CONVERGENCE OF A CLASS OF HANKEL-TYPE MATRICES

ARUP BOSE†, SREELA GANGOPADHYAY‡

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute
203 Barrackpore Trunk Road, Kolkata 700 108, India

(Received December 24, 2014)

Let \( H_n \) be the \( n \times n \) symmetric Hankel-type matrix whose \((i,j)\)th element on the \( k\)th anti-diagonal (where \( k = 0 \) denotes the main anti-diagonal) is defined as: \( H_{k,n}(i,j) = g_k\left(\frac{i-\lfloor n+k+1/2\rfloor}{n}\right) \) if \( i + j = n + 1 + k \) and 0 otherwise. Under suitable symmetry and summability conditions on \( \{g_k\} \), we show that as \( n \to \infty \), the limiting spectral distribution of \( \{H_n\} \) exists and is given by \( \sum_{k=-\infty}^{\infty} g_k(U)a_k \), where \( U \) is uniformly distributed on \([-1/2, 1/2]\) and is tensor-independent of the non-commuting variables \( \{a_k\} \) which are certain symmetric pair-wise free but not completely free Bernoulli variables.

DOI:10.5506/APhysPolB.46.1683
PACS numbers: 02.10.Yn, 02.05.Cw

1. Introduction

Let \( \{A_{k,n}\}, k = 1, 2, \ldots, K \) be \( K \) sequences of \( n \times n \) matrices. Then, as elements of the non-commutative probability space of \( n \times n \) complex matrices with the state as average trace, they are said to converge jointly (as \( n \to \infty \)), if for every polynomial \( P(A_{k,n}, A_{k,n}^*, k \leq K) \), the average trace converges. Here, \( A^* \) denotes the complex conjugate of \( A \). The limit non-commutative (polynomial) \(*\)-algebra is defined by the non-commutative indeterminates (limit variables) \( \{a_k\} \), where the state \( \phi \) satisfies \( \phi(P(a_k, a_k^*, k \leq K)) = \lim \frac{1}{n} \text{Tr}(P(A_{k,n}, A_{k,n}^*, k \leq K)) \) for all polynomials \( P \). The limit non-commutative joint distribution of \( \{a_k\} \) is defined as the collection of all the joint moments \( \phi(a_{i_1}^{\epsilon_1}a_{i_2}^{\epsilon_2} \ldots a_{i_n}^{\epsilon_n}) \) for all \( 1 \leq i_1, i_2 \ldots, i_n \leq K, n \geq 1 \) and \( \epsilon_i \in \{1, *\} \).

† bosearu@gmail.com
‡ gangopadhyay7@gmail.com

(1683)
When we have only one sequence of matrices, say \(\{A_n\}\) (which are, for simplicity, real symmetric), then there is a related notion of convergence. Let \(\lambda_1, \lambda_2, \ldots, \lambda_n\) be the eigenvalues of \(A_n\). Then, the Empirical Spectral Distribution Function (ESD) of \(A_n\) equals

\[
F_{A_n}(x) = n^{-1} \sum_{i=1}^{n} \mathbb{I}\{\lambda_i \leq x\}.
\]

As \(n \to \infty\), the Limiting Spectral Distribution (LSD) of \(\{A_n\}\) is defined as the weak limit \(F\) of \(\{F_{A_n}\}\), if it exists. We identify \(F\) with any random variable \(X\) whose distribution is \(F\). This definition extends to non-symmetric matrices with complex entries in the obvious way.

It is easy to construct examples of real symmetric matrices \(\{A_n\}\) where the LSD exists but there is no convergence in the non-commutative sense (that is, \(\lim \frac{1}{n} \text{Tr}(A_n^k)\) does not exist for some \(k\)). On the other hand, by using the moment-trace formula, it is also easy to see that if the real symmetric \(\{A_n\}\) converges in the non-commutative sense (that is, \(\lim \frac{1}{n} \text{Tr}(A_n^k) = \mu_k\) exists for all positive integers \(k\)), and if \(\{\mu_k\}\) is the moment sequence of a unique probability distribution \(F\), then the LSD of \(A_n\) equals \(F\).

Let \(U_1\) and \(U_2\) be i.i.d. random variables, uniformly distributed on the interval \((0, 1)\). The famous Szegő’s theorem implies that if \(T_n := ((t_{[i-j]})_{1 \leq i, j \leq n}\) is the Toeplitz matrix and \(\{t_k\}\) is square summable, then the LSD of \(T_n\) equals \(t_0 + 2 \sum_{k=1}^{\infty} t_k \cos(2\pi k U_2)\). This result was extended to the Toeplitz-type matrix \(T_{n,g}\) say, where the elements of the \(k\)th upper and lower diagonals, instead of being the constant \(t_k\), are of the form of \(g_k(i/n)\) in the \(i\)th row for some suitable functions \(g_k\) (see [1, 2]). The limit in this case equals \(g_0(2\pi U_1) + 2 \sum_{k=1}^{\infty} g_k(2\pi U_1) \cos(2\pi k U_2)\).

The related Hankel matrix \(H_n = ((h_{i+j}))\) and the corresponding Hankel operator has been extensively treated in the literature. See [3–6] for detailed information. Note that the elements on each anti-diagonal of \(H_n\) are identical. However, while in \(T_n\) the constant on the main diagonal does not change with \(n\), the main anti-diagonal in \(H_n\) is \(h_{n+1}\). We take a cue from this observation and the matrix \(T_{n,g}\), to consider the following class of Hankel-type matrices.

In our convention of labelling the anti-diagonals, \(k = 0\) refers to the main anti-diagonal and \(k = 1, 2, \ldots\) denote the successive anti-diagonals below the main anti-diagonal and, similarly, the negative integers label the upper anti-diagonals. For each \(k\), first consider the Hankel matrix \(D_{k,n}\) whose \(k\)th anti-diagonal elements equal one and the rest of the elements are zero. These matrices converge jointly. The non-commutative joint distribution of the limit variables \(\{a_k\}\) can be described in terms of the non-commutative moments as
\[ \phi(a_{i_1} \ldots a_{i_k}) = \begin{cases} \mathbb{I}_{\{i_1+i_3+\ldots+i_{2m-1}=i_2+i_4+\ldots+i_{2m}\}} & \text{if } k=2m \text{ for some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases} \] (1.1)

Interestingly, the above \(\{a_k\}\) are symmetric Bernoulli and are pair-wise free but not completely free. This is easy to check by using the above description.

Now generalise \(D_{k,n}\) as follows. Let \(g_k : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}\) be continuous and symmetric about 0; let \(H_{k,n}\) be the \(n \times n\) Hankel-type matrix whose \((i,j)\)th element is defined as

\[ H_{k,n}(i, j) = \begin{cases} g_k \left( i - \left\lceil \frac{n+k+1}{2} \right\rceil \right) & \text{if } i + j = n + 1 + k, \\ 0 & \text{otherwise.} \end{cases} \] (1.2)

Note that unlike the Hankel matrices considered usually in the literature, for example [5, 6], where the main anti-diagonal has the variable \(h_{n+1}\) which changes as \(n\) changes, in our case the main anti-diagonal has elements of \(g_k(\cdot)\) which is a fixed function. So the labelling is different.

We show that \(\{H_{k,n}\}\) converge jointly and the limit variables are \(\{g_k(U)a_k\}\), where \(U\) is uniformly distributed on \([-1/2, 1/2]\) and is tensor independent of \(\{a_k\}\). As a consequence, for any \(K \geq 1\), \(\sum_{|k| \leq K} H_{k,n}\) converges in the (algebraic sense) and the LSD of this real symmetric matrix exists and equals \(\sum_{|k| \leq K} g_k(U)a_k\) with distribution \(\tilde{F}_K\) say.

Finally, consider the full Hankel-type matrix \(H_n = \sum_{|k| \leq n} H_{k,n}\). By imposing suitable restrictions on the functions \(\{g_k\}\), \(H_n\) is approximated by the finite-diagonal matrix \(\sum_{|k| \leq K} H_{k,n}\) in an appropriate metric and this helps us to conclude that the LSD of \(H_n\) exists under these conditions on \(\{g_k\}\). The limit distribution function is the weak limit of \(\tilde{F}_K\) as \(K \to \infty\) and may be formally expressed as \(\sum_{k=-\infty}^{\infty} g_k(U)a_k\). There does not seem to be any analytic description of the limit distribution function.

The case when the \(\{g_k\}\) are not symmetric leads to a non-symmetric \(H_n\). Studying the LSD of this matrix is an extremely difficult problem. We have made some elementary remarks on some special cases at the end of the article.

2. Preliminaries

A non-commutative probability space is a pair \((\mathcal{A}, \phi)\) where \(\mathcal{A}\) is a unital algebra (with unity \(\mathbf{1}\)) and \(\phi : \mathcal{A} \to \mathbb{C}\) is a linear functional satisfying \(\phi(\mathbf{1}) = 1\). Elements of a non-commutative probability space will also be called (non-commutative) random variables. If an appropriate * operation is
defined on $\mathcal{A}$, then $(\mathcal{A}, \phi)$ is called a $\ast$-probability space\footnote{Since we shall, mostly, be dealing with only real symmetric matrices, all our algebras, unless otherwise stated, are real.}. A random variable $a \in \mathcal{A}$ is said to be self-adjoint if $a = a^\ast$, and unitary if $aa^\ast = a^*a = 1$. It is called Haar unitary if $\phi(a^k) = \mathbb{I}_{\{k=0\}}$.

For our purposes, we need the following $\ast$-probability space. Let $\mathcal{A}_n$ be the space of $n \times n$ symmetric random matrices with elements which are real numbers or are random variables with all moments finite. Then $\phi_n$ equal to $\frac{1}{n} \mathbb{E}_{\mu}[\text{Tr}(\cdot)]$ or $\frac{1}{n} \mathbb{E}[\text{Tr}(\cdot)]$ both yield a $\ast$-probability space.

For $\{b_i\}_{i \in J} \subset \mathcal{A}$, their joint moments is the collection $\{\phi(b_{i_1}b_{i_2}\ldots b_{i_k}) \mid k \geq 1\}$, where each $b_{ij} \in \{b_i\}_{i \in J}$.

Random variables $\{b_{i,n}\}_{i \in J} \subset (\mathcal{A}_n, \phi_n)$ are said to converge in law to $\{b_i\}_{i \in J} \subset (\mathcal{A}, \phi)$ (as $n \to \infty$) if each joint moment of $\{b_{i,n}\}_{i \in J}$ converges to the corresponding joint moment of $\{b_i\}_{i \in J}$. That is, if for $k \geq 1$,

$$\phi_n [P(b_{i_1,n}, b_{i_2,n}, \ldots, b_{i_k,n})] \to \phi [P(b_{i_1}, b_{i_2}, \ldots, b_{i_k})]$$

for all polynomials $P$. If this happens, we write

$$\{b_{i,n}\}_{i \in J} \xrightarrow{\phi_n} \{b_i\}_{i \in J}.$$

If the random variables $\{b_{i,n}\}_{i \in J}$ are $n \times n$ (non-random) matrices, then the above convergence is assumed to be with respect to $\phi_n = \frac{1}{n} \text{Tr}$. If, instead, they are random matrices, then the above convergence is in one of the following two senses:

(i) We say that $\{b_{i,n}\}_{i \in J}$ converges to $\{b_i\}_{i \in J}$, if the convergence holds with respect to $\phi_n = \frac{1}{n} \mathbb{E} \text{Tr}$.

(ii) We say $\{b_{i,n}\}_{i \in J}$ converges almost surely to $\{b_i\}_{i \in J}$, if the convergence holds with respect to $\phi_n = \frac{1}{n} \text{Tr}$, almost surely.

3. Hankel-type finite-diagonal matrices

Let $\{g_k\}_{-\infty < k < \infty}$ be a two-sided sequence of functions, such that for each $k$, $g_k : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$, $g_k$ is continuous and symmetric about 0. Let $H_{k,n}$ be the $n \times n$ Hankel-type matrix defined in (1.2). When $g_k \equiv 1$, $H_{k,n}$ is the Hankel matrix with all entries 0, except the entries on the $k^{th}$ anti-diagonal which are all assumed to be 1. Note that counted from the main anti-diagonal, $k$ positive (negative) refers to the lower (respectively upper) anti-diagonal. We call this matrix $D_{k,n}$.

To describe the joint limit of $H_{k,n}$, let $(\mathcal{A}, \phi)$ be a $\ast$-probability space, and let $\{a_i\}_{i \in \mathbb{Z}} \subset \mathcal{A}$ be a sequence of self-adjoint and unitary elements such that $\phi(a_{i_1} \ldots a_{i_k})$ is as defined in (1.1).
It is then not hard to see that, $a_i$'s are distributed as symmetric Bernoulli and are pair-wise freely independent but not totally free.

**Theorem 3.1.** For any $K$, $(H_{k,n}, |k| \leq K)$ jointly converge to $(g_k(U)a_k, |k| \leq K)$ where $\{a_k\}$ are elements of a $*$-probability space $(A, \phi)$, $\phi$ as defined in (1.1) and $U$ is a random variable, uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$ and independent (in the classical sense) of $A$. In particular, the LSD of $\sum_{|j| \leq K} H_{j,n}$ equals $\sum_{|j| \leq K} g_j(U)a_j$.

Before we prove the above theorem, we state and prove a corollary.

**Corollary 1.** $(D_{k,n}, |k| \leq K)$ converge jointly to $(a_k, |k| \leq K)$ where $a_k$ are as in (1.1). In particular, for real numbers $\{h_k, |k| \leq K\}$, the LSD of $\sum_{|k| \leq K} h_k D_{k,n}$ equals $\sum_{|k| \leq K} h_k a_k$. For any $s \neq t$, the LSD of $D_{s,n} + D_{t,n}$ is the arc-sine law and $D_{s,n}D_{t,n}$ is asymptotically Haar unitary.

**Proof.** The joint convergence follows from Theorem 3.1. By that theorem, all moments of the ESD converge. Note that these moments determine a distribution uniquely which is as given in the statement of the corollary. Finally, it just suffices to observe that for any $s \neq t$, the $*$-distribution of $a_s + a_t$ is the arcsine law as it is a free convolution of symmetric Bernoulli (see, [7, pp. 200–202]) and that $a_s a_t$ is Haar unitary.

**Proof of Theorem 3.1.** First note that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $D_{k,n}$, then $\forall i = 1, \ldots, n$, $\lambda_i \in \{-1, 0, 1\}$ and $0$ has algebraic multiplicity $|k|$ and multiplicity of $1$ and $-1$ are equal as $n \to \infty$. So ESD of $D_{k,n}$ converges to the random variable $a_k = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$, i.e., $a_k$ is symmetric Bernoulli.

Let, for any fixed $s$, $T_{s,n}$ denote the $n \times n$ Toeplitz matrix whose entries are all zero except those on the $s$th diagonal which equal $1$, index $s$ being counted from the main diagonal ($s = 0$) and $s = \pm 1, \ldots$ above and below the main diagonal respectively.

If $r$ and $s$ are any two integers, then (for large enough $n$), the product $D_{r,n}D_{s,n}$ equals $T_{s-r,n}$ except for $s$ many rows and $r$ many columns which are zero. Consequently, $D_{k,n}^2$ is an identity matrix except whose $k$ rows and $k$ columns are zero. Thus for asymptotic purposes, we may treat $D_{k,n}^2$ as an identity matrix.

Now, consider $T_{s,n}$ and $D_{r,n}$. Then, the $(i,j)^{th}$ entry of the product $T_{s,n}D_{r,n}$ equals $\sum_{j_1} t_{i,j_1} h_{j_1,j}$ which is $\neq 0$ iff $j = n + r + 1 - i - s$. Since there are only finitely many such possibilities, $\lim_{n \to \infty} \frac{1}{n} \text{Tr}(T_{s,n}D_{r,n}) = 0$.

Finally, note that $T_{r,n}T_{s,n} = T_{r+s,n}$ except for a finitely many entries.
Using the above facts repeatedly, it is easy to see that
\[
\lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( H_{k_1,n} H_{k_2,n} \cdots H_{k_s,n} \right) = 0, \quad \text{if } s = 2m - 1 \text{ for some } m \geq 1.
\]

So assume, \( s = 2m \). For convenience, we will write \( n_k \) for \( n + k + 1 \) for any integer \( k \)

\[
\frac{1}{n} \text{Tr} \left( H_{k_1,n} H_{k_2,n} \cdots H_{k_{2m},n} \right)
= \frac{1}{n} \sum_{i,j_1,\ldots,j_{2m-1}} (H_{k_1,n}(i,j_1)H_{k_2,n}(j_1,j_2)\cdots H_{k_{2m},n}(j_{2m-1},i))
\]

\[
= \frac{1}{n} \sum_i H_{k_1,n}(i,n_k - i)\cdots H_{k_{2m},n} \left( \sum_{j=1}^{2m-1} (-1)^{j+1} n_{k_j} - i, \sum_{j=1}^{2m} (-1)^j n_{k_j} + i \right)
\]

(to satisfy trace condition the last index must be \( i \), \( i.e. \), \( k_1 + k_3 + \ldots = k_2 + k_4 + \ldots \))

\[
= \frac{1}{n} \sum_i g_{k_1} \left( \frac{i - \left\lfloor \frac{n_{k_1}}{2} \right\rfloor}{n} \right) \cdots g_{k_{2m}} \left( \frac{i - \left\lfloor \frac{n_{k_{2m}}}{2} \right\rfloor}{n} \right) \mathbb{I}_{k_1+k_3+\cdots+k_{2m-1}=k_2+k_4+\cdots+k_{2m}}
\]

(this is a Riemann sum and using uniform continuity of \( g_k \)'s)

\[
\to \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} (g_{k_1} \cdots g_{k_{2m}}) (x) dx \right) \mathbb{I}_{k_1+k_3+\cdots+k_{2m-1}=k_2+k_4+\cdots+k_{2m}}.
\]

Thus, \( \lim \frac{1}{n} \text{Tr}(H_{k_1,n} H_{k_2,n} \cdots H_{k_s,n}) = \mathbb{E}_U \otimes \phi(g_{k_1}(U)a_1, g_{k_2}(U)a_2, \ldots, g_{k_s}(U)a_s) \), where \( \mathbb{E}_U \) is the usual expectation with respect to Lebesgue measure on \( [-\frac{1}{2}, \frac{1}{2}] \) and \( \phi \) is a linear functional on \( \mathcal{A} \) as defined in (1.1) and they act independently (classical sense) on

\[
\mathcal{C} \left( [-\frac{1}{2}, \frac{1}{2}] \right) \otimes \mathcal{A}
:= \{ f(U)a : f \text{ continuous real-valued function on } [-\frac{1}{2}, \frac{1}{2}], \ a \in \mathcal{A} \}.
\]

This completes the proof of the theorem.

**Remark 3.1.** Let us define an \( n \times n \) \( k \)-diagonal random Hankel matrix \( \tilde{H}_{k,n} \) whose \((i,j)\)th entry is \( g_k(U)\mathbb{I}_{i+j=n+k+1} \) where \( U \) is a random variable uniformly distributed on \( I := [-\frac{1}{2}, \frac{1}{2}] \). Suppose \( g_k(\cdot) \) are continuous even functions on \( I \). Following arguments similar to that given in the proof of Theorem 3.1, one can show that
Convergence of a Class of Hankel-type Matrices

(i) For fixed $K > 0$,
\[
\left( \tilde{H}_{-K, n}, \tilde{H}_{-K+1, n}, \ldots, \tilde{H}_{K-1, n}, \tilde{H}_{K, n} \right)^{1/n} \xrightarrow{E \text{ Tr}} (g_{-K}(U)a_{-K}, \ldots, g_K(U)a_K),
\]
where $\{a_i\}_{|i| \leq K} \subset (\mathcal{A}, \phi)$ are as defined in (1.1) and $U$ is independent of $(\mathcal{A}, \phi)$. As a consequence, the expected ESD of $\sum_{i=-K}^{K} \tilde{H}_{i, n}$ converges weakly to $\sum_{j=-K}^{K} g_j(U)a_j$.

(ii) For almost every value of $U$,
\[
\left( \tilde{H}_{-K, n}, \tilde{H}_{-K+1, n}, \ldots, \tilde{H}_{K, n} \right)^{1/n} \xrightarrow{\text{Tr}} (g_{-K}(U)a_{-K}, \ldots, g_K(U)a_K)
\]
and hence for fixed $K > 0$, for almost every given $\omega$, the ESD of $\sum_{i=-K}^{K} \tilde{H}_{i, n}$ converges weakly to $\sum_{j=-K}^{K} g_j(U(\omega))a_j$. Note that this is a random limit depending on $\omega$ (a typical point in the probability space where $U$ is defined).

4. When all diagonals are present

Now, for $U$ as previously defined, let $(\mathcal{C}(U), \mathbb{E}_U)$ be a classical probability space where $\mathcal{C}(U) := \{f(U) : f \text{ is continuous on } I\}$ and $\mathbb{E}_U$ is the usual expectation on $I$ with respect to Lebesgue measure. Then consider the non-commutative probability space $(\tilde{\mathcal{A}}, \tilde{\phi})$ where $\tilde{\mathcal{A}}$ is the algebra generated by $\{f(U)a : f(U) \in \mathcal{C}(U), a \in \mathcal{A}\}$ and $\tilde{\phi}$ acts on $f(U)a \in \tilde{\mathcal{A}}$ as $\tilde{\phi}(f(U)a) = (\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx)\phi(a)$ which is extended linearly on $\tilde{\mathcal{A}}$. $(\tilde{\mathcal{A}}, \tilde{\phi})$ is a $*$-probability space where $(f(U)a)^* = f(U)a^*$.

Let
\[
\sum_{j=-K}^{k} g_j(U)a_j =: b_k \in \tilde{\mathcal{A}}.
\]
We have seen that $\sum_{j=-k}^{k} H_{j, n}$ converges to $b_k$ which is self-adjoint. It is also easy to see that $\{\tilde{\phi}(b_k^m)\}_{m \geq 1}$ defines a unique distribution function $\tilde{F}_k$ (say) which is the LSD of $\sum_{j=-k}^{k} H_{j, n}$.

To deal with matrices which may have all diagonals non-zero, we need some additional conditions on $\{g_j\}$ and an appropriate metric which will allow such matrices to be approximated by Hankel-type matrices with finitely many non-zero anti-diagonals.

The Mallow’s metric is defined on the space of all probability distributions with finite second moment. Let $F$ and $G$ be two distribution functions
with finite second moment. Then, the Mallow’s distance between $F$ and $G$ is defined as

$$d_M^2(F, G) := \inf_{X \sim F, Y \sim G} E |X - Y|^2.$$  \hfill (4.1)

It is known that $d_M(F_n, F) \to 0$ if and only if $\int x^2 dF_n(x) \to \int x^2 dF(x)$ and $F_n$ converges to $F$ weakly.

We need the following upper bound of this metric between the ESD of two matrices: let $A, B$ be two $n \times n$ real symmetric matrices with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, respectively. Then,

$$d_M^2(F_A, F_B) \leq \frac{1}{n} \sum_{j=1}^n (\lambda_j - \beta_j)^2 \leq \frac{1}{n} \text{Tr}(A - B)^2.$$  \hfill (4.2)

The first inequality is obvious and the last inequality above is a standard result in matrix algebra; one can see a proof of this in Lemma 2.3 of [8].

**Theorem 4.1.** Suppose $\{g_j\}$ are continuous even functions on $I := [-\frac{1}{2}, \frac{1}{2}]$. Suppose

(i) $\sum_{k=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} g_j^2(x) dx < \infty$, and

(ii) $\sum_{j=-n+1}^{n-1} \sum_{i=1}^n \int_{i-\frac{n_j}{n}}^{i+\frac{n_j}{n}} \left( g_j^2 \left( \frac{i-\frac{n_j}{n}}{n} \right) - g_j^2(x) \right) dx \to 0$ as $n \to \infty$,

where $n_j := n + j + 1$.

Then, LSD of $H_n := \sum_{j:|j| \leq n} H_{j,n}$ exists and equals $\lim_{k \to \infty} \tilde{F}_k =: \tilde{F}_\infty$. The limiting random variable may be written as $\sum_{j=-\infty}^{\infty} g_j(U)a_j$ where $U$ and $a_j$ are as defined in Theorem 3.1 and has distribution function $\tilde{F}_\infty$.

**Proof.** Let $F_n$ and $F_{k,n}$ denote respectively the ESD of $H_n$ and $\sum_{j=-k}^{k} H_{j,n}$. First, we will show that $\{\tilde{F}_k\}$ is weakly convergent. For that, it is enough to show that $\{\tilde{F}_k\}$ is Cauchy in $d_M$. Let $n > k_2 > k_1$. Then,

$$d_M \left( \tilde{F}_{k_1}, \tilde{F}_{k_2} \right) \leq d_M \left( \tilde{F}_{k_1}, F_{k_1,n} \right) + d_M \left( \tilde{F}_{k_2}, F_{k_2,n} \right) + d_M \left( F_{k_1,n}, F_{k_2,n} \right)$$

$$= d_1 + d_2 + d_3 \ (\text{say}).$$
But by Theorem 3.1, \(d_1 + d_2 \to 0\) as \(n \to \infty\). For \(d_3\), observe that

\[
d_3^2 := d_M^2(F_{k1,n}, F_{k2,n}) \leq \frac{1}{n} \text{Tr} \left( \sum_{k_1 < |j| \leq k_2} H_{j,n} \right)^2 \text{ (by (4.2))}
\]

\[
\leq \frac{1}{n} \sum_{j:k_1 < |j| \leq k_2} \sum_i g_j^2 \left( \frac{i - \left\lfloor n_j \frac{n}{2} \right\rfloor}{n} \right)
\]

\[
\leq \sum_{j:k_1 < |j| \leq k_2} \left[ \frac{1}{n} \sum_{i=1}^n g_j^2 \left( \frac{i - \left\lfloor n_j \frac{n}{2} \right\rfloor}{n} \right) - \frac{1}{2} \int_{-1/2}^{1/2} g_j^2(x)dx \right]
\]

\[
+ \sum_{j:k_1 < |j| \leq k_2} \int_{-1/2}^{1/2} g_j^2(x)dx,
\]

\[\to 0 \text{ as } k_1, k_2 \to \infty \text{ (by Conditions (i) and (ii)).}\]

This implies that \(\tilde{F}_k\) converges weakly to a distribution function \(\tilde{F}_\infty\) (say).

Now, to prove the theorem, consider

\[
d_M(F_n, \tilde{F}_\infty) \leq d_M(F_n, F_{k,n}) + d_M(F_{k,n}, \tilde{F}_k) + d_M(\tilde{F}_k, \tilde{F}_\infty).
\]

Since \(\tilde{F}_k\) converges weakly to \(\tilde{F}_\infty\), for a fixed \(\varepsilon > 0\), there exists a \(K \in \mathbb{N}\) such that

\[
d_M(\tilde{F}_k, \tilde{F}_\infty) \leq \varepsilon \text{ for all } k \geq K.
\]

Now, for any fixed \(k \geq K\), by Theorem 3.1 \(d_M(F_{k,n}, \tilde{F}_k) \to 0\) as \(n \to \infty\).

Finally, again using (4.2), we have

\[
d_M^2(F_n, F_{k,n}) \leq \frac{1}{n} \text{Tr} \left( H_n - \sum_{j=-k}^{k} H_{j,n} \right)^2 \leq \frac{1}{n} \sum_{j:|j| > k} \sum_i g_j^2 \left( \frac{i - \left\lfloor n_j \frac{n}{2} \right\rfloor}{n} \right)
\]

\[
\leq \sum_{j:|j| > k} \left[ \frac{1}{n} \sum_{i=1}^n g_j^2 \left( \frac{i - \left\lfloor n_j \frac{n}{2} \right\rfloor}{n} \right) - \frac{1}{2} \int_{-1/2}^{1/2} g_j^2(x)dx \right]
\]

\[
+ \sum_{j:|j| > k} \int_{-1/2}^{1/2} g_j^2(x)dx,
\]
and due to Conditions (i), (ii), right-hand side goes to zero as $n \to \infty$. Hence, $d_M(F_n, \tilde{F}_\infty) \to 0$ as $n \to \infty$. This completes the proof of the theorem.

**Remark 4.1.** The study of the LSD of $H_{k,n}$ and $H_n$, when the symmetry assumption is removed, does not seem to be easy. This requires further investigation. The following simple observations though may be made. Let $N_{k,n} = (a_{ij})_{1 \leq i,j \leq n}$ be the $n \times n$ non-symmetric Hankel-type matrix,

$$a_{i,j} = \begin{cases} 1 & \text{if } i + j = n + k + 1 \text{ and } i \leq \lfloor (n + k)/2 \rfloor; \\ 0 & \text{otherwise} \end{cases}.$$ 

Then clearly, the LSD of both $N_{k,n}$ and $N_{k,n}^*$ are the point mass at zero, $\delta_0$. The LSD of the symmetric matrix $N_{k,n}N_{k,n}^*$ converges in distribution to the Bernoulli random variable $(1/2)\delta_0 + (1/2)\delta_1$. Since $N_{k,n} + N_{k,n}^* = D_{k,n}$, its LSD is the symmetric Bernoulli $(1/2)\delta_{-1} + (1/2)\delta_1$. The limiting moment of any monomial in $(N_{k,n}, N_{k,n}^*)$ is zero unless $N_{k,n}$ and $N_{k,n}^*$ appear alternately in the monomial. However, the limiting joint free cumulants of $(N_{k,n}, N_{k,n}^*)$ may be non-zero even if they do not appear alternately. For example, it can be checked that

$$\lim \kappa_4 \left( N_{k,n}, N_{k,n}, N_{k,n}^*, N_{k,n}^* \right) = \frac{1}{4}.$$

Research supported by J.C. Bose National Fellowship, Department of Science and Technology, Government of India.

**REFERENCES**


