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COUPLED NONLINEAR OSCILLATORS: METAMORPHOSES OF AMPLITUDE PROFILES FOR THE APPROXIMATE EFFECTIVE EQUATION — THE CASE OF 1 : 3 RESONANCE

Jan Kyzioł

Department of Mechatronics and Mechanical Engineering
Kielce University of Technology
Al. 1000-lecia PP7, 25-314 Kielce, Poland

Andrzej Okniński

Physics Division, Department of Management and Computer Modelling
Kielce University of Technology
Al. 1000-lecia PP7, 25-314 Kielce, Poland

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We study dynamics of two coupled periodically driven oscillators. A classic example of such a system is a dynamic vibration absorber which consists of a small mass attached to the primary vibrating system of a large mass. Periodic solutions of the approximate effective equation (derived in our earlier papers) are determined within the Krylov–Bogoliubov–Mitropolsky approach to compute the amplitude profiles $A(\Omega)$. In the present paper, we investigate metamorphoses of the function $A(\Omega)$ induced by changes of the control parameters in the case of 1 : 3 resonances.

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1. Introduction

In the present paper, we analyse two coupled oscillators, one of which is driven by an external periodic force $F(t)$. An important example of such a system is a dynamic vibration absorber which consists of a mass m_2, attached to the primary vibrating system of mass m_1 [1, 2]. Equations describing dynamics of this system are of form

$$\left.\begin{array}{l} m_1\ddot{x}_1 - V_1(\dot{x}_1) - R_1(x_1) + V_2(\dot{x}_2 - \dot{x}_1) + R_2(x_2 - x_1) = F(t) \\ m_2\ddot{x}_2 - V_2(\dot{x}_2 - \dot{x}_1) - R_2(x_2 - x_1) = 0 \end{array}\right\}, \tag{1.1}$$

where V_1, R_1 and V_2, R_2 represent (nonlinear) force of internal friction and (nonlinear) elastic restoring force for mass m_1 and mass m_2, respectively. In the present paper we shall consider a simplified model

$$F(t) = f\cos(\omega t)\ , \qquad R_1(x_1) = -\alpha_1 x_1\,, \qquad V_1(\dot{x}_1) = -\nu_1 \dot{x}_1\,. \tag{1.2}$$

Dynamics of coupled periodically driven oscillators is very complicated, see [3,4,5,6,7] and references therein. We simplified the problem described by equations (1.1), (1.2) deriving the exact fourth-order nonlinear equation for internal motion as well as approximate second-order effective equation in [8].

Moreover, applying the Krylov–Bogoliubov–Mitropolsky method to these equations we have computed the corresponding nonlinear resonances in the effective equation (*cf.* [8] and [9] for the cases of 1 : 1 and 1 : 3 resonances, respectively). Dependence of the amplitude A of nonlinear resonances on the frequency ω is significantly more complicated than in the case of Duffing oscillator and this leads to new nonlinear phenomena. We shall refer to functions $A(\omega)$ as amplitude profiles or resonance curves. In a recent paper, we investigated metamorphoses of amplitude profiles induced by changes of control parameters in the case of 1 : 1 resonance [10]. In the present paper, we continue this approach studying metamorphoses of $A(\omega)$ for 1 : 3 resonance.

In the next section, the exact 4th-order equation for the internal motion and approximate 2nd-order effective equations in nondimensional form are described. In Section 3 amplitude profiles for 1 : 3 resonances are determined within the Krylov–Bogoliubov–Mitropolsky approach for the approximate 2nd-order effective equation (and for the Duffing equation which follows from the effective equation if some parameters are put equal zero). In Section 4 theory of algebraic curves is used to compute singular points of effective equation amplitude profiles — metamorphoses of amplitude profiles occur in neighbourhoods of such points. In Section 5 examples of analytical and numerical computations are presented for the Duffing and effective equations. Our results are summarized and perspectives of further studies are described in the last section.

2. Exact equation for internal motion and its approximations

In new variables, $x \equiv x_1$, $y \equiv x_2 - x_1$, equations (1.1), (1.2) can be written as

$$\left.\begin{array}{l} m\ddot{x} + \nu\dot{x} + \alpha x + V_e(\dot{y}) + R_e(y) = f\cos(\omega t) \\ m_e(\ddot{x} + \ddot{y}) - V_e(\dot{y}) - R_e(y) = 0 \end{array}\right\}, \tag{2.1}$$

where $m \equiv m_1$, $m_e \equiv m_2$, $\nu \equiv \nu_1$, $\alpha \equiv \alpha_1$, $V_e \equiv V_2$, $R_e \equiv R_2$. It is possible to simplify the problem eliminating the variable x in (2.1) to obtain the exact fourth-order equation for the variable y only, describing relative motion of the mass m_e [8], see also Ref. [11], where separation of variables for a more general system of coupled equations was described.

In the present work, we assume

$$R_e(y) = \alpha_e y - \gamma_e y^3 \,, \qquad V_e(\dot{y}) = -\nu_e \dot{y} \,. \tag{2.2}$$

The exact equation for relative motion reads

$$\left.\begin{array}{l} \widehat{L}\left(\mu \frac{d^2 y}{dt^2} + \nu_e \frac{dy}{dt} - \alpha_e y + \gamma_e y^3\right) + \lambda m_e \left(\nu \frac{d}{dt} + \alpha\right) \frac{d^2 y}{dt^2} = F\cos(\omega t) \\ \widehat{L} \equiv M \frac{d^2}{dt^2} + \nu \frac{d}{dt} + \alpha \end{array}\right\} , \tag{2.3}$$

where $F = m_e \omega^2 f$, $\mu = m m_e / M$ and $\lambda = m_e / M$ is a nondimensional parameter [8].

Eq. (2.3) can be written in the following nondimensional form [10]

$$\left.\begin{array}{l} \widehat{\mathcal{L}}\left(\frac{d^2 z}{d\tau^2} + h \frac{dz}{d\tau} - z + z^3\right) + \kappa \left(H \frac{d}{d\tau} + a\right) \frac{d^2 z}{d\tau^2} = G \frac{\kappa}{\kappa+1} \Omega^2 \cos(\Omega \tau) \\ \widehat{\mathcal{L}} \equiv \frac{d^2}{d\tau^2} + H \frac{d}{d\tau} + a \end{array}\right\} , \tag{2.4}$$

where nondimensional time τ and nondimensional displacement z of the mass m_e are defined as

$$\tau = t\bar{\omega} \,, \qquad z = y\sqrt{\frac{\gamma_e}{\alpha_e}} \,, \qquad \left(\bar{\omega} = \sqrt{\frac{\alpha_e}{\mu}}\right) \,, \tag{2.5}$$

while nondimensional constants are given by

$$\begin{array}{lll} h = \dfrac{\nu_e}{\mu\bar{\omega}} \,, & H = \dfrac{\nu}{M\bar{\omega}} \,, & \Omega = \dfrac{\omega}{\bar{\omega}} \,, \\ G = \dfrac{1}{\alpha_e}\sqrt{\dfrac{\gamma_e}{\alpha_e}} f \,, & \kappa = \dfrac{m_e}{m} \,, & a = \dfrac{\alpha\mu}{\alpha_e M} \,. \end{array} \tag{2.6}$$

We shall consider hierarchy of approximate equations arising from (2.4) [10]. For small enough values of the parameters κ, H, a we can reject the second term on the left in (2.4) obtaining the approximate equation which can be integrated partly to yield the effective equation

$$\frac{d^2 z}{d\tau^2} + h \frac{dz}{d\tau} - z + z^3 = -\gamma \frac{\Omega^2}{\sqrt{\left(\Omega^2 - a\right)^2 + H^2 \Omega^2}} \cos(\Omega\tau + \delta) \,, \tag{2.7}$$

where transient states have been omitted, $\gamma \equiv G \frac{\kappa}{\kappa+1}$ and $\tan\delta = \frac{\Omega H}{\Omega^2 - a}$. And, finally, for $H = 0$, $a = 0$ we get the Duffing equation

$$\frac{d^2 z}{d\tau^2} + h \frac{dz}{d\tau} - z + z^3 = -\gamma \cos(\Omega\tau + \delta) \,. \tag{2.8}$$

3. Perturbation analysis of the 1 : 3 resonance

The 1 : 3 resonance, a solution of the effective equation (2.7) of form $z = A\cos\left(\frac{1}{3}\Omega\tau+\varphi\right)$, can be seen in the bifurcation diagram computed for the effective equation — see Fig. 4.1 in [8], $\omega < 3.1$. We apply the Krylov–Bogoliubov–Mitropolsky (KBM) perturbation approach [12, 13] to Eq. (2.7), working in the spirit of [6], to determine the corresponding amplitude profile, *i.e.* dependence of the amplitude A on frequency Ω. The KBM formula is approximate but the existence of the 1 : 3 attractor can be verified numerically with high precision — there are also rigorous mathematical proofs demonstrating existence of such (subharmonic) solutions in analogous dynamical systems [14].

To study subharmonic resonance 1 : 3 we cast equation (2.7) into form

$$\frac{d^2z}{d\tau^2}+\Theta^2 z+\varepsilon\left(\left(-a_0-\Theta_0^2+a_0z^2\right)z+h_0\frac{dz}{d\tau}\right)=\frac{-\gamma\Omega^2\cos(\Omega\tau+\delta)}{\sqrt{\left(\Omega^2-a\right)^2+H^2\Omega^2}} \tag{3.1}$$

with

$$\varepsilon\Theta_0^2=\Theta^2\,,\qquad \varepsilon a_0=1\,,\qquad \varepsilon h_0=h\,, \tag{3.2}$$

where we assumed that the external force is of order ε^0 rather than ε^1 (see [15] for discussion).

We substitute $z(\tau)=u(\tau)+u_0(\tau)$ into (3.1) to remove the external forcing term on the right-hand side. We thus get

$$\left.\begin{array}{l}\frac{d^2u}{d\tau^2}+\Theta^2u+\frac{d^2u_0}{d\tau^2}+\Theta^2u_0+\varepsilon g\left(u,u_0\right)=\frac{-\gamma\Omega^2\cos(\Omega\tau+\delta)}{\sqrt{\left(\Omega^2-a\right)^2+H^2\Omega^2}}\\ g\left(u,u_0\right)=h_0\frac{d(u+u_0)}{d\tau}+\left(-a_0-\Theta_0^2\right)\left(u+u_0\right)+a_0\left(u+u_0\right)^3\end{array}\right\}. \tag{3.3}$$

Now we put $u_0(\tau)=C\cos(\Omega\tau+\delta)$ into (3.3). It follows that for $C=\frac{-\gamma\Omega^2}{\sqrt{\left(\Omega^2-a\right)^2+H^2\Omega^2}}\frac{1}{\Theta^2-\Omega^2}$ two terms on the left-hand side, $\frac{d^2u_0}{d\tau^2}+\Theta^2u_0$, and the external forcing term on the right-hand side of (3.3) cancel out to yield

$$\frac{d^2u}{d\tau^2}+\Theta^2u+\varepsilon g\left(u,u_0\right)=0\,. \tag{3.4}$$

We shall now determine approximate form of Θ^2 following procedure described in [6]. Neglecting in (3.1) the damping term $h_0\frac{dz}{d\tau}$ and external forcing we get

$$\frac{d^2z}{d\tau^2}-z+z^3=0\,. \tag{3.5}$$

Substituting in (3.5) $z(\tau)=A\cos(\Theta\tau)$, applying identity $\cos^3(\Theta\tau)=\frac{3}{4}\cos(\Theta\tau)+\frac{1}{4}\cos(3\Theta\tau)$, and rejecting term proportional to $\cos(3\Theta\tau)$ we get finally the approximate expression $\Theta^2=\frac{3}{4}A^2-1$.

We have thus written the effective equation (2.7) in form (3.4) with $g(u, u_0)$ defined in (3.3) and

$$\left.\begin{array}{ll} u_0(\tau) = C\cos(\Omega\tau + \delta)\,, & C = \frac{-\gamma\Omega^2}{\sqrt{(\Omega^2-a)^2+H^2\Omega^2}\,(\Theta^2-\Omega^2)} \\ \Theta^2 = \frac{3}{4}A^2 - 1 & \end{array}\right\}. \tag{3.6}$$

Since we are looking for 1 : 3 resonances we have to consider frequencies Ω close to 3Θ. We thus put $\Theta^2 = \left(\frac{\Omega}{3}\right)^2 + \varepsilon\sigma$ with σ of order ε^0 into (3.4), obtaining finally

$$\frac{d^2u}{d\tau^2} + \left(\frac{\Omega}{3}\right)^2 u + \varepsilon\left(\sigma u + g(u, u_0)\right) = 0\,. \tag{3.7}$$

We assume the following form of the solution

$$u = A\cos\left(\frac{\Omega}{3}\tau + \varphi\right) + \varepsilon u_1(A, \varphi, \tau) + \ldots\,. \tag{3.8}$$

Substituting (3.8) into (3.7), eliminating secular terms and demanding that $\frac{dA}{d\tau} = 0$, $\frac{d\varphi}{d\tau} = 0$ to find stationary states we get finally [9]

$$\left.\begin{array}{l} \left(h\frac{\Omega}{3}\right)^2 + \left(\frac{3}{4}A^2 + \frac{3}{2}C^2 - \frac{1}{9}\Omega^2 - 1\right)^2 = \left(\frac{3}{4}AC\right)^2 \\ \tan(3\varphi - \delta) = \frac{-h\Omega}{3\left(\frac{3}{4}A^2+\frac{3}{2}C^2-\frac{1}{9}\Omega^2-1\right)} \end{array}\right\} \tag{3.9}$$

with C given by (3.6). If we put $H = 0$, $a = 0$ then we get implicit equation for the amplitude profile for the Duffing equation

$$\left.\begin{array}{l} \left(h\frac{\Omega}{3}\right)^2 + \left(\frac{3}{4}A^2 + \frac{3}{2}C^2 - \frac{1}{9}\Omega^2 - 1\right)^2 = \left(\frac{3}{4}AC\right)^2 \\ C = \frac{-\gamma}{\left(\frac{3}{4}A^2-\Omega^2-1\right)} \end{array}\right\}. \tag{3.10}$$

4. Metamorphoses of the amplitude profiles for the 1 : 3 resonance

Equations (3.9), (3.10) define the corresponding amplitude profiles (resonance curves) implicitly. Such amplitude profiles can be classified as planar algebraic curves, see [16] for a general theory. Let $L(X, Y; \lambda) = 0$ defines such a curve, where λ is a parameter. A singular point $(X, Y) = (X_*, Y_*)$ of the algebraic curve fulfils conditions

$$L(X, Y; \lambda) = 0\,, \qquad \frac{\partial L(X, Y; \lambda)}{\partial X} = 0\,, \qquad \frac{\partial L(X, Y; \lambda)}{\partial Y} = 0\,. \tag{4.1}$$

Assume that a solution (X_*, Y_*) of Eqs. (4.1) exists for $\lambda = \lambda_*$ and there are no other solutions in some neighbourhood of λ_*. Let $\lambda < \lambda_*$, then the curve $L(X, Y; \lambda) = 0$ for growing values of λ changes its form at $\lambda = \lambda_*$ and, again, for $\lambda > \lambda_*$. We shall refer to such changes as metamorphoses (*cf.* [10] for metamorphoses of amplitude profiles in the case of $1:1$ resonance in the effective equation).

In the case of the effective equation, the amplitude profile of $1:3$ resonance is given by Eqs. (3.9), (3.6) or, in new variables $X \equiv \Omega^2$, $Y \equiv A^2$, by the equation $L(X, Y; a, \gamma, h, H) = 0$, where

$$\begin{aligned} &L(X,Y;a,\gamma,h,H) = U^4\left(\tfrac{1}{9}h^2X + U^2\right)\left((X-a)^2 + H^2X\right)^2 \\ &+3\gamma^2U^2X^2\left(\tfrac{9}{16}Y - \tfrac{1}{9}X - 1\right)\left((X-a)^2 + H^2X\right) + \tfrac{9}{4}\gamma^4X^4\,, \\ &U \equiv \tfrac{3}{4}Y - X - 1\,. \end{aligned} \tag{4.2}$$

Equations for singular points of the amplitude profile for the $1:3$ resonance of the effective equation are given by (4.1), (4.2). We can construct a solution of these equations in the following way. We first solve the following cubic equation for U

$$\begin{aligned} &c_3U^3 + c_2U^2 + c_1U + c_0 = 0\,, \\ c_3 &= 567B_1^2\,, \qquad c_0 = 32B_2^2X\left(55X^2+9\right)\,, \\ c_2 &= -18B_1\left(28X^3 + (87q-81)X^2 + (146r-54q)X - 27r\right)\,, \\ c_1 &= B_2'\left(967X^3 + 639X^2 + (162 - 967r + 423q)X + 207r + 81q\right)\,, \\ B_1 &= 3X^2 + qX - r\,, \qquad B_2 = X^2 + qX + r\,, \qquad B_2' = -4B_2X\,, \\ q &= H^2 - 2a\,, \qquad r = a^2 \end{aligned} \tag{4.3}$$

for arbitrary $X > 0$, $a \geq 0$, $H \geq 0$, and next compute variables $p,\ s$

$$\left.\begin{aligned} p &= -\frac{2}{9}\frac{(9U+8X)\left(\left(3X^3+2X^2q+(-9q+r)X-18r\right)U-16X^3-16X^2q-16Xr\right)}{X\left((81X^2+27qX-27r)U-46X^3+(-18-46q)X^2+(-46r-18q)X-18r\right)} \\ s &= -\frac{4}{9}\frac{U^2\left(X^2+qX+r\right)\left(128X^2+162pX+360XU+243U^2\right)}{X^2(46X+81U-18)} \end{aligned}\right\}. \tag{4.4}$$

Finally, we find Y, h and γ from the definitions

$$U = \tfrac{3}{4}Y - X - 1\,, \qquad p = \tfrac{1}{9}h^2\,, \qquad s = \tfrac{3}{2}\gamma^2\,, \tag{4.5}$$

and choose physically acceptable solutions: $Y > 0,\ h > 0,\ \gamma > 0$.

The case of the Duffing equation is obtained by putting $a = H = 0$ in the above formulae. We thus get

$$L(X,Y;\gamma,h) = U^4\left(\tfrac{1}{9}h^2X + U^2\right) + 3\gamma^2U^2\left(\tfrac{9}{16}Y - \tfrac{1}{9}X - 1\right) + \tfrac{9}{4}\gamma^4\,, \tag{4.6}$$

and

$$5103U^3 + (-1512X + 4374)\,U^2 + \left(-3868X^2 - 2556X - 648\right)U$$
$$+1760X^3 + 288X = 0\,, \tag{4.7}$$

where X is arbitrary, $U = \frac{3}{4}Y - X - 1$ and

$$h^2 = 2\frac{24UX+27U^2-128X-144U}{46X-81U+18}\,, \qquad \gamma^2 = \frac{8}{27}U^2\frac{72UX+243U^2-128X^2}{46X-81U+18}\,. \tag{4.8}$$

5. Analytical and numerical computations

We shall find metamorphoses of bifurcation diagrams for the Duffing and effective equations (2.8), (2.7), respectively, using results of the preceding section. Bifurcation diagrams show projection of the numerical solution (a trajectory) $z(t)$ onto the Poincaré section $(z(t)\,,\ t = t_0 + nT)$, where $n = 0, 1, 2, \ldots$ and $T = \frac{2\pi}{\Omega}$ is the period of the external force. The 1 : 3 resonance in Fig. 1 (and in other bifurcation diagrams below) is thus represented by three lines — this shows that the subharmonic 1 : 3 solution is stable. On the other hand, quasi-periodic and/or chaotic attractors form complicated structures (they can be further distinguished by computing Lyapunov exponents).

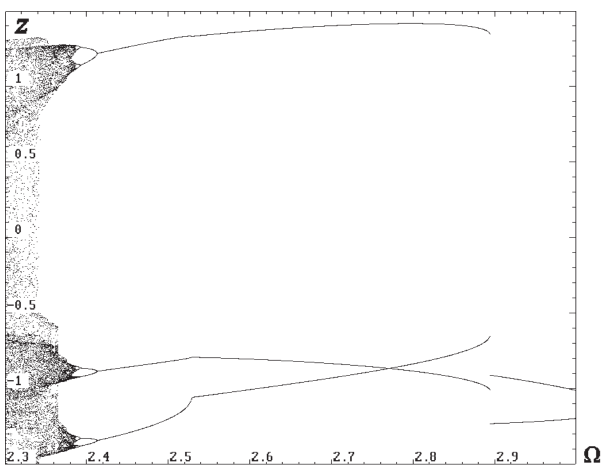

Fig. 1. Bifurcation diagram for the Duffing equation, $h = 0.4$, $\gamma = \gamma_*$.

5.1. The case of the Duffing equation

We shall compute a singular point of the amplitude profile (3.10). The solution of Eqs. (4.1) is given by equations (4.6), (4.7), (4.8), where X is arbitrary and $X = \Omega^2$. We thus choose, for example, $\Omega_* = 1.6$ $(X_* = 2.56)$. We compute from Eq. (4.7) for $X = X_*$ one physical solution: $U_* = -2.946\,247$, $h_* = 0.894\,934$ and next from (4.8) we obtain $\gamma_* = 2.235\,386$. Finally, $Y_* = 0.818\,338$ $(A_* = 0.904\,620)$ is computed from the definition, $U = \frac{3}{4}Y - X - 1$.

In Fig. 2 we plot amplitude profiles, *i.e.* variables A, Ω fulfilling (3.10), for the critical value of parameter γ, $\gamma = \gamma_*$, and $h = 0.4,\ 0.8,\ 0.85$ as well as for $h = h_*$.

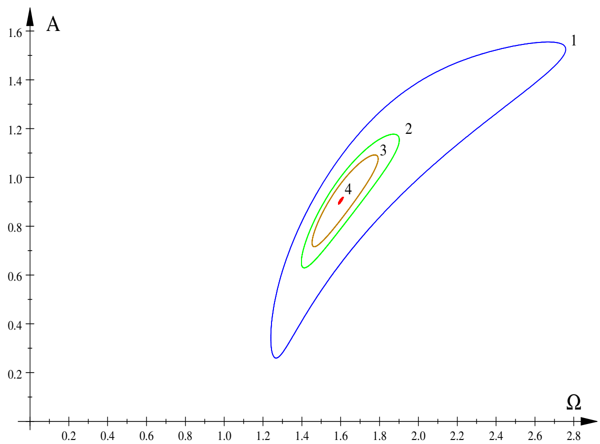

Fig. 2. Amplitude profiles $A(\Omega)$, $\gamma = \gamma_*$, $h = 0.4$ (1, blue), 0.8 (2, green), 0.85 (3, sienna), h_* (4, red).

Resonance 1 : 3 (three thin lines) is visible in Fig. 1, where bifurcation diagram for the Duffing equation (2.8) in $(z,\ \Omega)$ plane, computed for $h = 0.4$, $\gamma = \gamma_*$ and $\Omega \in [2.3,\ 3.0]$, is shown.

In Figs. 3, 4 below, bifurcations diagrams showing dependence of z on Ω for $\gamma = \gamma_*$ and $h = 0.90,\ 0.92$ are shown. It can be seen that the 1 : 3 resonance shrinks for growing h (*cf.* Figs. 1, 3, 4). We shall demonstrate below that it disappears completely for a slightly higher value of parameter h and at $\Omega \cong 1.67$ (*i.e.* near the middle of the resonance, *cf.* Fig. 4) in qualitative agreement with analytical computations.

Since the KBM method is approximate metamorphosis in the real system (*i.e.* complete disappearance of the resonance in the present case) may happen at a slightly different value of the parameter h than its analytical value h_*. The numerically exact critical value of this parameter was determined from the bifurcation diagram below shown in Fig. 5, where dependence of z on h is demonstrated for $\Omega = 1.67$, $\gamma = \gamma_*$.

It follows that the 1 : 3 resonance ends abruptly for growing h at $h = 0.92065$, *i.e.* three per cent above the critical magnitude $h_* = 0.894\,934$.

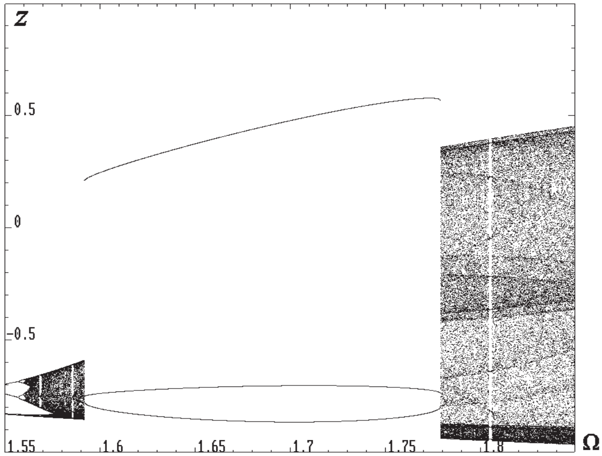

Fig. 3. Bifurcation diagram for the Duffing equation, $h = 0.9$, $\gamma = \gamma_*$.

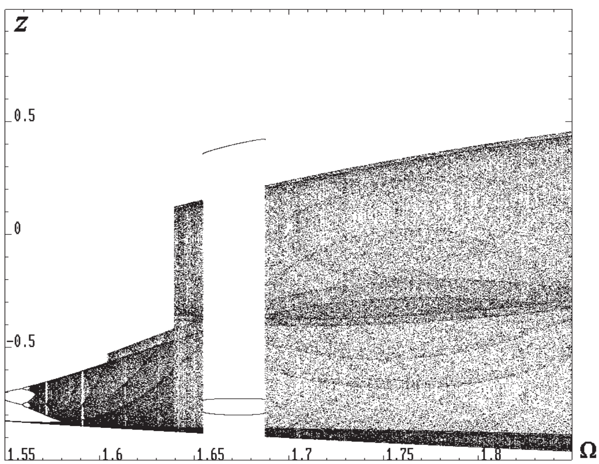

Fig. 4. Bifurcation diagram for the Duffing equation, $h = 0.92$, $\gamma = \gamma_*$.

5.2. The case of the effective equation

We start to study metamorphoses of amplitude profiles of 1 : 3 resonance in the effective equation computing a singular point of the resonance curve (3.9), (3.6). The solution of Eqs. (4.1), (4.2) is given by equations (4.3), (4.4), (4.5). It follows that X, H, a are arbitrary. We thus assume $\Omega_* = 2.4$ ($X_* = 5.76$), $H_* = 0.04$, $a_* = 22$. Next, we compute from Eqs. (4.3), (4.4), (4.5) values of the other variables and choose these which fulfil physical requirements. We get $U_* = -5.470\,783$, $p_* = 0.052\,589$, $s_* = 120.986\,336$ and, finally, $Y_* = 1.718\,956$, $h_* = 0.687\,966$, $\gamma_* = 8.980\,955$.

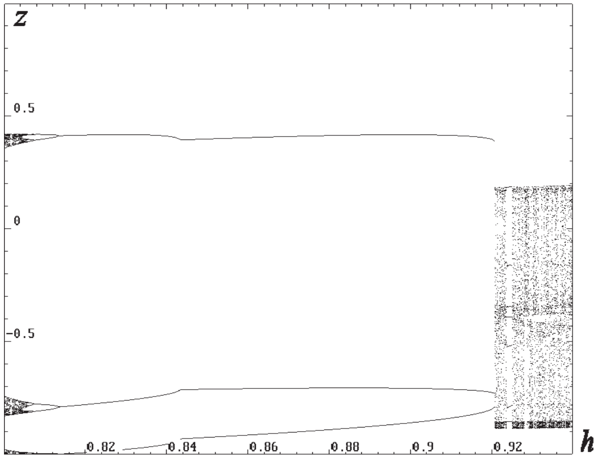

Fig. 5. Bifurcation diagram for the Duffing equation, $\Omega = 1.67$, $\gamma = \gamma_*$.

In Fig. 6 the resonance curves $A(\Omega)$ were plotted in the singular point and in its neighbourhood. For $h = 0.67$ ($h < h_*$, outer (green) curve) there are two branches of $A(\Omega)$, there is self-intersection in the left branch for $h = h_*$ (red curve) and for $h = 0.75$ the left branch splits into two subbranches ($h > h_*$, inner (blue) curve).

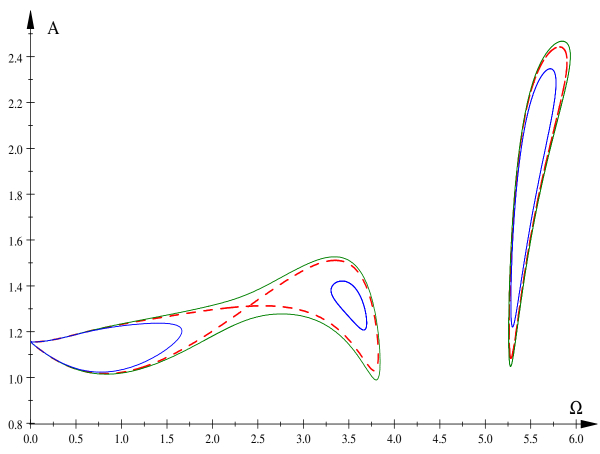

Fig. 6. Amplitude profiles $A(\Omega)$, $\gamma = \gamma_*$, $H = H_*$, $a = a_*$ and $h = h_*$ (dashed/red), $h = 0.67$ (outer/green curve), $h = 0.75$ (inner/blue curve).

In Figs. 7, 8 below, bifurcations diagrams showing dependence of z on Ω for the $1:3$ resonance (three thin lines) for $\gamma = \gamma_*$, $H = H_*$, $a = a_*$ and $h = 0.67$, $h = 0.75$ are shown, respectively (some other dynamical states present in this range of Ω were omitted).

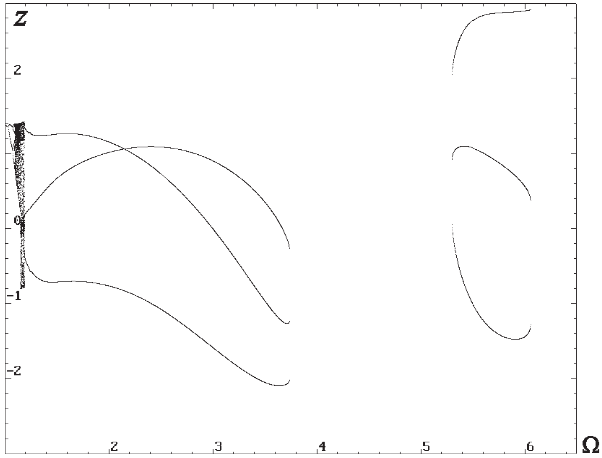

Fig. 7. Bifurcation diagram for the effective equation, $\gamma = \gamma_*$, $H = H_*$, $a = a_*$ and $h = 0.67$.

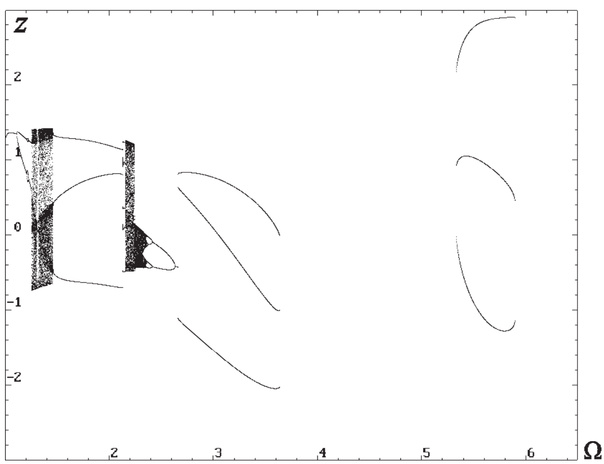

Fig. 8. Bifurcation diagram for the effective equation, $\gamma = \gamma_*$, $H = H_*$, $a = a_*$ and $h = 0.75$.

It follows that for $h = 0.67 < h_*$ the $1 : 3$ resonance exists in two intervals, $\Omega \in [1.20,\ 3.75]$ and $\Omega \in [5.21,\ 6.06]$, while for $h = 0.75 > h_*$ the resonance is present in three intervals, $\Omega \in [1.45,\ 2.13]$, $\Omega \in [2.66,\ 3.64]$ and $\Omega \in [5.34,\ 5.90]$, in agreement with Fig. 6.

6. Summary and discussion

In this work, we have studied metamorphoses of amplitude profiles for the 1 : 3 resonances of the effective equation (2.7), describing approximately dynamics of two coupled periodically driven oscillators, as well as for the Duffing equation (2.8). Our analysis has been analytical although based on the approximate KBM method. Theory of algebraic curves has been used to compute singular points on the corresponding amplitude profiles. It follows from general theory that metamorphoses of amplitude profiles occur in neighbourhoods of such points. The results obtained can be compared with our work on metamorphoses of 1 : 1 resonances in the effective equation [10].

In Section 3, resonance curve $A(\Omega)$ was computed within the KBM method for the effective equation. In Section 4 equations for singular points for $A(\Omega)$ were derived and solved. Appropriate equations and solutions for the Duffing equation are obtained by substituting $a = H = 0$ in the corresponding formulae.

In Section 5, computational results have been presented. We have computed numerically bifurcation diagrams in the neighbourhoods of singular points, computed analytically, and indeed dynamics of the Duffing equation (2.8) as well as of the effective equation (2.7) changes according to metamorphoses of the corresponding amplitude profiles.

More exactly, we have found only the case of isolated singular point for the Duffing equation and this corresponds to creation or destruction of the 1 : 3 resonance. The function $A(\Omega)$ for the effective equation is much more complex and hence we have documented presence of self-intersection only. We are going to study variety of singular points possible in the case of effective equation in our next paper.

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