

ERGOMETRIC THEORY OF THE ERGODIC HYPOTHESIS: SPECTRAL FUNCTIONS AND CLASSICAL ERGODICITY*

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The ergometric theory of the ergodic hypothesis is a physical theory for studying ergodicity in *quantum* many-body systems. It is based on the recurrence relations method, an exact dynamical formalism, well established and applied to numerous models both classical and quantum mechanical. In this work we show that the frequency spectra, in particular the distributions of the frequency, have properties which appear equivalent to the invariant measure and transitivity of phase space in classical ergodicity. We also show that the ergometric statement of the ergodic hypothesis can be reduced to Boltzmann's statement cast in microcanonical ensembles. This reduction indicates that the ergometric theory is a general theory of the ergodic hypothesis.

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1. Introduction

The ergodic hypothesis is fundamental to statistical mechanics, first put forth by Boltzmann more than a hundred years ago, well before the time of quantum mechanics. It is thus a concept couched in classical ideas, intended for classical many-body systems. Would a hypothesis born in classical mechanics be valid in its form in quantum many-body systems? Surely some generalizations on the original statement would be necessary. The purpose of our paper is to show how Boltzmann's definition must be generalized for modern problems of interest.

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First we will briefly review here the classical theory as it will be needed when we discuss ergodicity in quantum many-body systems. For reviews on the classical theory, see Refs. [1-3]. Consider a classical many-body Hamiltonian defined by

$$H = H(\mathbf{p}, \mathbf{q}), \quad (1)$$

where \mathbf{p} and \mathbf{q} are sets of canonical momenta and coordinates, $\mathbf{p} = \{p_i\}$ and $\mathbf{q} = \{q_i\}$, $i = 1, 2, \dots, 3N$, where N is the total number of particles. They are independent classical variables which depend on time t , determined by Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \quad (2)$$

Using these canonical variables, one can construct an element of phase space $\Delta\Gamma = \Pi\Delta p_i\Delta q_i$. By the Liouville theorem this space is a measure preserving space. The average of an arbitrary positive phase function $\phi = \phi(\mathbf{p}, \mathbf{q})$ is assumed to exist in this phase space. That is, $\bar{\phi} < \infty$, where

$$\bar{\phi} = \int d\Gamma \phi(\mathbf{p}, \mathbf{q}) \delta(H - E) / \int d\Gamma \delta(H - E), \quad (3)$$

evaluated on the surface of constant energy E . The average is also known as microcanonical ensemble average.

The classical ergodic hypothesis states that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi(\mathbf{p}(t), \mathbf{q}(t)) dt = \bar{\phi}. \quad (4)$$

Is it true? Or conditionally true? This is a physical question tied to the thermodynamic limit and other physical properties and processes. Yet it has drawn the attention of many mathematicians for nearly a hundred years, even today. Almost all the important progress has been made by mathematicians, hardly anything comparable by physicists. See *e.g.* Refs. [3] and [4].

The mathematical approach by its own nature tends to be abstract and universal, not tied to some specific models or systems. They have given us theorems on invariant measures and transitivity of phase space. A theorem due to Khinchin [4], for example, says that a variable A is ergodic for almost all trajectories starting from any initial point on the surface of constant energy except those of a set of measure 0. Another theorem due to Birkhoff [5] says that a variable A is ergodic if the phase space of motions of the phase points of A is metrically transitive. See Ref. [3].

They appear devoid of physical content. It is unclear where and how some of these abstract ideas bear on the essential physical properties. It is after all a physical problem and we need to know where the thermodynamic limit enters into. We need some physical basis to understand Boltzmann's hypothesis.

The classical definition of the ergodicity is restrictive. Most problems of modern interest are not formulated in microcanonical ensembles, but in canonical or grand canonical ensembles [1,2]. It is not known how or even whether the math theorems drawn from microcanonical ensembles could be translated for larger and less restrictive ensembles.

Some years ago Kubo [6] argued that "for a very large system, the canonical distribution is almost equivalent to a microcanonical distribution, the relative fluctuation being very small". Thus he concluded that the time average over canonical ensembles is almost like the time average over an ergodic surface, *i.e.* a surface of constant energy. This view seems to be more of a throwback, rather than an attempt to create something new. Given the weight of mathematical advances in classical ergodic theory, the reluctance to tamper with its very foundation is understandable.

In fact, it may very well be that these math theorems are universal enough to perhaps transcend the ensembles. Could they transcend the systems too? To answer this question we first need to see whether the classical definition of the ergodicity is tenable if quantum behavior rules. If not tenable, we would need to put forth a more general definition if to study ergodicity in quantum many-body systems. A new general definition must encompass the classical one. If, for example, the classical limit is taken on it, the classical definition must emerge. Let us now see whether the classical definition can hold up in face of quantum mechanics.

2. Limitations of the classical definition

The concept of phase space, whose element is $\Delta\Gamma$, is central to the classical theory. The phase element is built up with $\Delta p_i \Delta q_i, i = 1, 2, \dots, 3N$. If they are not classical variables, each of them is bound by the uncertainty principle, $\Delta p_i \Delta q_i \sim \hbar$ for every i . Thus $\Delta\Gamma$ is no longer a well defined quantity. It is a statement that the canonical variables (\mathbf{p}, \mathbf{q}) are not independent and commuting variables, implicitly assumed in classical mechanics. Thus from the very outset it is evident that the phase space concept, so critical to the classical formulation of the ergodic hypothesis, runs into difficulty.

Moreover, the classical ergodicity is premised on time averaging of precisely described trajectories on the surface of constant energy E , the ergodic surface. This basic concept is no more on secure ground as soon as the uncertainty principle is operative. Since $\Delta E \Delta t \sim \hbar$, at a precise moment of

time where a trajectory lies in phase space is not ascertainable. Quantum mechanics undermines the very foundations of the classical ergodicity. It brings to mind the challenges posed to classical mechanics by the black-body radiation before the advent of old quantum theory.

That is not the end of difficulty as there is a technical issue that must also be addressed: In quantum mechanics, variables are in general operators, not functions. Variables of interest in quantum many-body systems are typically the current or velocity operators, or the number density or spin operators. Let us denote these operators by A . If $A(t)$ is the time evolution of A , one would not time-average $A(t)$ for they are pure operators. There would be no physical content to them.

One would need to make functions out of them by bringing in the time-dependent density matrix $\rho(t)$. That is, we would time-average $\langle A(t) \rangle$, where

$$\langle A(t) \rangle = \text{Tr } A\rho(t). \quad (5)$$

The presence of the density matrix here implies that the system whose variable A is to be time-averaged in some manner would attain a thermal equilibrium once a perturbing field were turned off. See below for more details.

3. Ergometric theory

Quantum many-body systems are of primary contemporary interest. Knowing whether these systems are ergodic adds to our understanding of these models. The ergometric theory provides essential means for gaining this knowledge. Being a physical theory, it is system-specific. It provides a tool known as an *ergometer* with which to probe a particular variable of a system to see whether it is ergodic.

We will sketch below the essentials of the ergometric theory developed in the 2000s. For more details, see Refs. [7–17]. This development seems to have come at a time when there has been a growing active interest by physicists in this old abstract problem, still fundamental to statistical mechanics. See *e.g.* Refs. [18–45].

Let a large body be described by a *Hermitian* Hamiltonian $H(A)$ where A is a dynamical variable of interest. Place this body in a weak external field h , which couples linearly to the body via A . The external field may be either time-independent or time-dependent. The total energy is described by

$$H' = H(A) + hA. \quad (6)$$

The relevant density matrix is to be described by the above total energy H' . If $h = h_0$ a time-independent field, the ensemble average of A is

$$\langle A \rangle_{H'} = \text{Tr } A\rho. \quad (7)$$

It is a standard expression except in that the density matrix ρ is defined by H' not H . If $h = h_0(t)$ a time-dependent field, the observable or measurable is

$$\langle A(t) \rangle_{H'} = \text{Tr } A\rho(t), \tag{8}$$

where $\rho(t)$ is again defined by H' with $h = h_0(t)$.

What we have here is a large body initially in thermal equilibrium, perturbed by a weak external field h . This process is to be described by linear response theory, a quantum mechanical many-body perturbation theory.

According to the ergometric theory [7,12,15], the ergodic hypothesis on dynamical variable A of $H(A)$ takes the following form:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle A(t) \rangle_{H'} dt = \langle A \rangle_{H'}. \tag{9}$$

In the physical setting described by (6), h the external field (neutron beams, X-rays *etc.*) scatters off from a macroscopic target $H(A)$. In this scattering picture, the right-hand side of (9) may be viewed as elastic scattering at a fixed value of the momentum transfer while the left-hand side of (9) as inelastic scattering at the same value of the momentum transfer but at vanishing energy transfer. Thus, in our view, ergodicity is a measurable quantity, an observable.

If h is a very weak field, by the linear response theory [46–49] we have proved [7,12,15] that (9) is equivalent to:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \chi_A(t - t') dt' dt = \chi_A, \tag{10}$$

where $\chi_A(t)$ and χ_A are, respectively, the dynamic and static susceptibilities in zero external field, hence defined by $H(A)$, *i.e.*, system-specific. As a result, the corresponding density matrices are also defined by $H(A)$. Thus these scattering response functions are calculable by the formalism of the recurrence relations method for a given model H . See Sec. 4 below.

Proved in Refs. [7,12,15] is that the equality (10), hence also (9), holds if and only if $W_A \neq 0$ or ∞ , where W_A , dubbed an ergometer, is defined below:

$$W_A = \int_0^\infty r_A(t) dt, \tag{11}$$

where $r_A(t)$ is the autocorrelation function of A :

$$r_A(t) = \frac{(A(t), A)}{(A, A)}. \tag{12}$$

Here the inner product means the Kubo scalar product defined by H [50], which is thus system-specific. The inner product space of A is realized by the Kubo scalar product, so that it is not an abstract space. We shall now bring in the recurrence relations formalism to show how $r_A(t)$ is calculated which goes into W_A .

If $h = 0$ in (6), so that $H' = H$

$$\rho(t) = e^{-iHt} \rho e^{iHt}, \quad (13)$$

that is, $\dot{\rho}(t) = -i[H, \rho(t)]$. See *e.g.* Ref. [51]. Thus, now suppressing the subscript on the right bracket

$$\langle A(t) \rangle = \text{Tr} A \rho(t) = \text{Tr} A(t) \rho, \quad (14)$$

where

$$A(t) = e^{iHt} A e^{-iHt}, \quad (15)$$

which satisfies the Heisenberg equation of motion:

$$\dot{A}(t) = i[H, A(t)]. \quad (16)$$

The recurrence relations formalism calculates $A(t)$ on a realized space. For different but related approaches, see *e.g.* Refs. [52–60].

4. Recurrence relations formalism

This is an exact dynamical formalism, developed in the early 1980s, to obtain $A(t)$ and related physical functions like the autocorrelation function $r_A(t)$ defined by (12). For a recent review, see Ref. [61]. The formalism in addition yields conditions for admissible solutions for the autocorrelation function. For example, a simple exponential decay is found not admissible for a Hermitian system [92]. During the ensuing years it has been widely applied to a variety of problems by many workers, cited in [61]. For more recent works, see Refs. [62–83].

The solution to the equation of motion may be described geometrically as follows [84]: If $A(t)$ is a vector in an inner product space, it is spanned by d orthogonal basis vectors. Hence d denotes the number of dimensions of the inner product space, itself system-specific. As time t evolves, the projection of $A(t)$ onto different basis vectors changes. If H is Hermitian, the norm $\|A(t)\| = \|A\|$, where $A = A(0)$, meaning that the norm of $A(t)$ is an invariant of time. Thus as t evolves the magnitude of $A(t)$ does not change only its direction. More formally it is given by an orthogonal expansion

$$A(t) = \sum_{k=0}^{d-1} a_k(t) f_k, \quad (17)$$

where $\{f_k\}, k = 0, 1, 2, \dots, d - 1$, is a set of basis vectors satisfying orthogonality

$$(f_k, f_{k'}) = 0 \quad \text{if } k' \neq k \tag{18}$$

and $\{a_k(t)\}$ is a set of autocorrelation functions

$$a_k(t) = \frac{(A(t), f_k)}{(f_k, f_k)}. \tag{19}$$

If $f_0 = A$ for $k = 0$, we obtain the basal function

$$a_0(t) = \frac{(A(t), A)}{(A, A)}, \tag{20}$$

which corresponds to $r_A(t)$ of W_A . See (11) and (12). The boundary conditions are: $a_k(0) = 1$ if $k = 0$ and $= 0$ if $k \neq 0$. Both $\{f_k\}$ and $\{a_k(t)\}$ satisfy 3-term recurrence relations of their own known as *rr1* and *rr2*, respectively. In all this, the value of d is an important property. It too is system-specific, depending on the parameters of H , *e.g.* spin statistics, number of degrees of freedom, *etc.* and on its environment through the realized inner product space.

If \mathcal{L} is the Laplace transform operator,

$$\tilde{a}_0(z) = \mathcal{L}[a_0(t)] = \frac{1}{z + \frac{\Delta_1}{z + \frac{\Delta_2}{z + \dots \frac{\Delta_{d-1}}{z}}}}, \tag{21}$$

a continued fraction, where $\Delta_k = \|f_k\|/\|f_{k-1}\|, k = 1, \dots, d - 1$.

If H and A are given, one can determine d by calculating f_k 's term by term by *rr1* up to $f_d = 0$. If $\{f_k\}$ is known, $\{\Delta_k\}$ is also calculable. Thus, $\tilde{a}_0(z)$ is explicitly given. For a number of quantum and classical many-body models, $\{\Delta_k\}$ has been calculated. See the review paper by Balucani *et al.* [61].

5. Frequency spectra

If $z = \pm i\omega$, $\tilde{a}_o(z)$ gives the spectral or frequency distribution. Observe that since $W = \tilde{a}_o(z = 0)$, it refers to the frequency distribution at the origin or zero frequency. But not all distributions are admissible for W . In deriving W it was proved [7,12,15] that $d \rightarrow \infty$ is a necessary condition. For this limit to exist, it is necessary that $N \rightarrow \infty$ where N is the number of particles, a parameter of H .

If $d \rightarrow \infty$ in (21), $\tilde{a}_0(z)$ becomes an *infinite* continued fraction. It is no longer a meromorphic function but an irrational or multi-valued function with a branch cut or cuts. If H is Hermitian, the branch cuts must lie on the imaginary axis of z symmetrically above and below the real axis of z . We will consider two examples:

5.1. Example 1

$$\begin{aligned} \Delta_k &= 2 \quad \text{if } k = 1, \\ &= 1 \quad \text{if } k = 2, 3, \dots, \infty. \end{aligned} \quad (22)$$

This particular structure (known as a *hypersurface* in the language of the recurrence relations method) is realized in both quantum and classical models: (i) In the ground state of a 2D ideal electron gas at long wavelengths if A is the density fluctuation operator [85]. (ii) In a 1D nn coupled classical harmonic oscillator (HO) chain with periodic boundary conditions if A is the momentum of any particle in the chain [86]. The two inequivalent systems (one quantum mechanical and the other classical) necessarily must have dynamically equivalent behavior.

For this hypersurface the infinite continued fraction (21) is summable to:

$$\tilde{a}_0(z) = \frac{1}{\sqrt{z^2 + 4}}. \quad (23)$$

There is a branch cut from $-2i$ to $+2i$.

5.2. Example 2

$$\Delta_k = \frac{k^2}{(4k^2 - 1)}, \quad k = 1, 2, 3, \dots, \infty. \quad (24)$$

This hypersurface is realized in the ground state of a 3D ideal electron gas at long wavelengths if A is the density fluctuation operator [87]. The infinite continued fraction is summable to

$$\tilde{a}_0(z) = \arctan \frac{1}{z} = \frac{1}{2}i \log \frac{(z+i)}{(z-i)}. \quad (25)$$

There is a branch cut between branch points $z = +i$ and $z = -i$.

Mathematically, one may draw a branch cut any line connecting the two branch points, *e.g.* a line going through $\pm i\infty$. But to be physically relevant it must be drawn a line on the imaginary axis going through the origin since this is where the frequency spectra lie.

In the literature of the recurrence relations method are found many other solutions of this kind, all derived from some Hermitian many-body models both classical and quantum. But these two examples should suffice for our purposes of illustration. Since the above two solutions are realized in physical models, the following analysis is not abstract but physically pertinent.

The presence of branch cuts is significant. On a branch cut between a pair of branch points are contained all possible irrational values of the frequency. There are indenumerably many such that the frequency spectrum is dense or has measure $\mu = 1$. That is, $d\mu(\omega) = P(\omega)d\omega$, where $P(\omega)$ is the distribution or density of the frequency ω 's. This is analogous to an invariant measure with which ergodicity in classical chaos is characterized [88,89]. Thus we can conclude that the ergometric theory also contains the same element of classical ergodicity, that of an invariant measure.

But we must remember that in the ergometric theory this requirement of $d \rightarrow \infty$ is only a necessary condition for ergodicity. The sufficient condition is provided by $W = \text{finite}$ (not zero, not infinity). Since $W = \tilde{a}_0(z = 0)$, it refers to the frequency distribution at the origin. If $W = 0$, we see that the branch cut is divided into two halves. If $W = \infty$, $\tilde{a}_0(z)$ must be singular as $z \rightarrow 0$ from both sides, again splitting the branch cut into two halves. If W is finite, it is as if there is a bridge over the origin connecting the two halves of the spectrum.

The two end points $W = 0$ and ∞ can occur in a single model as duality points. For example, consider a classical 1D nn coupled equal-mass m HO chain with periodic boundary conditions. Now make the mass of one oscillator different from the rest as if it were an impurity mass m_0 . This introduces a new parameter say $\lambda = m/m_0$. See Ref. 86. If $\lambda \rightarrow 0$ (Brownian limit), $W \rightarrow \infty$. If $\lambda \rightarrow \infty$ (vacancy limit), $W \rightarrow 0$. If λ is finite, the dynamical variable A (the momentum of the impurity mass) is ergodic.

If a spectrum is divided into two halves, it is equivalent to saying that the classical phase space is metrically *intransitive*. If a spectrum is not divided, it is like the phase space is *transitive*. The ergometric theory has a property very much like the transitivity in classical ergodic theory. Birkhoff's theory [3,5,13] states that a phase function of a classical system is ergodic if it is metrically transitive provided that the phase average exists almost everywhere, *i.e.* no zero measure. We can thus conclude that the ergometric theory contains these essential properties of classical ergodic theory.

6. Classical limit

The ergometric theory is constructed for ergodicity in quantum many-body systems. As already stated, this theory is based on the recurrence relations formalism. The formalism applies to quantum many-body models

in the classical limit such as the *spin-1/2* Ising and *XY* chains at $T = \infty$ [64,65,76–79,90]. It also applies to classical many-body systems such as classical *nn* coupled harmonic oscillator chains [63,86] and classical plasmas [79–83]. Thus the ergometric theory should also be applicable to ergodicity in quantum many-body models in the classical limit as well as classical many-body systems themselves. Below we shall show how the ergometric formulation of the ergodic hypothesis reduces to Boltzmann's ergodic hypothesis postulated in microcanonical ensembles.

To obtain the classical version of the recurrence relations formalism, we need to express two basic quantities in the classical form:

6.1. Equation of motion

The starting point of the recurrence relations formalism is the Heisenberg equation of motion, see (16), given in terms of H and A . Let H represent a classical many-body model. Now A being a dynamical variable of a classical model, it is a classical function: $A = A(\mathbf{p}, \mathbf{q})$, where \mathbf{p} and \mathbf{q} denote relevant sets of canonical variables.

For the equation of motion, the right-hand side of (16) is replaced by the Poisson brackets (pb). Letting C.F. to mean the classical form

$$A(t)|_{\text{C.F.}} = e^{Lt} A, \quad (26)$$

where L is the Liouville operator,

$$LA = [H, A]_{\text{pb}} = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j} \frac{\partial A}{\partial p_j} \right). \quad (27)$$

6.2. Kubo scalar product

According to the recurrence relations formalism, the time evolution of A takes place on the surface of an inner product space realized by the Kubo scalar product. The classical form of the inner product of a pair of dynamical variables A and B say is

$$(A, B)|_{\text{C.F.}} = \int d\Gamma \rho AB, \quad (28)$$

where $d\Gamma$ is an element of phase space and ρ is the density matrix in appropriate ensembles. All the quantities on the right-hand side of (28) are the classical analogs of the quantum mechanical quantities.

If the density matrix is defined in grand ensembles (ge), the classical form is given by

$$\rho_{\text{ge}} = \exp\{\alpha N - \beta H\} / \int d\Gamma \exp\{\alpha N - \beta H\}, \quad (29)$$

where $\alpha = \beta\mu$, μ the chemical potential, N and H are number and energy variables, respectively. If the variable N is fixed at $N = N_1$ say,

$$\rho_{ge}|_{N=N_1} = \exp(-\beta H) \int d\Gamma \exp(-\beta H), \tag{30}$$

where the right-hand side of (30) is ρ_{ce} the density matrix in canonical ensembles (ce) in the classical form. If the energy is fixed at $H = E$,

$$\rho_{ce}|_{H=E} = \delta(H - E) / \int d\Gamma \delta(H - E), \tag{31}$$

where the right-hand side of (31) is ρ_{me} the density matrix in microcanonical ensembles (me) in the classical form.

Now let us see how the ergometric form of the ergodic hypothesis (9) reduces to the classical form (4). Since the classical form has no external field, we let $h = 0$ in H' , see (6), so that $H' = H$.

6.3. Classical ensemble average in microcanonical ensembles

$$\begin{aligned} \langle A \rangle|_{\text{C.F./me}} &= \text{Tr} A \rho|_{\text{C.F./me}} \\ &= \int d\Gamma A(\mathbf{p}, \mathbf{q}) \delta(H - E) / \int d\Gamma \delta(H - E) = \bar{A}. \end{aligned} \tag{32}$$

6.4. Time evolution in classical form

$$A(t)|_{\text{C.F.}} = e^{Lt} A = A(\mathbf{p}(t), \mathbf{q}(t)). \tag{33}$$

6.5. $\langle A(t) \rangle|_{\text{C.F.}}$ in microcanonical ensembles

$$\begin{aligned} \langle A(t) \rangle|_{\text{C.F./me}} &= \text{Tr} A(t) \rho|_{\text{C.F./me}} \\ &= \int d\Gamma A(\mathbf{p}(t), \mathbf{q}(t)) \delta(H - E) / \int d\Gamma \delta(H - E). \end{aligned} \tag{34}$$

We now return to the ergometric statement of the ergodic hypothesis (9).

If A is a classical function and not an operator, the left-hand side of (9) may be written as (suppressing the lim sign for simplicity),

$$\text{l.h.s. of (9)} = \left\langle \frac{1}{T} \int_0^T A(t) dt \right\rangle. \tag{35}$$

That is, the order of the two averages is exchanged. This exchange is possible if and only if A is a classical function. See Ref. [91]. The right-hand of (9) is a number. Hence its value is unchanged if it is averaged again:

$$\text{r.h.s. of (9)} = \langle\langle A \rangle\rangle. \quad (36)$$

Thus, for classical systems only,

$$\left\langle \frac{1}{T} \int_0^T A(t) dt \right\rangle = \langle\langle A \rangle\rangle. \quad (37)$$

It must also be true if the outer ensemble average is removed from both sides giving us:

$$\frac{1}{T} \int_0^T A(t) dt = \langle A \rangle. \quad (38)$$

If $A(t)$ is a classical function of t and $\langle \dots \rangle$ means microcanonical ensembles, by Secs. 6.1–6.5 the above is expressed as (with the limit sign now restored),

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T A(\mathbf{p}(t), \mathbf{q}(t)) dt = \int d\Gamma A(\mathbf{p}, \mathbf{q}) \delta(H - E) / \int d\Gamma \delta(H - E), \quad (39)$$

where on the left-hand side the trajectories of $A(\mathbf{p}(t), \mathbf{q}(t))$, now a phase function, are to be constrained to move on the surface of constant energy E . Evidently (39) is precisely the ergodic hypothesis postulated by Boltzmann.

7. Concluding remarks

In an earlier paper [13] we showed that if a variable A in a system is measured *ergodic* by the ergometer, it also satisfies all the ergodic conditions of Birkhoff's theorem. If it is measured *not ergodic*, one or more of the ergodic conditions were not satisfied. In that paper we have noted that this correspondence is probably not a coincidence. Although Birkhoff's theorem is based on classical physics, pitched in microcanonical ensembles, the underlying principles must be universal to transcend the ensembles and systems. In this present work we have demonstrated that these same principles are also contained in the ergometric theory, although in a different form, which explains the correspondence earlier observed.

The ergometric theory contains additional properties beyond classical ergodic theory. It was shown in another paper [14] that Khinchin's condition for ergodicity, which we have termed *irreversibility*, is not a sufficient condition. According to the recurrence relations formalism, the autocorrelation function in a Hermitian system must be irreversible if d (the dimensions of the realized inner product space) increases without limit. That $d \rightarrow \infty$ is a necessary condition for ergodicity in the ergometric theory. An irreversible autocorrelation function does not necessarily ensure that W be finite. In such a case, an ergometric analysis would show that there is an incomplete delocalization of energy were a system perturbed by an external field.

Because of the physical basis of our formulation, we believe that the ergometric theory is built on sound ground. It contains essential elements of the classical ergodic theory albeit in different guise. It lends a physical understanding to what has been long viewed abstruse. Had Boltzmann however not postulated the ergodic hypothesis, statistical mechanics probably would have gone in a far different direction.

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