

THREE-PHOTON ENTANGLEMENT FROM ORTHO-POSITRONIUM REVISITED*

M. NOWAKOWSKI, D. BEDOYA FIERRO

Departamento de Física, Universidad de los Andes
Cra. 1E, 18A-10, Bogotá, Colombia

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Entanglement of the three-photon from the decay of ortho-positronium is re-analyzed. We use the full three-body phase space to write down the entangled states classified according to the spin directions of the ortho-positronium. Even in the case when the spin is perpendicular to the decay plane, we find non-negligible phases entering the entangled state. This has not been noticed before. We advocate a fixed quantization axis of the spin for the sake of generality. A brief discussion of a three-dimensional formalism for photons, including correlations, *versus* a two-dimensional one is given.

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1. Introduction

It is not often that a device designed principally for medical applications turns out to be useful also in probing fundamental issues of physics. J-PET (Jagiellonian Positron Emission Tomograph) is a tomography machine based on positron–electron annihilation with novel technological asseoire. It can be used to test CP/T and CPT violation in the purely leptonic sector of positronium [1] and probably also in probing non-locality aspects of quantum mechanics by studying the three-photon entanglement from the decay of ortho-positronium (which is a spin-1 bound state of electron–positron) [1, 2]. It is the latter topic which we will discuss in this paper. Of course, many tests of locality have been already performed which confirm the non-local nature of quantum mechanics. That still other tests are being suggested (among other using the J-PET) lies in the underlying positivistic philosophy of any natural theory which cannot be verified to one hundred per cent, but falsified

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by only one experimental result. Coming back to the topic of three-photon entanglement from ortho-positronium, we will follow the seminal paper [3] which paved the ground for the subject. In doing so, we recover most of the results in [3], but find also important differences. One of our motivations to revisit the subject is the interplay between relativistic dynamics/kinematics and quantum entanglement.

2. Kinematics and decay dynamics

The spin-1 ortho-positronium decays into three photons where every photon is characterized by its momentum and polarization, $\gamma_i = \gamma[\epsilon(\hat{\mathbf{k}}_i, \lambda), \mathbf{k}_i]$. For circular polarization, we have the polarization vectors

$$\epsilon\left(\hat{\mathbf{k}}_i, \lambda\right) = -\frac{\lambda}{\sqrt{2}}\left(\cos\theta_i \cos\Phi_i - i\lambda \sin\Phi_i, \cos\theta_i \sin\Phi_i + i\lambda \cos\Phi_i, -\sin\theta_i\right) \quad (1)$$

with $\lambda = \pm 1$ and the angles define the direction of each momentum by $\hat{\mathbf{k}}_i = (\cos\Phi_i \sin\theta_i, \sin\Phi_i \sin\theta_i, \cos\theta_i)$. The energy-momentum conservation at rest reads $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ and $k_1 + k_2 + k_3 = m \simeq 2m_e$. The momentum conservation defines a plane in which the photons move. The plane changes its orientation from event to event. The unit vector $\hat{\mathbf{n}} = \frac{\mathbf{k}_1 \times \mathbf{k}_2}{|\mathbf{k}_1 \times \mathbf{k}_2|}$ is perpendicular to the plane. Often one chooses $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and the question arises if we can do it for one event or for the whole sample.

Briefly, the matrix element \mathcal{M} is $\mathcal{M} = -\sqrt{2}V_3$ for $S_z = 0$ and $\mathcal{M} = \pm V_1 + iV_2$ for $S_z = \pm 1$. The vector function \mathbf{V} is a lengthy expression [3, 4], but an important one for the entanglement. We, therefore, give it in full length here

$$\begin{aligned} & \mathbf{V}(\mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2; \mathbf{k}_3, \lambda_3) \\ &= (\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3)\epsilon^*\left(\hat{\mathbf{k}}_1, \lambda_1\right)\left[\epsilon^*\left(\hat{\mathbf{k}}_2, \lambda_2\right) \cdot \epsilon^*\left(\hat{\mathbf{k}}_3, \lambda_3\right)\right] \\ &+ (\lambda_2 - \lambda_3)(\lambda_3 + \lambda_1)\epsilon^*\left(\hat{\mathbf{k}}_2, \lambda_2\right)\left[\epsilon^*\left(\hat{\mathbf{k}}_3, \lambda_3\right) \cdot \epsilon^*\left(\hat{\mathbf{k}}_1, \lambda_1\right)\right] \\ &+ (\lambda_3 - \lambda_1)(\lambda_1 + \lambda_2)\epsilon^*\left(\hat{\mathbf{k}}_3, \lambda_3\right)\left[\epsilon^*\left(\hat{\mathbf{k}}_1, \lambda_1\right) \cdot \epsilon^*\left(\hat{\mathbf{k}}_2, \lambda_2\right)\right]. \end{aligned}$$

This function encodes the whole dynamics including the entanglement. For instance, $V(\mathbf{k}_1, \pm; \mathbf{k}_2, \pm; \mathbf{k}_3, \pm) = 0$.

3. The three-photon entanglement

On the other hand, we find for instance $\mathbf{V}(\mathbf{k}_1, +; \mathbf{k}_2, +; \mathbf{k}_3, -) = 2\epsilon^*(\hat{\mathbf{k}}_3, -)f_{12}$ with $f_{ij} = [1 - \hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_j]$. This leads directly to the coefficients of the entangled states classified according to the spin projections.

For $S_z = 0$ and $\mathcal{M} = -\sqrt{2}V_3$, we obtain $\epsilon_3^*(\hat{\mathbf{k}}_i, \lambda = \pm) = -\sin \theta_i$. Therefore, in this case, the unnormalized three parties entangled state comes out as

$$|\Psi\rangle_{S_z=0} = \gamma_0 [|++\rangle - |--\rangle] + \beta_0 [|+-\rangle - |-+-\rangle] + \alpha_0 [| - ++\rangle - | + --\rangle] \tag{2}$$

with the coefficients

$$\gamma_0 = \sin \theta_3 f_{12}, \quad \beta_0 = \sin \theta_2 f_{13}, \quad \alpha_0 = \sin \theta_1 f_{23}. \tag{3}$$

On the other hand, for $S_z = \pm 1$ and $\mathcal{M} = \pm V_1 + iV_2$, we have

$$\epsilon_1^*(\hat{\mathbf{k}}_i, \lambda = \pm) + i\epsilon_2^*(\hat{\mathbf{k}}_i, \lambda = \pm) = e^{i\Phi_i} (\cos \theta_i \pm 1). \tag{4}$$

This leads to the entangled state of the form of

$$|\Psi\rangle_{S_z=\pm 1} = \gamma_{\pm}^{(1)} |++\rangle + \gamma_{\pm}^{(2)} |--\rangle + \beta_{\pm}^{(1)} |+-\rangle + \beta_{\pm}^{(2)} |-+-\rangle + \alpha_{\pm}^{(1)} |-++\rangle + \alpha_{\pm}^{(2)} |+--\rangle \tag{5}$$

with

$$\begin{aligned} \gamma_{\pm}^{(1)} &= e^{\pm i\Phi_3} (\cos \theta_3 - 1) f_{12}, & \gamma_{\pm}^{(2)} &= e^{\pm i\Phi_3} (-\cos \theta_3 - 1) f_{12}, \\ \beta_{\pm}^{(1)} &= e^{\pm i\Phi_2} (\cos \theta_2 - 1) f_{13}, & \beta_{\pm}^{(2)} &= e^{\pm i\Phi_2} (-\cos \theta_2 - 1) f_{13}, \\ \alpha_{\pm}^{(1)} &= e^{\pm i\Phi_1} (\cos \theta_2 - 1) f_{23}, & \alpha_{\pm}^{(2)} &= e^{\pm i\Phi_1} (-\cos \theta_2 - 1) f_{23}. \end{aligned}$$

For the first case ($S_z = 0$), it is only if we choose a coordinate system such that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ ($\sin \theta_i = 1$) that the expression becomes simpler and coincides with the one given in [3] for $S_z = \pm 1$. In the case of $S_z = \pm 1$ we do not recover the expression in [3] for $S_z = 0$. Even if we choose the spin quantization axis in the $\hat{\mathbf{n}}$ direction, we differ by the phases $e^{\pm i\Phi_i}$. Apart from that, we have a different assignation for $S_z = 0, \pm 1$. This means that our coefficients depend explicitly on the coordinates (Φ_i) even if we take $\theta_i = \pi/2$. Dependence on Φ_i has also been noticed in [2] which appeared after the talk presented in this paper was given.

How important are the phases? If, in addition to choosing the z -axis perpendicular to the three-photon plane, we make a rotation of the x - y coordinates, we can get rid of one phase. Factorizing a second phase (which becomes global), we are certainly left with one relative phase. One can demonstrate already in a simpler system (two particle entanglement) that a phase will play a role in correlation functions which enter Bell’s inequalities. Deforming a spin-singlet by writing

$$|00\rangle_{\alpha} = \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - e^{i\alpha} |\downarrow\rangle|\uparrow\rangle), \tag{6}$$

the correlation function becomes

$$\begin{aligned} & {}_{\alpha}\langle 00 | \left(\hat{\mathbf{a}} \cdot \mathbf{S}^{(1)} \hat{\mathbf{b}} \cdot \mathbf{S}^{(2)} \right) | 00 \rangle_{\alpha} \\ &= -\frac{1}{4} [\cos \alpha \cos \theta + a_z b_z (1 - \cos \alpha) - (\mathbf{a} \times \mathbf{b})_z \sin \alpha] \end{aligned}$$

with the internal variable $\cos \theta = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ and (external) variable a_i and b_i (coordinates of the unit vectors). We can then safely state that in correlation functions, the phases $e^{\pm i\Phi_i}$ which we encountered will certainly play a role. Incidentally, with linearly polarized photons reference [5] writes a two-photon entanglement as $\frac{1}{\sqrt{2}}[|H\rangle|V\rangle + e^{i\alpha}|V\rangle|H\rangle]$, where $|H\rangle = (1, 0)$ and $|V\rangle = (0, 1)$ and the matrices used in the correlations are the Pauli matrices. A related question comes then into mind. Shall we choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$? For one single event this is certainly possible. If we do this for every subsequent event, we will not be able to classify the entanglements according to the spin projections of the positronium $S_z = 0, \pm 1$ since we keep on changing the quantization axis then. If we group together, say $|\Psi\rangle_{S_z=0}$ with $|\Psi\rangle_{S_z \neq 0}$, we might be comparing and classifying wrongly. The condition that the state factorizes (giving essentially rise to two-particle entanglement) is given by

$$\beta = 0, \quad \gamma = \pm\alpha; \quad \gamma = 0, \quad \beta = \pm\alpha; \quad \alpha = 0, \quad \gamma = \pm\beta. \quad (7)$$

For $\theta_i = \pi/2$, choosing one of the three cases above, we get a configuration at the edge of the allowed phase space: two collinear momenta and with third one anti-parallel, *e.g.* for $\gamma = 0$ and $\beta = \alpha$. For an arbitrary spin quantization axis for the spin-1 ortho-positronium, there are other configurations of momenta for which the state factorizes.

In a three-party entanglement, there are two non-biseparable classes: W-class and GHZ-class (Greenberger–Horne–Zeilinger) which cannot be transformed into each other by local operations [6]. Generically, one writes

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} [|000\rangle + |111\rangle], \quad |\text{W}\rangle = \frac{1}{3} [|001\rangle + |010\rangle + |100\rangle]. \quad (8)$$

If we have a three-party entanglement $|\phi\rangle = \sum_{ijk} c_{ijk} |ijk\rangle$, an invariant measure of the entanglement is the so-called hyper-determinant

$$0 \leq \text{Hdet}(c_{ijk}) = c_{000}^2 c_{111}^2 + \dots + \dots c_{000} c_{110} c_{001} + \dots \leq 1/4. \quad (9)$$

If $\text{Hdet} = 0$ and the state does not factorize, then we have a W-class entanglement. If we choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, the condition $\text{Hdet} = 0$ leads to factorization, *i.e.*, the configuration of two collinear and one anti-parallel three momentum which we had before. Since by our choice of the z -axis, we might lose some generality, could it be that we can reach the W-class in accordance with the energy-momentum conservation? The constraint of the energy-momentum

conservation makes the problem more complex than anticipated. We just outline the first few steps. First, we perform a change of basis, from circular to linear polarization $|\pm\rangle = |R/L\rangle$, $|H/V\rangle = |0/1\rangle$ with $|\pm\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle]$. Then, for instance,

$$|\Psi\rangle_{S_z=0} = (\alpha_0 + \beta_0 - \gamma_0) |010\rangle + (\alpha_0 - \beta_0 - \gamma_0) |100\rangle + (-\alpha_0 - \beta_0 + \gamma_0) |001\rangle + (\alpha_0 + \beta_0 + \gamma_0) |111\rangle.$$

It is easy to calculate the hyper-determinant on this basis. The result is

$$\text{Hdet} = (-\alpha_0 + \beta_0 - \gamma_0) (\alpha_0 - \beta_0 - \gamma_0) \cdot (-\alpha_0 - \beta_0 + \gamma_0) (\alpha_0 + \beta_0 + \gamma_0). \tag{10}$$

The main question regarding $\text{Hdet} = 0$ is whether this is possible in accordance with energy-momentum conservation (see also [2]). This remains to be our future task.

4. The two-dimensional versus the three-dimensional formalism

What do we mean exactly when we write, say $|+-\rangle$? Obviously, it is the tensor product $|+-\rangle = |+\rangle \otimes |+\rangle \otimes |-\rangle$. But what exactly is, say $|+\rangle$ or what is its representation in a finite dimensional Hilbert space? Consider the following: (1) we have a two-level system given by the two degrees of freedom of the photon polarization $\lambda = \pm 1$, and (2) the photons carry momenta and are entangled in their polarizations which depend on the directions of the momenta, hence $|+\rangle = |+\rangle_1 = |\hat{\mathbf{k}}_1, +\rangle = |\boldsymbol{\epsilon}(\hat{\mathbf{k}}_1, +1)\rangle$. But the polarization vectors are three-dimensional objects. The mismatch between the number of qubits \pm and the dimension of the state vector comes from the fact that photons are vector particles, but strictly speaking, do not have spin (defined only in the rest frame). The standard correlation functions are of the form

$$\langle \Psi | \left(\hat{\mathbf{a}} \cdot \boldsymbol{\sigma}^{(1)} \right) \left(\hat{\mathbf{b}} \cdot \boldsymbol{\sigma}^{(2)} \right) \left(\hat{\mathbf{c}} \cdot \boldsymbol{\sigma}^{(3)} \right) | \Psi \rangle \tag{11}$$

with σ_i the two-dimensional Pauli matrices (this correlation enters inequalities like the Mermin or Svetlichny inequality). This implies that the state vectors are also two dimensional. This goes back to the so-called Jones formalism where all polarization states are represented as two-dimensional objects. In general, $|\psi\rangle = (\cos \phi e^{i\alpha_x}, \sin \phi e^{i\alpha_y})$. If the difference between the phases is $\pi/2$ and $\cos \phi = \sin \phi$, we get the circular polarization states $\frac{1}{\sqrt{2}}(1, i)$, $\frac{1}{\sqrt{2}}(1, -i)$. This is in accordance with the polarization vectors if $\theta = 0$ (and not $\pi/2$) *i.e.* we choose $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. Indeed, we have then

$$\boldsymbol{\epsilon}(\hat{\mathbf{k}}, +) = \frac{e^{-i\Phi}}{\sqrt{2}}(1, i, 0), \quad \boldsymbol{\epsilon}(\hat{\mathbf{k}}, -) = \frac{e^{-i\Phi}}{\sqrt{2}}(1, -i, 0) \tag{12}$$

which is effectively two dimensional. However, we cannot choose for all photons $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ especially when we have chosen once $\theta_i = \pi/2$. For two photons with $\mathbf{k}_1 + \mathbf{k}_2 = 0$, say in the decay of para-positronium with the entanglement $|\Psi\rangle_{\text{para}} = \frac{1}{\sqrt{2}}[|++\rangle - |--\rangle]$, we can do that if we choose $\hat{\mathbf{k}}_i = \hat{\mathbf{z}}$. With three photons, we will have to choose a more appropriate formalism and operators (corresponding to, say circular polarizations). We can do that by going to the adjoint representation of $SU(2)$ *i.e.*, $(S_i)_{jk} = -i\epsilon_{ijk}$. The eigenvectors to S_3 are $\frac{1}{\sqrt{2}}(1, \pm i, 0)$ which are the circular polarizations states in a plane perpendicular to the z -axis. A third eigenvector $(0, 0, 1)$ is possible, but the photon will not have it. The eigenvectors to S_1 and S_2 are $\frac{1}{\sqrt{2}}(0, \pm i, 1)$, $\frac{1}{\sqrt{2}}(\pm i, 0, 1)$, respectively. The first state describes circular polarized photons in a plane perpendicular to the x -axis *etc.* It makes then sense to consider $\hat{\mathbf{a}} \cdot \mathbf{S}$ and to calculate expectation values of the form of

$$\langle \Psi | \left(\hat{\mathbf{a}} \cdot \mathbf{S}^{(1)} \right) \left(\hat{\mathbf{b}} \cdot \mathbf{S}^{(2)} \right) \left(\hat{\mathbf{c}} \cdot \mathbf{S}^{(3)} \right) | \Psi \rangle. \quad (13)$$

5. Conclusions

The physics of the three-photon entanglement from positronium has all the physical ingredients which makes the twentieth and the beginning of the twenty first century physics: the positron necessary to form the positronium is a decay product of a nuclear decay, it forms a leptonic bound state whose physics is well-understood in the framework of Quantum-Electro-Dynamics and, finally, the three-body entanglement has been a subject of intense studies in the last two decades. In this contribution, we have re-analyzed the entangled state of three photons from ortho-positronium in full generality and found some differences as compared to [3].

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