COHERENT STATES AND QUANTUM NUMBERS FOR TWIST-DEFORMED OSCILLATOR MODEL

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The coherent states for twist-deformed oscillator model provided in article by M. Daszkiewicz, C.J. Walczyk [Acta Phys. Pol. B 40, 293 (2009)] are constructed. Besides, it is demonstrated that the energy spectrum of considered model is labeled by two quantum numbers — by the so-called main and azimuthal quantum numbers respectively.

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The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2, 3] and String Theory models [4, 5], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [6, 7]) as well as with field theoretical models (see e.g. [8, 9]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [10] and nonrelativistic [11] symmetries, one can distinguish three basic types of space-time noncommutativity:

1. The canonical (soft) deformation

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu},
\]

(1)
with constant and antisymmetric tensor $\theta_{\mu\nu}$. The explicit form of corresponding Poincare Hopf algebra has been provided in [12, 13], while its nonrelativistic limit has been proposed in [14].

2. The Lie-algebraic case

$$[x_\mu, x_\nu] = i\theta^0_{\mu\nu} x_\rho,$$

with particularly chosen constant coefficients $\theta^0_{\mu\nu}$. Particular kind of such space-time modification has been obtained as representations of $\kappa$-Poincare [15, 16] and $\kappa$-Galilei [17] Hopf algebras. Besides, the Lie-algebraic twist deformations of relativistic and nonrelativistic symmetries have been provided in [18, 19] and [14], respectively.

3. The quadratic deformation

$$[x_\mu, x_\nu] = i\theta^\rho_{\mu\nu} x_\rho x_\tau,$$

with constant coefficients $\theta^\rho_{\mu\nu}$. Its Hopf-algebraic realization was proposed in [20, 21] and [19] in the case of relativistic symmetry, and in [22] for its nonrelativistic counterpart.

Besides, it has been demonstrated in [23], that in the case of the so-called $N$-enlarged Newton–Hooke Hopf algebras $U_0^{(N)}(NH_\pm)$, the twist deformation provides the new space-time noncommutativity of the form

$$[t, x_i] = 0, \quad [x_i, x_j] = i f_{\kappa \pm} (t) \theta^{ij}_{x} (x),$$

with time-dependent functions

$$f_{\kappa +} (t) = \kappa f \left( \sinh \left( \frac{t}{\tau} \right), \cosh \left( \frac{t}{\tau} \right) \right),$$

$$f_{\kappa -} (t) = \kappa f \left( \sin \left( \frac{t}{\tau} \right), \cos \left( \frac{t}{\tau} \right) \right),$$

$\theta^{ij}_{x} (x) \sim \theta^{ij}_{x} = \text{const}$ or $\theta^{ij}_{x} (x) \sim \theta^{k}_{ij} x_k$ and $\tau$ as well as $\kappa$ denoting the cosmological constant and deformation parameter respectively. It should be also noted that different relations between all mentioned above quantum spaces 1, 2, 3 and 4 have been summarized in article [23].

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1 $x_0 = ct$.

2 The discussed space-times have been defined as the quantum representation spaces, the so-called Hopf modules (see e.g. [12, 13]), for quantum $N$-enlarged Newton–Hooke Hopf algebras.
Let us now turn to the quantum oscillator model defined on the twist-deformed phase space \[24]\3

\[ [ t, \bar{x}_i ] = 0, \quad [ \bar{x}_i, \bar{x}_2 ] = i f_\kappa(t), \quad [ \bar{x}_i, \bar{p}_j ] = i \hbar \delta_{ij}, \quad [ \bar{p}_i, \bar{p}_j ] = 0. \] (5)

Its dynamic is given by the following Hamiltonian function with constant mass \(m\) and frequency \(\omega\)

\[ \hat{H}(\bar{p}, \bar{x}) = \frac{1}{2m} (\bar{p}_1^2 + \bar{p}_2^2) + \frac{1}{2} m \omega^2 (\bar{x}_1^2 + \bar{x}_2^2). \] (6)

In order to analyze the above system, we represent the noncommutative variables \((\bar{x}_i, \bar{p}_i)\) on classical phase space \((x_i, p_i)\) as follows (see e.g. \[25, 26\])

\[ \bar{x}_1 = \hat{x}_1 - \frac{f_\kappa(t)}{2\hbar} \hat{p}_2, \quad \bar{x}_2 = \hat{x}_2 + \frac{f_\kappa(t)}{2\hbar} \hat{p}_1, \] (7)

where

\[ [ \hat{x}_i, \hat{x}_j ] = 0 = [ \hat{p}_i, \hat{p}_j ], \quad [ \hat{x}_i, \hat{p}_j ] = i \hbar \delta_{ij}. \] (8)

Then, the Hamiltonian (6) takes the form\(^4\)

\[ H_f(t) = \frac{(\hat{p}_1^2 + \hat{p}_2^2)}{2M_f(t)} + \frac{1}{2} M_f(t) \Omega_f^2(t) (\hat{x}_1^2 + \hat{x}_2^2) - \frac{f_\kappa(t)}{2\hbar} m \omega^2 \hat{L}, \] (9)

with symbol

\[ \hat{L} = \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1 \] (10)

denoting angular momentum of particle. Besides, the coefficients \(M_f(t)\) and \(\Omega_f(t)\) present in the above formula denote the time-dependent functions given by

\[ M_f(t) = \frac{m}{1 + m^2 \omega^2 f_\kappa^2(t) / 4\hbar^2}, \quad \Omega_f(t) = \omega \sqrt{1 + m^2 \omega^2 f_\kappa^2(t) / 4\hbar^2}, \] (11)

respectively, such that

\[ M_f(t) \Omega_f^2(t) = m \omega^2 = \text{const}. \] (12)

Further, we introduce a set of time-dependent creation \((a_A^\dagger(t))\) and annihilation \((a_A(t))\) operators

\[ \hat{a}_\pm(t) = \frac{1}{2\sqrt{\hbar}} \left[ \frac{\hat{p}_1 \pm i \hat{p}_2}{\sqrt{M_f(t) \Omega_f(t)}} - i \sqrt{M_f(t) \Omega_f(t)} (\hat{x}_1 \pm i \hat{x}_2) \right], \] (13)

\(^3\) See type 4 of quantum space-time.

\(^4\) It should be noted that for \(f_\kappa(t) = \theta\), we get the canonically deformed oscillator model provided in [26].
satisfying the standard commutation relations

$$\left[ \hat{a}_A, \hat{a}_B \right] = 0, \quad \left[ \hat{a}^\dagger_A, \hat{a}^\dagger_B \right] = 0, \quad \left[ \hat{a}_A, \hat{a}^\dagger_B \right] = \delta_{AB} ; \quad A, B = \pm . \quad (14)$$

Then, one can easily check that in terms of the operators (13) the Hamiltonian function (9) looks as follows

$$\hat{H}_f(t) = \Omega_+(t) \left( \hat{N}_+(t) + \frac{1}{2} \right) + \Omega_-(t) \left( \hat{N}_-(t) + \frac{1}{2} \right) , \quad (15)$$

with

$$\Omega_{\pm}(t) = \Omega_f(t) \equiv \frac{\kappa(t) m \omega^2}{2 \hbar} , \quad (16)$$

and number operators in $\pm$ direction given by

$$\hat{N}_{\pm}(t) = \hat{a}^\dagger_{\pm}(t) \hat{a}_{\pm}(t) , \quad (17)$$

respectively. Moreover, we see that the energy eigenvectors can be generated in a standard way as follows

$$|n_+, n_-, t\rangle = \frac{1}{\sqrt{n_+!}} \frac{1}{\sqrt{n_-!}} \left( \hat{a}^\dagger_+(t) \right)^{n_+} \left( \hat{a}^\dagger_-(t) \right)^{n_-} |0\rangle , \quad (18)$$

while the corresponding (parameterized by $n_+$ and $n_-$) eigenvalues are

$$E_{n_+, n_-}(t) = \Omega_+(t) \left( n_+ + \frac{1}{2} \right) + \Omega_-(t) \left( n_- + \frac{1}{2} \right) , \quad n_+, n_- = 0, 1, 2, \ldots \quad (19)$$

Besides, using operator representation (13), one finds

$$\left( \Delta \hat{x}_i \right)^2_{n_+, n_-} \left( \Delta \hat{p}_i \right)^2_{n_+, n_-} = \frac{\hbar^2}{4} \left( 1 + n_+ + n_- \right)^2 , \quad (20)$$

where symbol $\left( \Delta \hat{a} \right)_{|\varphi\rangle}$ denotes the uncertainty of observable $\hat{a}$ in quantum state $|\varphi\rangle$. The above result means that momentum-position uncertainty relations for eigenstates (18) become saturated only for $n_+ = n_- = 0$, i.e. only for vacuum vector $|0\rangle$. Apart from that, it is easy to see that the momentum operator (10) can be written as follows

$$\hat{L} = \hbar \left( \hat{a}_-(t) \hat{a}_+(t) - \hat{a}^\dagger_-(t) \hat{a}^\dagger_+(t) \right) , \quad (21)$$

while its action on quantum states (18) is given by

$$\hat{L} |n_+, n_- , t\rangle = \hbar (n_- - n_+) |n_+, n_-, t\rangle . \quad (22)$$
Consequently, the energy spectrum (19) can be written in terms of eigenvalues (22) as follows

\[ E_{n_+,n_-}(t) = \hbar \Omega_f(t)(n_+ + n_- + 1) + \frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2} (n_- - n_+) \].

(23)

Let us now solve two problems. First of them concerns the construction of the so-called coherent states for considered model, \textit{i.e.} the quantum vectors which saturate the momentum-position Heisenberg uncertainty relations. The second problem applies to the proper interpretation of quantum numbers \( n = n_+ + n_- \) and \( l = n_- - n_+ \) labeling the energy spectrum (23).

Hence, let us consider the quantum states of the form

\[ |c_+,c_-,t\rangle = \sum_{n_+,n_-} e^{-\frac{1}{2}|c_+|^2} e^{-\frac{1}{2}|c_-|^2} \sqrt{n_+!} \sqrt{n_-!} |n_+,n_-,t\rangle, \]

(24)

which play the role of the eigenfunctions for annihilation operators (13)

\[ \hat{a}_\pm(t)|c_+,c_-,t\rangle = c_\pm |c_+,c_-,t\rangle. \]

(25)

By direct calculation, one may check that

\[ (\Delta p_i)_{|c_+,c_-,t\rangle}^2 = \frac{\hbar M_f(t)\Omega_f(t)}{2}, \quad (\Delta x_i)_{|c_+,c_-,t\rangle}^2 = \frac{\hbar}{2M_f(t)\Omega_f(t)}, \quad i = 1,2, \]

(26)

what leads to the saturated momentum-position Heisenberg uncertainty relations

\[ (\Delta p_i)_{|c_+,c_-,t\rangle}^2 (\Delta x_i)_{|c_+,c_-,t\rangle}^2 = \frac{\hbar^2}{4}, \quad i = 1,2. \]

(27)

Consequently, we see that the vectors (24) are, in fact, nothing else than the coherent states for twist-deformed oscillator model, satisfying

\[ \left\langle \hat{H}_f \right|_{|c_+,c_-,t\rangle} = E_{\{0,0,t\}} + \frac{\Omega_f(t)}{\hbar} (\Delta L)^2_{|c_+,c_-,t\rangle} + \frac{M_f(t)\Omega_f^2(t)f_\kappa(t)}{2\hbar} \langle L \rangle_{|c_+,c_-,t\rangle} \]

(28)

with

\[ \langle L \rangle_{|c_+,c_-,t\rangle} = \hbar (|c_-|^2 - |c_+|^2), \]

(29)

\[ (\Delta L)^2_{|c_+,c_-,t\rangle} = \hbar^2 (|c_-|^2 + |c_+|^2). \]

(30)

In the case of second problem, one should solve the eigenvalue equation for Hamiltonian (9) written in terms of polar coordinates

\[ \hat{H}_f(t)\psi(r,\varphi,t) = E(t)\psi(r,\varphi,t), \]

(31)
where
\[ \hat{H}_f(t) = -\frac{\hbar^2}{2M_f(t)} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{\hbar^2 r^2} \right) \]
\[ + \frac{M_f(t)\Omega_f^2(t)}{2}r^2 - \frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2\hbar} \hat{L}, \]
(32)
and
\[ \hat{L} = -i\hbar \frac{\partial}{\partial \varphi}, \quad [\hat{H}, \hat{L}] = 0. \]
(33)

To this aim, it is convenient to take the corresponding eigenfunctions in the form
\[ \psi(r, \varphi, t) = \phi(\varphi)R(r, t), \]
(34)
with its azimuthal part \( \phi(\varphi) \) satisfying
\[ \hat{L}\phi_l(\varphi) = \hbar l\phi_l(\varphi), \quad \phi_l(\varphi) = \frac{1}{\sqrt{2\pi}}e^{il\varphi}, \quad l = 0, \pm 1, \pm 2, \ldots \]
(35)

Then, the proper equation for radial function \( R(r, t) \) looks as follows
\[ \left( -\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{l^2}{\rho^2} + \frac{\rho^2}{4} - E_l(t) \right) R_l(\rho(t)) = 0, \]
\[ E_l(t) = \frac{E(t) - \frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2}}{\hbar\Omega_f(t)}, \]
(36)

where \( \rho(t) = r\sqrt{2M_f(t)\Omega_f(t)/\hbar} \) plays the role of dimensionless variable. Its physical solution can be written as
\[ R_l^{(n)}(\rho(t)) = w_l^{(n)}(\rho(t))e^{-\rho^2(t)/4}, \]
(37)
with \( w_l^{(n)}(\rho(t)) \) denoting the polynomial of degree \( n \). Then, equation (36) reduces to the following one
\[ -\frac{\partial^2 w_l^{(n)}(\rho(t))}{\partial \rho^2} + \frac{\rho^2 - 1}{\rho} \frac{\partial w_l^{(n)}(\rho(t))}{\partial \rho} \]
\[ + \frac{l^2}{\rho^2} w_l^{(n)}(\rho(t)) - (E_l(t) - 1)w_l^{(n)}(\rho(t)) = 0, \]
(38)
for which the solution (this time) is given by
\[ w_l^{(n)}(\rho(t)) = a_l^{(n)} \left( 1 + \sum_{k=1}^{(n-|l|)/2} \left[ \prod_{s=1}^{k} \frac{n + 2 - (2s + |l|)}{l^2 - (2s + |l|)^2} \right] \rho^{2k}(t) \right) \rho^{|l|}(t), \]
(39)

\footnote{The symbol \( a_l^{(n)} \) denotes the normalization factor.}
only when
\[ \mathcal{E}_l(t) \to \mathcal{E}_l^{(n)}(t) = n+1, \quad l \in \{-n, -n+2, \ldots, n-2, n\}, \quad n = 0, 1, 2, 3, \ldots, \quad (40) \]
or (equivalently)
\[ E_l^{(n)}(t) = \hbar \Omega f(t)(n+1) + \frac{f_\kappa(t)M_f(t)\Omega^2_f(t)}{2} l. \quad (41) \]
Consequently, after substitution \( n = n_+ + n_- \) and \( l = n_- - n_+ \) into eigenvalues (41), we get, in fact, the energy spectrum (23) labeled by \( n_+ \) and \( n_- \) parameters. For this reason as well as due to the formulas (35), (37) and (41), the quantities \( n \) and \( l \) may be called the “main” and “azimuthal” quantum numbers respectively.

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