Vol. 36 (2005) *ACTA PHYSICA POLONICA B* No 8

INVESTIGATING MULTIFRACTALITY OF STOCK MARKET FLUCTUATIONS USING WAVELET AND DETRENDING FLUCTUATION METHODS*

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(Received April 21, 2005)

We apply the Multifractal Detrended Fluctuation Analysis and the Wavelet Transform Modulus Maxima to investigate multifractal properties of stock price fluctuations. By applying both methods to the same data sets coming from the German and the American stock markets and based on our earlier knowledge of how these methods detect multifractality while employed to well-known mathematical models, we compare the results given by both methods and infer which one can be preferable in the case of the financial data. We argue that the Multifractal Detrended Fluctuation Analysis acts better for a global detection of multifractal behavior, while the Wavelet Transform Modulus Maxima method is the optimal tool for the local characterization of the scaling properties of signals.

PACS numbers: 89.75.–k, 89.75.Da, 89.75.Fb, 89.65.Gh

1. Introduction

After introduction of the concept of multifractality [1, 2] and of the methods of its mathematical characterization *e.g.* in terms of the scaling exponents spectrum $\tau(q)$ and the singularity spectrum $f(\alpha)$ [2], much effort has been devoted to the investigation of multifractal properties of real data coming from diverse sources. Earth and atmospheric science [3–6], human physiology [7–9], molecular biology [10–12] and finance [13–18] are only few among many such examples. In particular, the multifractal behavior of the financial data from stock, currency and commodity markets was detected quite early after discovering its close similarity to the behavior of

* Presented at the First Polish Symposium on Econo- and Sociophysics, Warsaw, Poland, November 19–20, 2004.

fluid turbulence [19]. The subsequent introduction of a mathematical model based on the binomial multiplicative cascade (Multifractal Model of Asset Returns [13, 32–34]) comprising the multifractality as its inherent property, allowed one to explain the observed properties of the financial data also from a theoretical point of view. This approach was largely successful also due to the fact that it easily accounts for the observed stylized facts [20–23] of the data like non-Gaussian tails of p.d.f. and long-range temporal correlations in volatility, the issues which were beyond the reach of other models like the fractional Brownian motion and the GARCH processes.

However, despite a multitude of the real-data analyses, a proper detection of the multifractality in the experimental data still presents much difficulty and is not always reliable [24]. In principle, there are two methods of dealing with this subject: the Multifractal Detrending Fluctuation Analysis (MF-DFA) [25] and the Wavelet Transform Modulus Maxima method (WTMM) [26,27]. MF-DFA is a generalization of the Detrended Fluctuation Analysis [10] and is based on identification of scaling of the qth order moments of the data segments of varying length. On the other hand, WTMM allows one to detect scaling by means of the maxima lines of the continuous wavelet transform on different scales. Both methods remove trends present in nonstationary signals and analyze the fluctuations. Both methods have their advantages as the recent study [25] has shown, but our motivation behind the present work is the apparent lack of a systematic comparative study of these two competitive methods in the context of the financial data.

2. Formalism

2.1. Multifractal Detrended Fluctuation Analysis

MF-DFA has recently gained much popularity owing to its simple implementation and significant ability to describe a multifractal structure of data. From a technical point of view it is a Rényi-like generalization of the older DFA approach [10] commonly used to detect long-range temporal correlations and to assess Hurst exponent for nonstationary data. In the standard DFA procedure we consider the so-called signal profile $Y(j)$ which is derived from a time series $x(i),\ i = 1,\dots,N$ by the following expression

$$Y(j) = \sum_{i=1}^{j} (x(i) - \langle x \rangle),\ j = 1,\dots,N\,, \tag{1}$$

where N is the time series length. $Y(j)$ has to be divided into M_n segments of length n $(n < N)$ starting from both the beginning and the end of the time series and, thus, one obtains $2M_n$ segments. A local trend of each segment ν has to be subtracted from the signal by approximating it by an l^{th} order polynomial $P_\nu^{(l)}$. The variances for all the segments ν and all segment lengths n are then evaluated

$$F^2(\nu, n) = \frac{1}{n}\sum_{j=1}^{n}\left\{Y[(\nu-1)n+j] - P_\nu^{(l)}(j)\right\}^2 . \tag{2}$$

$F^2(\nu, n)$ has to be averaged over ν's and, in the final step being the multifractal generalization of DFA, the q^{th} order fluctuation function is calculated for all possible segment lengths n:

$$F_q(n) = \left\{\frac{1}{2M_n}\sum_{\nu=1}^{2M_n}[F^2(\nu, n)]^{q/2}\right\}^{1/q}, \quad q \in \boldsymbol{R} . \tag{3}$$

It can be shown that for a signal with clear fractal properties, the function $F_q(n)$ scales within a range of n's according to a power-law

$$F_q(n) \sim n^{h(q)} , \tag{4}$$

where $h(q)$ denotes the generalized Hurst exponent. For a monofractal signal, $h(q)$ is independent of q and thus equals $H = h(2)$, while for a multifractal signal $h(q)$ form a monotonously decreasing function of q. There is a unique relation between the generalized Hurst exponents and the singularity spectrum $f(\alpha)$

$$\alpha = h(q) + qh'(q) \quad \text{and} \quad f(\alpha) = q[\alpha - h(q)] + 1 . \tag{5}$$

α is called Hölder exponent and characterizes the strength of a singularity; $f(\alpha)$ is the Hausdorff dimension of the fractal subset with the exponent α. In general, a multifractal can consist of two or more convoluted monofractals or an infinite set of monofractals with the continuously varying α. How rich the multifractal is can be expressed by the spectrum width $\Delta\alpha := \alpha_{\max} - \alpha_{\min}$; the larger $\Delta\alpha$, the richer multifractal dynamics of the signal.

2.2. Wavelet Transform Modulus Maxima

The WTMM method [26–28] inherits the advantages of the wavelet transform analysis [29] and was developed to deal with strongly nonstationary data. It has an important ability to reveal hierarchical structure of singularities and therefore proves useful in analyzing self-similar structures like fractals. The wavelet transform, allowing one to decompose a signal in the time-scale plane, is a convolution of the signal $x(i)$ and a wavelet ψ:

$$T_\psi(n, s') = \frac{1}{s'}\sum_{i=1}^{N}\psi\left(\frac{i-n}{s'}\right)x(i) , \tag{6}$$

where s'-scale ψ is shifted by n. The wavelet transformed signal can be represented graphically by a color-coded map of the coefficients $T_\psi(n, s')$; an example is presented in figure 1.

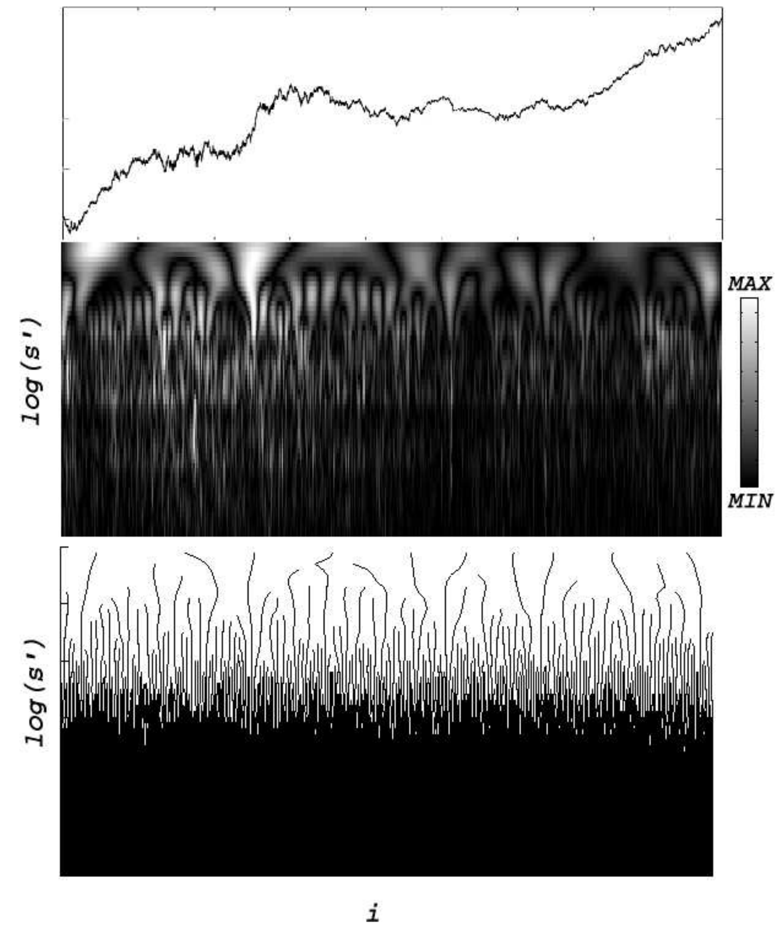

Fig. 1. Exemplary result of wavelet analysis: tick by tick data for Deutsche Telekom (DTE, top), map of coefficients of the wavelet transform (middle), and its maxima lines (bottom).

Out of a rich variety of available wavelets, we choose the third derivative of a Gaussian $\psi^{(3)}(x) = \frac{d^3}{dx^3}(e^{-x^2/2})$, which is insensitive to trends up to a quadratic one. What is important in this context, in the presence of a singularity of the strength α at a point n_0, the coefficients T_ψ show the power law behavior $T_\psi(n_0, s') \sim s'^{\,\alpha(n_0)}$. Due to the data structure, however, this relation can be unstable; therefore it is recommended to identify the local maxima of T_ψ and use their moduli

$$Z(q,s') = \sum_{l \in L(s')} |T_\psi(n_l(s'),s')|^q\,; \tag{7}$$

here $L(s')$ denotes the set of all maxima for the scale s' and $n_l(s')$ is the position of a particular maximum. An additional supremum condition is indispensable in order to preserve the necessary monotonicity of $Z(q,s')$

$$Z(q,s') = \sum_{l \in L(s')} \left(\sup_{s'' \leq s'} |T_\psi(n_l(s''),s'')| \right)^q . \tag{8}$$

If the signal under study is fractal we require the existence of scaling $Z(q,s') \sim s'^{\,\tau(q)}$. For a multifractal signal $\tau(q)$ is nonlinear. It is straightforward to relate this quantity to the singularity spectrum [2]

$$\alpha = \tau'(q) \quad \text{and} \quad f(\alpha) = q\alpha - \tau(q) \tag{9}$$

and to the generalized Hurst exponents $\tau(q) = qh(q) - 1$.

3. Data analysis

We applied both MF-DFA and WTMM to experimental signals coming from the two large stock markets: Deutsche Börse in Frankfurt, Germany and New York Stock Exchange. Our data [30] was the high-frequency tick-by-tick recordings of stock prices spanning the time interval 1 Dec 1997–31 Dec 1999. The analyzed sets of stocks comprised the 30 highly-capitalized Dow Jones Industrials (DJI) and the 30 companies included in the German DAX index. In econophysics, one typically studies the logarithmic price returns sampled with a constant frequency over certain time interval. Instead, here we analyze the transaction-to-transaction price increments $p_s(t_i) = \ln(P_s(t_{i+1})) - \ln(P_s(t_i))$, where t_i denotes the transaction moments ($i = 1, \ldots, N$). A reason for such a choice is that as our earlier studies proved [17, 18], the temporal evolution of a stock price can be represented by a two-dimensional fractal, *i.e.* a fractal process of the price fluctuation spread over a fractal support of the inter-transaction time intervals $\Delta T_s(t_i) := t_{i+1} - t_i$. In this situation, an analysis based only on the price returns omits an important part of the fractal structure of the data. Figure 2 illustrates this conclusion by presenting the rich multifractal spectra ($\Delta\alpha \gg 0$) of both the price increments $p_s(t_i)$ and the time intervals ΔT_s. All the spectra were evaluated using the MF-DFA procedure.

Although the spectra in figure 2 are multifractal, the ones for p_s differ much from the ones for ΔT_s. Maxima of the spectra for the price increments are placed near $\alpha = 0.5$ ($H \simeq 0.5$) while the spectra for the time

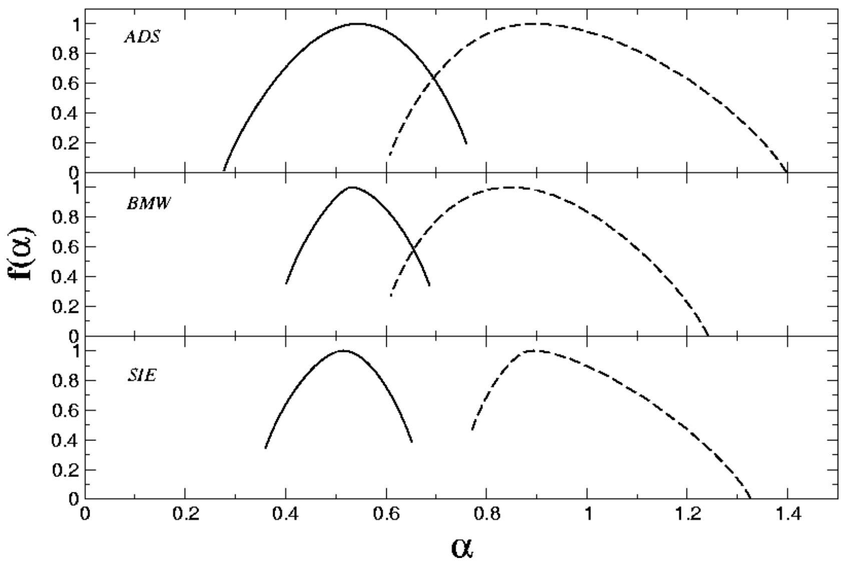

Fig. 2. Singularity spectra $f(\alpha)$ for price increments (solid) and for inter-transaction time intervals (dashed) for three typical German stocks: Adidas Solomon (ADS, top), BMW (middle), and Siemens (SIE, bottom).

intervals are placed at $\alpha = 0.8$. This result expresses the linearly uncorrelated dynamics of p_s and the persistent behavior of the time series of ΔT_s (see [17] for a more detailed discussion). However, despite the fact that MF-DFA clearly identifies the multifractality in the above example, our another study based on WTMM ([31], it will be published elsewhere) indicates that each of the methods of detecting the multifractality gives us slightly different results. We applied both methods to the same data sets obtained according to a few well-known theoretical models like Brownian motion, truncated Lévy flights, and binomial cascade. The results from that analysis suggest that the WTMM method can act better in the presence of strong temporal correlations while it gives significantly-biased results if the signals are characterized by broad p.d.f.'s. and weak correlations [31]. Since both the correlations and the broad p.d.f.'s can be sources of the observed multifractal behavior, the reliability of outcomes strongly depends on the properties of the analyzed signals.

In order to illustrate the MF-DFA and the WTMM performance if applied to our financial time series, we calculated the $\tau(q)$ multifractal spectra for all the 30 German and for all the 30 American stocks and averaged them separately within each of the two markets. Then we derived the singularity spectra $f(\alpha)$ from such mean $\tau(q)$ functions. We also constructed benchmark spectra from the randomly-shuffled time series which were characterized by lack of any temporal correlations and by the same p.d.f.'s as the original signals. Figure 3 displays what we obtained along this way for both the DAX (top panel) and the DJI stocks (bottom panel) and for the actual data (left) and the benchmarks (right). The spectra in each panel are located near the same $\alpha = 0.5$ or slightly above this value, but we cannot see any systematic difference in the maxima location between MF-DFA (full circles) and WTMM (empty squares). This result agrees with the earlier outcomes that the consecutive price increments are linearly uncorrelated.

The main difference between the results of MF-DFA and WTMM consists in distinct widths $\Delta\alpha$, with the width for WTMM being much larger than the one for MF-DFA. Indeed, for DAX we obtain: $\Delta\alpha = 0.26$ (WTMM) and $\Delta\alpha = 0.16$ (MF-DFA), and for the DJIs we have: $\Delta\alpha = 0.25$ (WTMM) and $\Delta\alpha = 0.13$ (MF-DFA). If interpreted straightforwardly, these results suggest that the wavelet-based method detects more sophisticated dynamics than the detrended fluctuation method. These results should be investigated with more care, though. As our experience teaches us, if the sig-

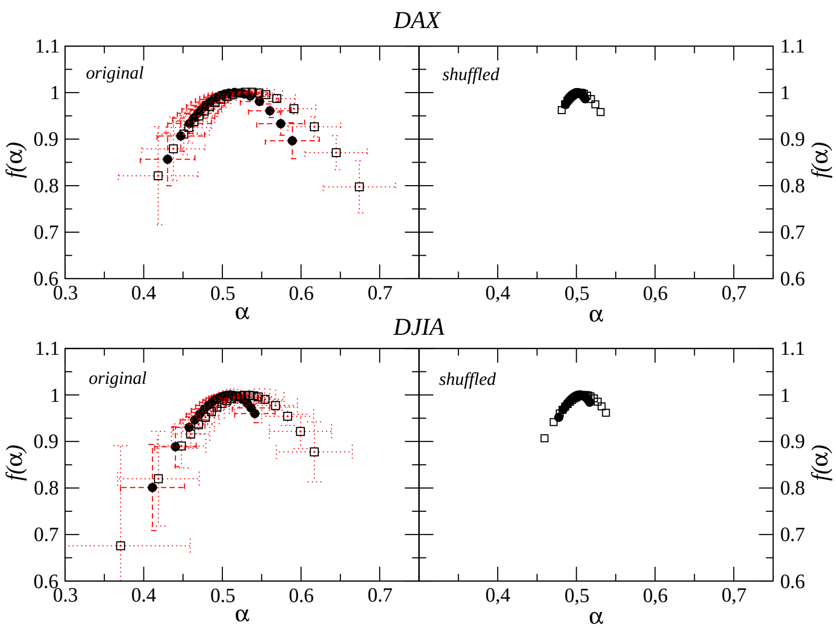

Fig. 3. Mean singularity spectra for all 30 stocks from the German (top) and from the American stock market (bottom); both the actual and the randomized signals are shown (left and right, respectively). MF-DFA (full circles) and WTMM (empty squares) are presented in each panel. Error bars on the right denote standard deviation of α and $f(\alpha)$ calculated from 30 stocks.

nal under study is monofractal or its multifractality is poor (*e.g.* fractional Brownian motion, truncated Lévy processes), the WTMM approach fails to detect its properties correctly and instead in that case one obtains spurious multifractal spectra of a significant width [31]. The same refers to the situation in which the probability density of a signal has fat non-Gaussian tails: WTMM overestimates $\Delta\alpha$ in this case, either. Now it is not surprising that the wavelet-based tool gives wider spectra for the randomized data (right panels). On the other hand, if the computer-generated model data is persistent, both methods produce spectra that are too wide when compared to the theoretical ones, but the bias is less evident for MF-DFA, thus preferring this method of analysis. By comparing the relative widths of the spectra for the randomized and the actual signals, we found that the relative widths are independent of the method. Taking all these pieces of information into consideration, we conclude that MF-DFA offers results which are less biased and therefore more reliable than the ones offered by WTMM.

The above discussion was related to the global properties of the signals. However, if one would like to investigate the local scaling structure of the signals and to look at the Hölder exponents in the neighborhood of each data point, it is possible only by means of the wavelet transform, because in DFA we loose the information (in favor of the result stability) while averaging the moments in Eq. (3). Numerically, one can calculate the approximation of the Hölder exponent called the effective Hölder exponent $\hat{h}(n_0, s')$. First, the mean Hölder exponent $\bar{h}$ has to be computed

$$\ln[M(s')] = \bar{h}\ln(s') + C\,, \tag{10}$$

where $M(s') = \sqrt{Z(s',q)/Z(s',0)}$ and s' is scale (for $q = 2$ we obtain the local version of the Hurst exponent). The effective Hölder exponent can be derived according to the following relation

$$\hat{h}(n_0, s') = \frac{\ln(T(n_0, s')) - (\bar{h}\ln(s'_{\max}) + C)}{\ln(s') - \ln(s'_{\max})}\,, \tag{11}$$

where $s'_{\max}$ denotes the scale corresponding to the time series length. An example of such analysis for a randomly selected stock (McDonald's, MCD) can be viewed in figure 4, where the effective Hölder exponent $\hat{h}$ is shown as a function of time expressed in data points (lower panel). The values fluctuate around 0.5, which is in a perfect agreement with the results from the global analysis (Fig. 3). However, we observe an apparent trend in the course of $\hat{h}(t)$, suggesting the existence of long intervals with the strong singularities $\hat{h} < 0.5$ and the intervals with slightly weaker ones $\hat{h} > 0.5$. Interestingly, such intervals can be associated with the specific behavior of the price (upper panel of figure 4). It is tempting therefore to relate the local changes of the effective Hölder exponent with the changes of the direction in which the price goes. This issue requires, however, a more detailed investigation and we shall not discuss it here.

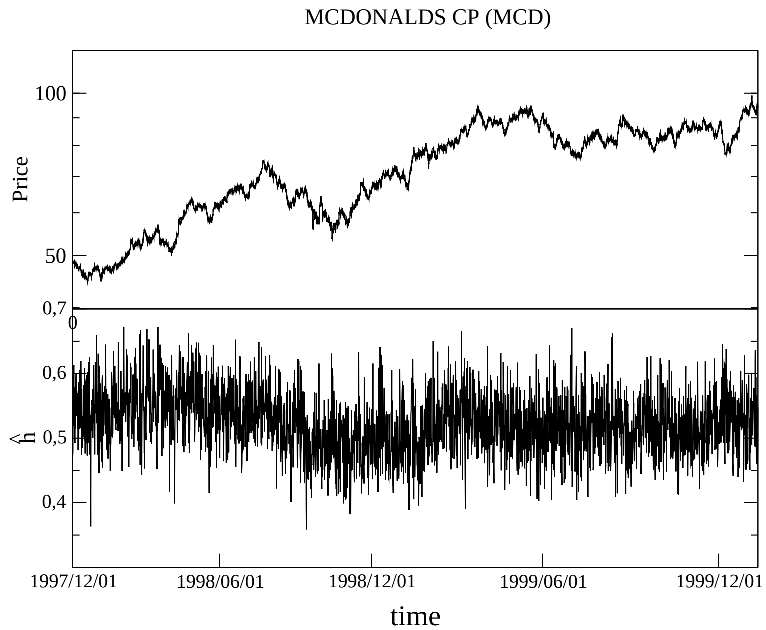

Fig. 4. McDonald's (MCD) stock price (top) and the effective Hölder exponents (bottom) as functions of time. Every 50th data point is shown.

4. Concluding remarks

The focus of this work was to compare the performance of the MF-DFA and the WTMM methods in the task of identification of the multifractal character of experimental data originating from the financial market. The financial data is known to be highly nonstationary and this makes the multifractal analysis difficult. Our results show that both methods are capable of detecting multifractality in time series of the stock price increments but their outcomes differ from each other if applied to the same data sets. Due to the lack of any knowledge of the exact processes underlying the financial dynamics, it is impossible to judge which of the methods is better basing only on the experimental results. Only by employing our previous experience on the subject, in which we applied MF-DFA and WTMM to investigate data from the theoretical models for which the exact results were known [31], we were able to interpret the results given by each of the methods. Our final conclusion is that in an attempt to a global characterization of the fractal properties of the signals, the wavelet-based method overestimates the widths of the $f(\alpha)$ spectra more than the MF-DFA one does. Thus we recommend using the detrended fluctuation method being more reliable, while the results from its wavelet-based counterpart should be interpreted more carefully. In contrast, WTMM is an optimal tool for a more locally-oriented analyses, which concentrate on local values of the effective Hölder exponents in close vicinity of some data point. This methods is also superior to other methods of the local analysis of the Hurst exponents like the DFA and the Detrended Moving Average method needing much longer signals than WTMM does.

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