THE TRUE FACE OF QUANTUM DECAY PROCESSES: UNSTABLE SYSTEMS AT REST AND IN MOTION*

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We analyze properties of unstable systems at rest and in motion.

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1. Introduction

Since the discovery of the radioactive decay law by Rutherford and Sody, the belief that the decay law has the exponential form has become common. This conviction was upheld by the Wesisskopf–Wigner theory of spontaneous emission [1]. Further studies of the quantum decay process showed that basic principles of the quantum theory led to rather widespread belief that a universal feature of the quantum decay process is the presence of three time regimes of the decay process: The early time (initial), exponential (or "canonical"), and late time having inverse-power law form [2]. The question arises, if indeed this is the true picture of quantum decay processes.

From the standard textbook considerations, one finds that if the decay law of the unstable particle at rest has the exponential form of $\mathcal{P}_0(t) = \exp\left[-\frac{\mathit{\Gamma}_0\, t}{\hbar}\right]$, then the decay law of the moving particle looks as follows:

$$\mathcal{P}_p(t) = \exp\left[-\frac{\mathit{\Gamma}_0\, t}{\hbar\,\gamma}\right]\,, \tag{1}$$

where t denotes time, $\mathit{\Gamma}_0$ is the decay rate (time t and $\mathit{\Gamma}_0$ are measured in the rest reference frame of the particle) and γ is the relativistic Lorentz factor. Formula (1) is the classical physics relation. It is almost common belief that this formula is valid also for any t in the case of quantum decay processes

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and does not depend on the model of the unstable particles considered. The problem seems to be extremely important because from some theoretical studies it follows that in the case of quantum decay processes, this relation is valid to a sufficient accuracy only for not more than a few lifetimes $\tau_0 = \hbar/\mathit{\Gamma}_0$ [3–6]. All the above problems will be analyzed in the next parts of this paper.

2. Unstable states in the rest system

The main information about properties of quantum unstable systems is contained in their decay law, that is in their survival probability. If one knows that the system in the rest frame is in the initial unstable state $|\phi\rangle \in \mathcal{H}$ ($\mathcal{H}$ is the Hilbert space of states of the considered system), which was prepared at the initial instant $t_0 = 0$, one can calculate its survival probability (the decay law), $\mathcal{P}_0(t)$, of the unstable state $|\phi\rangle$ decaying in vacuum, which equals

$$\mathcal{P}_0(t) = |a_0(t)|^2 \,, \tag{2}$$

where $a_0(t)$ is the probability amplitude of finding the system at the time t in the rest frame in the initial unstable state $|\phi\rangle$

$$a_0(t) = \langle\phi|\phi(t)\rangle \equiv \langle\phi|\exp\left[-itH\right]|\phi\rangle \,, \tag{3}$$

H is the selfadjoint Hamiltonian of the system considered and $|\phi(t)\rangle$ is the solution of the Schrödinger equation for the initial condition $|\phi(0)\rangle = |\phi\rangle$. Here, the system units $\hbar = c = 1$ is used. From basic principles of the quantum theory, it follows that the amplitude $a_0(t)$ can be represented by the Fourier transform of the mass (energy) distribution function $\omega(m)$ as follows [7–9]:

$$a_0(t) \equiv \int_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\,\mu\,t} \mathrm{d}\mu \,, \tag{4}$$

where $\omega(\mu) \geq 0$ for $\mu \geq \mu_0$ and $\omega(\mu) = 0$ for $\mu < \mu_0$.

The simplest way to compare the decay law $\mathcal{P}_0(t)$ with the exponential (canonical) decay law $\mathcal{P}_c(t) = |a_c(t)|^2$, where $a_c(t) = \exp\left[-i\frac{t}{\hbar}(m_\phi - \frac{i}{2}\mathit{\Gamma}_\phi)\right]$, m_ϕ is the rest mass of the particle ϕ, and $\mathit{\Gamma}_\phi$ is its decay width, is to analyze properties of the following function:

$$\zeta(t) \stackrel{\mathrm{def}}{=} \frac{a_0(t)}{a_c(t)} \,. \tag{5}$$

There is $|\zeta(t)|^2 = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)}$. Analysis of properties of this function allows one to visualize all the more subtle differences between $\mathcal{P}_0(t)$ and $\mathcal{P}_c(t)$.

3. Numerical studies: The Breit–Wigner model

Results of studies of numerous models presented in the literature show that decay curves obtained for these models are very similar in form to the curves calculated for $\omega(\mu)$ having a Breit–Wigner form $\omega(\mu) \equiv \omega_{\rm BW}(\mu)$ (see [10] and analysis in [8])

$$\omega_{\rm BW}(\mu) = \frac{N}{2\pi}\,\Theta(\mu-\mu_0)\;\frac{\mathit{\Gamma}_0}{(\mu-m_0)^2+\left(\frac{\mathit{\Gamma}_0}{2}\right)^2}\,, \tag{6}$$

where N is a normalization constant and $\Theta(\mu)$ is a step function. So to find the most typical properties of the decay curve, it is sufficient to make the relevant calculations for $\omega(\mu)$ modeled by the Breit–Wigner distribution of the mass (energy) density $\omega_{\rm BW}(\mu)$. The typical form of the survival probability $\mathcal{P}_0(t)$ is presented in Fig. 1. The form of the decay curves depends on the ratio $s_R = \frac{m_R}{\mathit{\Gamma}_0}$, where $m_R = m_0 - \mu_0$: The smaller s_R, the shorter time when the late time deviations from the exponential form of $\mathcal{P}_0(t)$ begin to dominate. Within the considered model, the standard canonical form

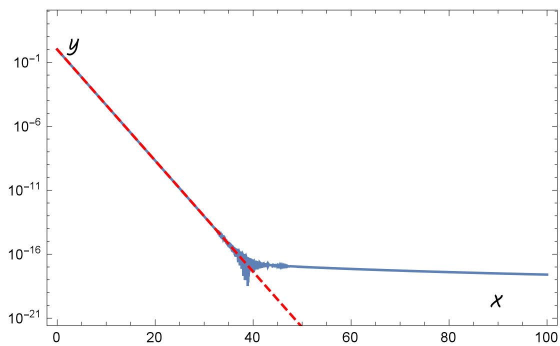

Fig. 1. Decay curves obtained for $\omega_{\rm BW}(E)$. Axes: $x = t/\tau_0$; y: $\mathcal{P}_0(t) = |a_0(t)|^2$ (solid line), $\mathcal{P}_c(t) = |a_c(t)|^2$ (dotted line). The case $s_R = \frac{m_R}{\mathit{\Gamma}_0} = 1000$.

of the survival amplitude $a_c(t)$, is given by the following relation, $a_c(t) = \exp\left[-i\frac{t}{\hbar}\left(m_0 - \frac{i}{2}\,\mathit{\Gamma}_0\right)\right]$. $\mathit{\Gamma}_0$ is the decay rate and $\frac{\hbar}{\mathit{\Gamma}_0} \equiv \frac{1}{\mathit{\Gamma}_0} = \tau_0$ is the lifetime within the assumed system of units $\hbar = c = 1$ (time t and $\mathit{\Gamma}_0$ are measured in the rest reference frame of the particle). The typical form of $|\zeta(t)|^2$ is presented in Fig. 2.

From results of the model calculations presented in Fig. 2, it follows that at the initial stage of the "exponential" (or "canonical") decay regime, the amplitude of these oscillations may be much less than the accuracy of detectors. Then with increasing time the amplitude of oscillations grows, which increases the chances of observing them. This is a true quantum picture of the decay process at the so-called "exponential" regime of times.

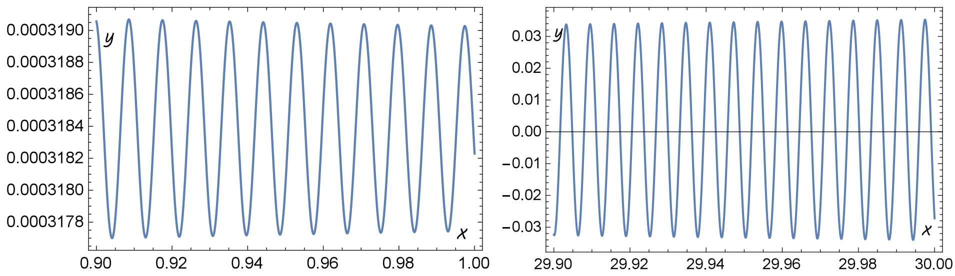

Fig. 2. A comparison of decay curves obtained for $\omega_{\rm BW}(\mu)$ with canonical decay curves. Axes: $x = t/\tau_0$, y: the function $f(t) = (|\zeta(t)|^2 - 1) = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)} - 1$, ($\mathcal{P}_0(t) = |a_0(t)|^2$, $\mathcal{P}_c(t) = |a_c(t)|^2$). The case $s_R = 1000$.

4. Moving unstable systems

Analyzing moving unstable systems, one can follow the classical physics results and assume that the unstable systems move with the constant velocity $\vec{v}$, or guided by conservations laws assume that the momentum $\vec{p}$ of the moving unstable system is constant in time. The assumption $\vec{v} = \text{const}$ was used, *e.g.* by Exner [5] and also by Alavi and Giunti [12]. Exner obtained a result that coincides with the classical result $\mathcal{P}_v(t) \simeq \mathcal{P}_0(t/\gamma)$ but detailed analysis shows that this result was obtained assuming that the velocity $\vec{v}$ is very small. Alavi and Giunti use this assumption and claim that their result is the general one but more detailed analysis of their considerations shows that their conclusion cannot be true. They use definition (2) of the survival probability mentioned earlier: $P_0(t) = |a_0(t)|^2$ of the unstable system in rest. The final result is obtained in [12] for states connected with the *reference frame in which the system is in motion with velocity* $\vec{v}$. In this new reference frame, the momentum of the particle equals $\vec{k}_m$ and $\vec{k}_m \neq \vec{p}$, where $\vec{p}$ is the momentum of the same particle but in the rest frame of the observer. The state of the moving unstable particle is described by a vector $|\Phi_{\vec{v}}\rangle$ which should be an element of the Hilbert space $\mathcal{H}_v$ connected with this new reference frame in which the system is in motion but this problem is not explained in [12]. Using states $|\Phi_{\vec{v}}\rangle$, authors of [12] define the amplitude (see (21) in [12]), $a_{\vec{v}}(t;\vec{x}) = \langle \Phi_{\vec{v}}| \exp[-itH + i\vec{P}\cdot\vec{x}]|\Phi_{\vec{v}}\rangle$, where $\vec{x}$ is a coordinate and $\vec{P}$ is the momentum operator. The interpretation of the amplitude $a_{\vec{v}}(t;\vec{x})$ is unclear: The vector $\exp[-itH + i\vec{P}\cdot\vec{x}]\,|\Phi_{\vec{v}}\rangle$ does not solve the Schrödinger evolution equation for the initial condition $|\Phi_{\vec{v}}\rangle$.

Searching for the properties of the amplitude $a_{\vec{v}}(t;\vec{x})$, authors of [12] use the integral representation of $a_{\vec{v}}(t;\vec{x})$ as the Fourier transform of the energy or, equivalently mass distribution function $\omega(m)$ (see, *e.g.* [7, 8]) and obtain that (see (39) in [12])

$$a_{\vec{v}}(t;\vec{x}) = \int \mathrm{d}m \left[\omega(m) \int \mathrm{d}^3\vec{p}\, |\phi(\vec{p})|^2\, e^{-iE_m\left(\vec{k}_m\right)\, t \,+\, i\vec{k}_m\cdot\vec{x}}\right], \tag{7}$$

where $\omega(m) = |\rho(m)|^2$ and $\rho(m)$ are the expansion coefficients of $|\Phi_{\vec{v}}\rangle$ in the basis of eigenvectors $|E_m(\vec{k}_m), \vec{k}_m, m\rangle$ for the Hamiltonian H (see (37) in [12]). $\phi(\vec{p})$ is the momentum distribution such that $\int \mathrm{d}^3\vec{p}\, |\phi(\vec{p})|^2 = 1$. The energy $E_m(\vec{k}_m)$ and momentum $\vec{k}_m$ in the new reference frame mentioned are connected with $E_m(\vec{p})$ and $\vec{p}$ in the rest frame by Lorentz transformations (see (33)–(35) in [12])

$$E_m\left(\vec{k}_m\right) = \gamma\left(E_m(\vec{p}) + v\, p_{\parallel}\right), \qquad k_{m\,\parallel} = \gamma\left(p_{\parallel} + vE_m(\vec{p})\right) \tag{8}$$

and $\vec{k}_{m\,\perp} = \vec{p}_{\perp}$, where $k_{m\,\parallel}$ ($\vec{k}_{m\,\perp}$) and $p_{\parallel}$ ($\vec{p}_{\perp}$) are components of $\vec{k}_m$ and $\vec{p}$ parallel (orthogonal) to the velocity $\vec{v}$, and $E_m(\vec{p}) = \sqrt{m^2 + \vec{p}^{\,2}}$.

Using the amplitude $a_{\vec{v}}(t;\vec{x})$, authors of [12] define the survival probability $\mathcal{P}_{\vec{v}}(t)$ of the moving relativistic unstable particle as (see (40) in [12])

$$\mathcal{P}_{\vec{v}}(t) = \frac{\int \mathrm{d}^3x\, |a_{\vec{v}}(t,\vec{x})|^2}{\int \mathrm{d}^3x\, |a_{\vec{v}}(t=0,\vec{x})}, \tag{9}$$

then they present main steps of calculations of this probability. In conclusion, they claim that the result of performed calculations shows that

$$\mathcal{P}_{\vec{v}}(t) = |a_0(t/\gamma)|^2 \equiv \mathcal{P}_0(t/\gamma), \tag{10}$$

where $\gamma = 1/\sqrt{1-v^2}$ within the system of units used.

To prove this last relation, authors of [12] limited their considerations to the case when for the decay width Γ, for mass of the particle M and for the momentum uncertainty $\sigma_p^2 = \int \mathrm{d}^3\vec{p}\, |\phi(\vec{p})|^2 (p_i)^2$ ($i = 1,2,3$), the condition $\Gamma \ll \sigma_p \ll M$ is assumed to hold. This is a crucial condition which allowed them to approximate the energy $E_m(p)$ for all m from the spectrum of H as follows:

$$E_m(\vec{p}) \simeq m \tag{11}$$

neglecting terms of the order of $\vec{p}^{\,2}/m^2$. Note that integral (7) is taken over all m from the spectrum $\sigma(H)$ of H. This means that approximation (11) has to hold for every $m \in \sigma(H)$. Approximation (11) was used in [12] to replace relations (8) by the following approximate one:

$$E_m\left(\vec{k}_m\right) \equiv \gamma\left(E_m(\vec{p}) + v\, p_{\parallel}\right) \simeq \gamma\left(m + v\, p_{\parallel}\right), \tag{12}$$

$$k_{m\,\parallel} \equiv \gamma\left(p_{\parallel} + vE_m(\vec{p})\right) \simeq \gamma\left(p_{\parallel} + vm\right). \tag{13}$$

A discussion of the admissibility of the mentioned conditions and approximations uses arguments similar to those one can find, *e.g.* in [5]. The difference is that in [5] the approximation $E_p(m) \simeq m + \vec{p}^{\,2}/2m$ is used instead of (11).

Finally, replacing $E_m(\vec{k}_m)$ and $\vec{k}_m$ under the integral sign in (7) by (12) respectively (or in [12], in (41) by (33) and (34)) after some algebra, authors of [12] obtain their relation (46) that was needed, that is the relation denoted as (10) in this section. This result obtained within the conditions and approximations described above was the basis of the all conclusions presented in [12].

Unfortunately, in [12], there is not any analysis of physical consequences of assumed conditions and approximations used. Note that

$$(E_m(\vec{p}) \simeq m \ \text{ for all } \ m \in \sigma(H)) \quad \Leftrightarrow \quad |\vec{p}\,| \simeq 0\,, \tag{14}$$

and $|\vec{p}\,| \simeq 0 \ \Leftrightarrow \ (|\vec{p}_{\perp}| \simeq 0$ and $p_{\|} \simeq 0)$. Note also that within the system of units used, $|v| < c = 1$. This means that $|vp_{\|}| \leq |v|\,|p_{\|}| < |p_{\|}| \simeq 0$. This is why approximations (12) cannot be considered as the correct and consistent with the assumed in [12] relation (11). From the above analysis, it follows that the only correct and self-consistent approximations are

$$E_m\left(\vec{k}_m\right) \simeq \gamma\, m\,, \qquad k_{m\,\|} \simeq \gamma v m\,. \tag{15}$$

The truth is that such approximations lead to the result $\mathcal{P}_{\vec{v}}(t) = \mathcal{P}_0(\gamma t)$, which was never met in experiments. So, in the light of the above analysis, the correctness of the final conclusions drawn in [12] is rather questionable.

Another possibility is to assume that $\vec{p} =$ const. This approach was used by, *e.g.* Stefanovich [3] or Shirokov [4]. It leads to the results which do not depend on that whether the assumed momentum $\vec{p} =$ const is small or not. So let us consider now the case of moving quantum system with definite momentum $\vec{p}$. We need the probability amplitude $a_p(t) = \langle \phi_p|\phi_p(t)\rangle$, (where $|\phi_p\rangle$ corresponds to the moving unstable system with definite momentum $\vec{p}\,$), which defines the survival probability $\mathcal{P}_p(t) = |a_p(t)|^2$. There is (see [3, 4, 11])

$$a_p(t) \;\equiv\; \int\limits_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\sqrt{p^2 + \mu^2}\;t} \mathrm{d}\mu\,. \tag{16}$$

Results of numerical calculations are presented in Fig. 3, where calculations were performed for $\omega(\mu) = \omega_{\mathrm{BW}}(\mu)$ and $\mu_0 = 0$, $E_0/\mathit{\Gamma}_0 \equiv m_0/\mathit{\Gamma}_0 = 1000$ and $cp/\mathit{\Gamma}_0 \equiv p/\mathit{\Gamma}_0 = 1000$. Values of these parameters correspond to $\gamma = \sqrt{2}$. According to the literature, for laboratory systems, a typical value of the ratio $m_0/\varGamma_0$ is $m_0/\varGamma_0 \geq O(10^3\text{–}10^6)$ (see *e.g.* [13]) therefore the choice $m_0/\varGamma_0 = 1000$ seems to be reasonable minimum. Decay curves obtained numerically are presented in Fig. 3.

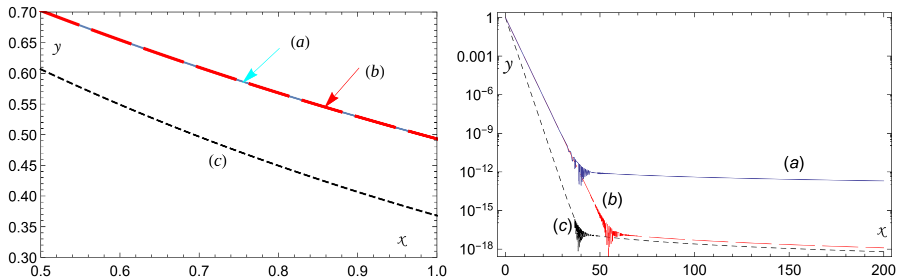

Fig. 3. Decay curves obtained for $\omega_{\rm BW}(\mu)$. Axes: $x = t/\tau_0$; y — survival probabilities: (a) $\mathcal{P}_p(t)$, (b) $\mathcal{P}_0(t/\gamma)$, (c) $\mathcal{P}_0(t)$.

Similarly to the case of quantum unstable systems at rest, one can calculate the ratio $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$ in the case of moving particles. Results of numerical calculations of this ratio are presented in Fig. 4, where calculations were performed for $\omega(\mu) = \omega_{\rm BW}(\mu)$ and for $\mu_0 = 0$, $m_0/\varGamma_0 = 1000$, $cp/\varGamma_0 \equiv p/\varGamma_0 = 1000$ and $\gamma = \sqrt{2}$.

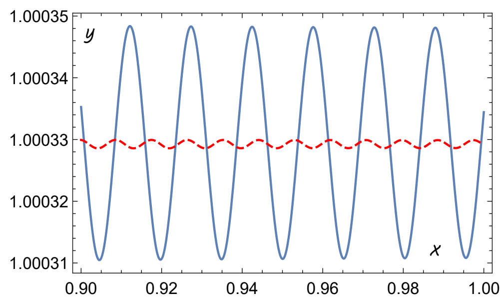

Fig. 4. Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — ratio of probabilities; solid line: $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$; dashed line: $\mathcal{P}_0(t/\gamma)/\mathcal{P}_c(t/\gamma)$.

5. Summary

From the results presented in Sec. 3, it follows that there is not any time interval in which the survival probability (decay) law could be a decreasing function of time of the purely exponential form: In the case of the Breit–Wigner model, in the so-called "exponential regime", the decay curves are oscillatory modulated with smaller or large amplitude of oscillations depending on the parameters of the model. In Sec. 4, it has been shown that in the case of relativistic quantum unstable system moving with constant momentum $\vec{p}$, when unstable systems are modeled by the Breit–Wigner mass distribution $\omega(\mu)$, only at times of the order of lifetime τ_0, it can be $\mathcal{P}_p(t) \simeq \mathcal{P}_0(t/\gamma)$ to a better or worse approximation. At times longer than a few lifetimes, the decay process of moving particles observed by an observer in his rest system is much slower that it follows from the classical physics relation $\mathcal{P}_p(t) \stackrel{?}{=} \exp\,[-\frac{t}{\gamma}\,\varGamma_0]$: There is $\mathcal{P}_p(t) > \mathcal{P}_0(t/\gamma)$, for $t \gg \tau_0$ in such a case. It also appears that in the case of moving relativistic quantum unstable system with constant momentum $\vec{p}$, decay curves are also oscillatory modulated but the amplitude of these oscillations is higher than in the case of unstable systems at rest. The general conclusion is that there is a need to test the decay law of moving relativistic unstable system for times much longer than the lifetime.

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