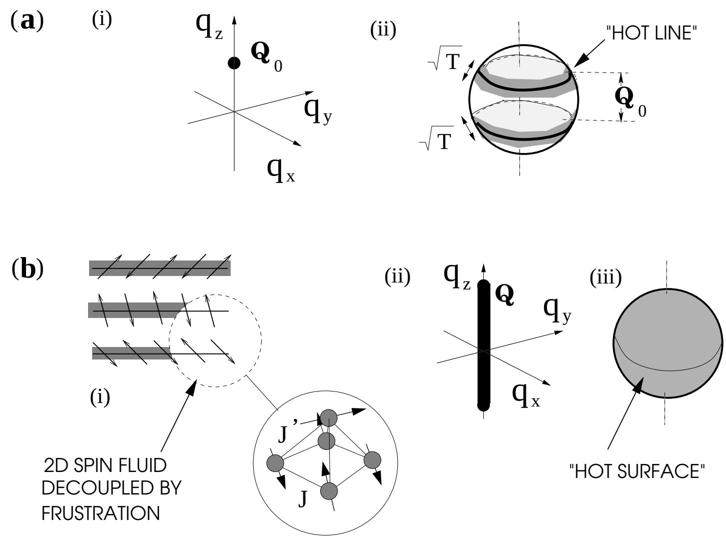

Fig. 3. (a) 3D QSDW scenario in which *(i)* the critical fluctuations focus around a point in momentum space giving rise to *(ii)* hot lines around the Fermi surface. (b) 2D QSDW scenario, in which *(i)* frustration leads to layers of decoupled spin fluid, *(ii)* rods of critical scattering in momentum space and *(iii)* non Fermi liquid behavior across the entire Fermi surface.

In the 3D QSDW scenario, the spin correlation time $\tau = \Gamma_{Q_0}\xi^2$ so time scales as $z = 2$ spatial dimensions. The effective spatial dimensionality of the phase space is $D = d + z$, and since $D_{\rm u} = 4$ is the upper-critical dimension of this kind of "ϕ^4" field theory, naïve scaling behavior is only expected for $d \le d_{\rm u} = 4 - z = 2$. The 3D QSDW model is thus inconsistent with

- E/T scaling in the spin correlations with a non-trivial exponent.
- A divergence in the specific heat.
- The cross-over to a linear resistivity at $T > T_0(x)$.

It is worth noting that the resistivity of $CeIn_3$ follows a $T^{1.6}$ variation that is said to be consistent with the 3D scenario [30]. However, recent NMR measurements suggest that this material has a *first order* transition, so that electrons never feel the full force of quantum criticality [31].

4. Is the spin fluid two-dimensional?

The failure of the 3D QSDW scenario has stimulated the proposal that magnetic frustration causes the spins to decouple into layers of independent two dimensional spin fluids [12, 32] (Fig. 3(b)). This scenario does predict a logarithmic divergence in the specific heat of the form

$$\frac{C_v}{T} \sim \ln\left(\frac{T_{\rm sf}}{\max(T_{\rm 3D}, T)}\right), \tag{7}$$

where $T_{\rm sf}$ is the characteristic scale of spin fluctuations and $T_{\rm 3D}$ is the scale at which the planes become coupled. Furthermore, the critical spin fluctuations are then critical along "rods" in momentum space[2], and in this situation large regions of the Fermi surface become "hot".

Part of the problem with the 2D QSDW, is that we know of no mechanism to produce such perfectly decoupled 2D spin fluids within three dimensional metals. Even in lattices where frustration decouples spin layers to first order in the interlayer coupling J', zero point spin fluctuations couple the spin layers to second order in the coupling via the mechanism called "order-from-disorder" [34–36], $T_{\rm 3D} \sim (J')^2/J$ and for this reason, it is difficult to suppress $T_{\rm 3D}$ more than an order of magnitude smaller than $T_{\rm sf}$ using frustration. Yet no such crossover has been observed, indeed, $YbRh_2Si_2$, the specific heat diverges faster than logarithmically at low temperatures.

Conventional heavy electron materials form Landau–Fermi liquids which are characterized by local scattering amplitudes. One of the consequences of this local scattering, is the constancy of the called Kadowaki–Woods ratio $K = A/\gamma^2$ between the quadratic temperature coefficient of the resistivity and the square of the specific heat coefficient [37]. This is not expected in the 2D QSDW picture, which will produce strongly momentum dependent scattering. Experimentally, the quadratic A coefficient of the resistivity diverges in the approach to quantum criticality. From the scaling results on field-tuned criticality in $YbRh_2Si_2$ mentioned above, $A \sim 1/(T_0(b)) \sim 1/b$ $(B = B - B_c)$. Such behavior can be obtained in a two dimensional spin fluid model in which the inverse squared correlation length is assumed to be proportional to b, $\xi^{-2} \propto b$ [7, 38]. The same model predicts a weak dependence of the linear specific heat on magnetic field $\gamma_{\rm th} \propto \log(1/b)$, so that

[2] In quantum critical $CeCu_{6-x}Au_x$ $(x = 0.1)$ there is evidence for rod-like regions of critical fluctuations [32, 33].

$$K_{\rm th} = \left.\frac{A}{\gamma^2}\right|_{\rm th} \sim \frac{1}{b\log^2(b)}\,.$$

Early experiments by Gegenwart *et al.*, [15] suggested that the Kadawaki–Woods ratio is independent of field. More extensive scaling results at lower fields and temperatures [28] indicate that $\gamma \sim 1/b^{1/3}$ so that

$$K_{\rm exp} = \left.\frac{A}{\gamma^2}\right|_{\rm exp} \sim \frac{1}{b^{1/3}}\,.$$

This weak field dependence of the Kadowaki–Woods ratio indicates that the scattering amplitudes in the Fermi liquid do not develop a strong momentum dependence in the approach to the QCP, arguing against the exchange soft magnetic fluctuations in a 2D spin fluid as the predominant origin of the scattering.

5. The search for new mean field theories

Traditionally, theories of critical fluctuations are built upon an underlying mean-field theory, which becomes exact above the upper critical dimension. The spin density wave scenario is a consequence of examining fluctuations about the Stoner and Slater mean-field theory for itinerant magnetism. The failure of this starting point may indicate that we should search for a new kind of mean-field theory. Two ideas have been recently explored:

- **Local spin criticality.** The momentum independence of the spin damping at the QCP point [21] has led to the suggestion that the spin correlations are critical in time, yet spatially local [39–41] permitting their treatment via the "extended dynamical mean field theory" (EDMFT). This is a bold departure from the Wilson–Kadanoff approach to criticality, for ultimately only one dimension time is active in the critical fluctuations.

- **Traditional RG approach on a new Lagrangian.** If we embrace a Wilson–Kadanoff approach to quantum criticality, then we must seek a new Lagrangian description to of magnetism, and the way it couples to the Fermi liquid. One idea here, is that at the quantum critical point, the heavy electron breaks-up into its spin and charge components [42].

The momentum-independent scaling term in the inverse dynamic susceptibility (6) certainly does suggests that the critical behavior associated with the heavy fermion QCP contains some kind of *local* critical excitation [21].

Si, Rabello, Ingersent and Smith [27] *et al.* have pursued the idea that the locally critical degree of freedom is spin itself. In their picture, in order that the characteristic energy scale of local spin fluctuations goes to zero at the QCP, there must be a divergent *local* spin susceptibility $\chi_{\rm loc} = \sum_{\vec{q}} \chi(\vec{q}, \omega)|_{\omega=0}$. The phenomenological form (3) appears naturally as part of the EDMFT scheme adopted by Si *et al.*, and by using this form to compute the local spin susceptibility,

$$\chi_{\rm loc}(T) \sim \int d^d q \frac{1}{(\boldsymbol{q} - \boldsymbol{Q})^2 + T^\alpha} \sim T^{(d-2)\alpha/2} . \qquad (8)$$

Si *et al.*, conclude that if a divergence of the local spin response requires a two dimension spin fluid. They find, based on this assumption, that it is possible to reproduce the anomalous frequency dependence seen in neutron scattering [27, 43].

This intriguing proposal for heavy electron quantum criticality still has some important technical hurdles to clear. In one interesting development, Pankov, Kotliar and Motome recently reported that the finite temperature solutions to the EDMFT give rise to a first order phase transition between the antiferromagnetic and paramagnetic phases [44]. The transition might become second order at zero temperature, but it is not clear how any scenario with a first order line can be simply reconciled with the finite temperature scaling behavior and the "fan" of quantum criticality observed in the vicinity of a heavy fermion QCP (Fig. 4.)

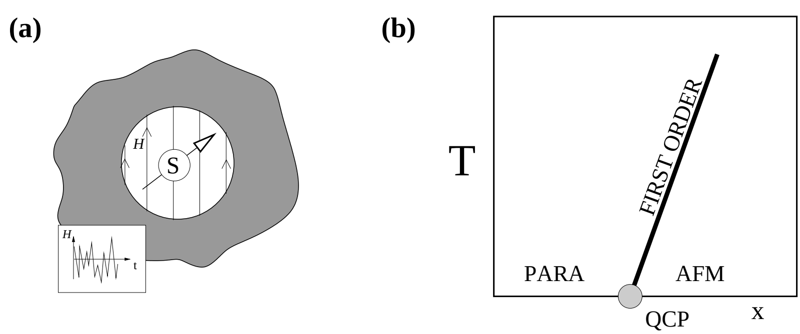

Fig. 4. In the extended dynamical mean-field theory description of local quantum critical theory, each spin behaves as a local moment in a fluctuating Weiss field. Recent work [44] indicates that the phase transition predicted by this approach may be first order at finite temperature, with a possible QCP at zero temperature as shown in (b).

6. A new Lagrangian for the emergence of magnetism?

Another alternative, is that the heavy fermion quantum criticality is a simply three-dimensional phenomenon. In this case we need to begin a search for a new class of critical Lagrangian with an upper critical spatial dimension $d_{\rm u} > 3$ [45]. There are a number of elements that might be expected in such a theory:

- ◇ First, to produce a qualitative departure from conventional spin fluctuation theory, we should in all probability seek a new description of the coupling between the magnetic modes and the heavy electrons.
- ◇ Second, there is a suspicion that in order to obtain a break-down of the quasiparticles over the whole Fermi surface, some aspect of the quantum criticality should be local.

One idea that the current authors have explored, is the notion that the critical magnetic modes in a heavy fermion system and their coupling to the Fermi fluid may be spinorial in character. We know, from various lines of reasoning that in a Kondo lattice the Luttinger sum rule [46–48] governing the Fermi surface volume $\mathcal{V}_{\rm FS}$ "counts" both the electron density $n_{\rm e}$ and the number of the number of local moments per unit cell n_s:

$$2\frac{\mathcal{V}_{\rm FS}}{(2\pi)^3} = n_{\rm e} + n_s\,. \tag{9}$$

The appearance of the spin density in the Luttinger sum rule reflects the composite nature of the heavy quasiparticles, formed from bound-states between local moments and high energy electron states. Suppose the spinorial character of the magnetic degrees of freedom seen in the paramagnet *also* manifests itself in the decay modes of the heavy quasiparticles. This would imply that at the QCP, the staggered magnetization factorizes into a spinorial degree of freedom $\vec{M}(x) = b^\dagger(x)\vec{\sigma}b(x)$, where b is a two-component bosonic spinor. "Spinorial magnetism" affords a direct coupling between the magnetic spinor b_α and the heavy electron quasi-particle fields $\psi_{\vec{k}\alpha}$ via an inner product, over the spin indices

$$L^{(2)}_{F-M} = g\sum_{\boldsymbol{k},\boldsymbol{q}}\left[\phi^\dagger_{\boldsymbol{q}}\, b^\dagger_{\boldsymbol{k}-\boldsymbol{q}\sigma}\psi_{\boldsymbol{k}\sigma} + \text{h.c.}\right]\,, \tag{10}$$

where conservation of exchange statistics obliges us to introduce of a spinless charge e fermion ϕ. This would imply that the composite heavy electron decays into a neutral "spinon" and a spinless charge e fermion $e^-_\sigma \rightleftharpoons s_\sigma + \phi^-$.

This line of reasoning leads suggests that the break-up of the heavy fermion QCP may involve *spin–charge separation*. In the antiferromagnet, the magnetic spinors will condense, and the ϕ fermion will propagate coherently.

At the QCP, the vanishing of the ordered magnetic moment will mean that all coherent motion of this object will cease. In such a scenario, it is then a "locally critical" fermion rather than spin that drives the non-Fermi liquid behavior.

Additional support for this line of reasoning comes from a quite unexpected direction: from the re-examination of a venerable model of magnetism, the underscreened Kondo model (UKM). The underscreened Kondo effect, whereby a spin is partially quenched from spin S to $S^* = S - 1/2$. occurs in an impurity model when the number of screening channels is insufficient to quench the local moment. In impurity models, this only arises when $S > 1/2$. In the Kondo lattice, underscreening may be an intrinsic feature of the quantum critical point for $S = 1/2$. The Curie-like power-law dependence $\chi^{-1}(T) - \chi_0^{-1} \sim T^a$, of the spin susceptibility [15, 21] seen at criticality might indeed be interpreted as circumstantial evidence for the existence of partially quenched moments at criticality.

The UKM model is written $H = H_0 + H_I$, where H_0 describes the conduction sea and

$$H_I = J\vec{S} \cdot \psi^\dagger{}_\alpha \vec{\sigma}_{\alpha\beta} \psi_\beta \, , \tag{11}$$

where S denotes a spin $S > 1/2$ and $\psi^\dagger{}_\alpha = \sum_k c^\dagger{}_{k\alpha}$ creates a conduction electron at the impurity site.

In recent work, we have found that the essential physics of the UKM is captured by a Schwinger boson representation of the local moments [49]. In an unexpected surprise, we have also found that the model exhibits a unique kind of field-tuned criticality, forming a tunable Fermi liquid in a magnetic field, but a non-Fermi liquid at $B = 0$. In this approach, the interaction between magnetism and the Fermi fluid in the UKM takes the form

$$H_I = J \sum_{\alpha,\beta} \psi^\dagger{}_\alpha b_\alpha b^\dagger{}_\beta \psi_\beta \rightarrow \left[\bar{\phi}\; b^\dagger{}_\sigma \psi_\sigma + \psi^\dagger{}_\sigma b_\sigma \phi\right] - \frac{1}{J}\bar{\phi}\phi \, , \tag{12}$$

where ϕ is a Grassman field. The Gaussian fluctuations of this field describe a fermionic resonance which couples to the conduction sea. The form of this coupling is suggestively close to the phenomenological form (10) discussed above.

In a magnetic field B, the Schwinger boson condenses, $\langle b_\sigma \rangle = \sqrt{2M}\delta_{\sigma\uparrow}$, where M is the magnetization, so that

$$H_I \rightarrow \sqrt{2M}\left[\bar{\phi}\; \psi_\uparrow + \text{h.c.}\right] + \text{fluctuations} \tag{13}$$

giving rise to a resonance in the Fermi sea. What is unexpected about this resonance, is that its characteristic weight Z or wave-function renormaliza-

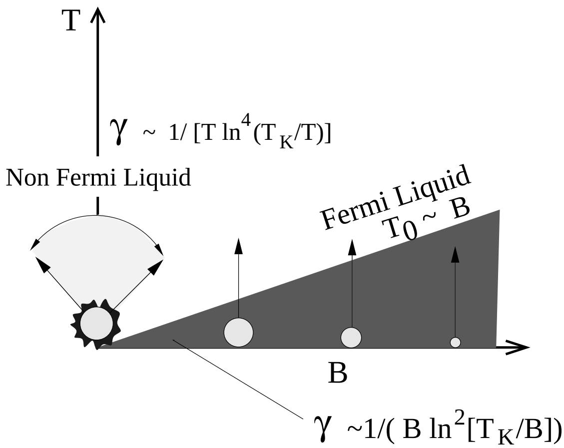

Fig. 5. Schematic phase diagram of the underscreened Kondo model. In a finite field, the ground-state is a Fermi liquid with a field-tuned Fermi energy. In zero field, a residual ferromagnetic coupling between the electron sea and the untethered moment leads to a break-down of Fermi liquid behavior and a divergence of $\gamma = C_V/T$ with temperature.

tion scales with the magnetic field B, $Z \propto B$, giving rise to a resonance with a characteristic energy scale $T_0(B) \propto B$. As the field is reduced to zero, so the width of the resonance narrows and the linear specific heat can be shown to diverge as

$$\gamma \sim \frac{1}{B \ln^2\left(\frac{B}{T_\mathrm{K}}\right)} .$$

At zero field the specific heat actually develops a divergence

$$\gamma \sim \frac{1}{T \ln^4\left(\frac{T_\mathrm{K}}{T}\right)} ,$$

which is reminiscent of the low-temperature upturn in the specific heat seen in $YbRh_2\,Si_2$. This field-tunability of the Fermi temperature curiously went unnoticed in the Bethe Ansatz solutions of this model for two decades [50]. In the new context it is fascinating because it provides a concrete example of a system of field-tuned criticality in a model where the coupling between the magnetism and the Fermi sea exhibits an explicit spinorial character. One of the open questions about this model, is whether the temperature dependent inelastic scattering it gives rise to will mimic the a cross-over between quadratic and T-linear scattering behavior around $T \sim B$ seen in real heavy electron systems.

Finally, we should note that if the transition between the antiferromagnet and the paramagnet involves the formation (or destruction) of new kinds of fermionic resonance at the Fermi surface, then the geometry of the Fermi surface will change far radically at the heavy electron QCP. This kind of behavior is expected to give rise to discontinuities in the Hall conductivity and the extrapolated de Haas–van Alphen frequencies at the QCP. This is clearly an area where we could benefit immensely from further experimental study.

7. Summary

We have reviewed the basic physics of heavy electron quantum criticality. The various properties of the antiferromagnetic heavy electron quantum critical point, most notably the observation of E/T scaling and the appearance of a single scale $T_0(x)$governing the cross-over from Fermi liquid, to non-Fermi liquid behavior in both the resistivity and the thermodynamics, suggest the existence of a new universality class of critical electronic behavior that lies beyond the reach of quantum spin density wave theories of quantum criticality. This motivates a search for a new class of theory for the emergence of magnetism in heavy electron systems. One idea, is that the heavy electron quantum critical point involves spin correlations that are singular and critical in time, but only weakly correlated in space, but this leads to the conclusion that non-trivial behavior requires a frustrated, quasi-two dimensional spin fluid. Alternatively, heavy electron quantum criticality may be intrinsically three dimensional in character, but involve a kind of spin–charge decoupling that develops as the spins bound within composite heavy electrons emerge into ordered magnetism. Our theoretical and experimental explorations of this phenomenon are still very much in their infancy, and it is clear that much work remains to be done.

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