SYMMETRY ANALYSIS AND SOME NEW EXACT SOLUTIONS OF THE (2+1)-DIMENSIONAL BURGERS EQUATIONS

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In this paper, the Lie point symmetry analysis method is used to investigate the (2+1)-dimensional Burgers equations. We have obtained the optimal system of Lie subalgebras. Some new exact solutions for the (2+1)-dimensional Burgers equations are obtained.

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1. Introduction

The solutions of nonlinear differential equations are an essential tool for many physical and engineering applications. There are many methods to solve nonlinear partial differential equations (PDEs) such as the Weierstrass function method [1], Jacobi elliptic function method [2, 3], Hirota bilinear method [4], the inverse scattering method [5], the tanh method [6], the extended mapping transformation method [7], the truncated expansion method [8], the simplest equation method [9], the bifurcation method [10] and Lie symmetry method [11–14]. The latter is considered as the most powerful method for getting exact solutions of PDEs.

In this paper, we use the Lie symmetry method to investigate the (2+1)-dimensional Burgers equations [15]

\[
\begin{align*}
    u_t &= uu_y + \lambda vu_x + \mu u_{yy} + \lambda \mu u_{xx}, \\
    u_x &= v_y,
\end{align*}
\]

(1.1)

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where \( u = u(x, y, t) \) and \( v = v(x, y, t) \), \( \mu \) and \( \lambda \) are real constants. When \( x = y \) and \( u = v \), Eq. (1.1) degenerates to the famous one-dimensional Burgers equation

\[
u_t = \mu uu_x + \nu u_{xx}, \tag{1.2}\]

where \( m = \lambda + 1 \) and \( n = \mu(\lambda + 1) \). Burgers’ equation (1.2) is widely used for describing physical phenomena in fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. For example, it is considered as the lowest order approximation for the one-dimensional propagation of weak shock waves in fluids [16]. Burgers’ equations (1.1) are a generalization of Burgers’ equation (1.2) and its equivalent form is derived from the Painleve integrability classification in [17].

Many types of exact solutions for Eq. (1.1) are obtained in [15, 18–25]. Soliton and soliton-like solutions are obtained in [15, 18]. Periodic and doubly periodic solutions are obtained in [19–21]. In [22, 23], variable separation solutions are obtained. Interaction between kink solitary wave and rogue wave is investigated in [24]. Residual symmetry analysis is investigated in [25]. In this paper, we concentrate on finding new similarity solutions of Eq. (1.1).

The sequence of this paper is as follows: In Section 2, we use the symmetry analysis of (1.1) to find all Lie algebra of symmetry generators. In Section 3, we obtain the optimal one-dimensional system of these subalgebras. In Section 4, we obtain exact solutions of the reduced equation that is produced from the infinitesimal transformations.

### 2. Symmetry analysis of Eq. (1.1)

The infinitesimal generator \( \Gamma \) of the Lie point transformations is given by

\[
\Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u} + V \frac{\partial}{\partial v}. \tag{2.1}
\]

We use Maple to obtain the infinitesimal symmetry generators by solving the determining equations that are produced from the invariant condition \( \Gamma(2) \Delta|_{\Delta=0} = 0 \) (this condition is defined in [26]). We obtain
\[ T = \frac{1}{2}c_5t^2 + c_4t + c_1, \]
\[ X = F(t) + \frac{1}{2}(c_5t + c_4)x, \]
\[ Y = \frac{1}{2}(c_5y + 2c_3)t + \frac{1}{2}c_4y + c_2, \]
\[ U = -\frac{1}{2}c_5(tu + y) - \frac{1}{2}c_4u - c_3, \]
\[ V = -\frac{1}{2}(c_5t + c_4)v - \frac{1}{\lambda}F'(t) - \frac{1}{2\lambda}c_5x, \]

(2.2)

where \( c_1, c_2, c_3, c_4 \) and \( c_5 \) are constants, \( F(t) \) is an arbitrary function and \( F'(t) \) is its derivative with respect to \( t \). In [27], the nonlocal symmetry analysis of Eq. (1.1) is investigated. The authors in [27] have obtained infinitesimals (2.2), however, they considered only the case of \( F(t) = c_6t + c_7 \). Here, we consider \( F(t) \) as an arbitrary function of \( t \) in order to obtain some new similarity solutions of Eq. (1.1). The Lie algebra of infinitesimal symmetry generators is spanned by five-dimensional and the infinite-dimensional subalgebras

\[ v_1 = \partial_t, \]
\[ v_2 = \partial_y, \]
\[ v_3 = t\partial_y - \partial_u, \]
\[ v_4 = 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v, \]
\[ v_5 = t^2\partial_t + tx\partial_x + ty\partial_y - (tu + y)\partial_u - \left( tv + \frac{x}{\lambda} \right) \partial_v, \]
\[ v_f = F(t)\partial_x - \frac{F'(t)}{\lambda} \partial_v. \]

(2.3)

3. Classification of group invariant solutions

To find the one-dimensional optimal system of the five-dimensional subalgebras (2.3), we follow the procedure described in [28]. To achieve this task, we use the commutator table shown in Table I and the table of adjoint shown in Table II.

Let \( v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \) be an arbitrary element of the subalgebra (2.3). The invariant function such that \( \phi(\text{Ad}_g(v)) = \phi(v) \), where \( v \in \mathcal{G} \) (\( \mathcal{G} \) is a five-dimensional Lie algebra of (1.1) generated by \( v_1, \cdots, v_5 \)), \( g \in G \) (\( G \) is the corresponding symmetry group of \( \mathcal{G} \)) and the adjoint action defined in [28] is given by

\[ \phi(v) = a_4^2 - a_1a_5. \]

(3.1)
Table of commutators.

<table>
<thead>
<tr>
<th>[v_i,v_j]</th>
<th>v_1</th>
<th>v_2</th>
<th>v_3</th>
<th>v_4</th>
<th>v_5</th>
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<td>0</td>
<td>v_2</td>
<td>2v_1</td>
<td>v_4</td>
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<tr>
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<td>0</td>
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<td>v_2</td>
<td>v_3</td>
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<tr>
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<td>0</td>
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<td>-v_2</td>
<td>v_3</td>
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<td>0</td>
<td>-2v_5</td>
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</table>

Table of adjoint.

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<tr>
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<th>v_2</th>
<th>v_3</th>
<th>v_4</th>
<th>v_5</th>
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</thead>
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<td>v_2</td>
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<td>v_4-2εv_1</td>
<td>v_5-εv_4+ε^2v_1</td>
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<tr>
<td>v_2</td>
<td>v_1</td>
<td>v_2</td>
<td>v_3</td>
<td>v_4-εv_2+1/2ε^2v_2</td>
<td>v_5-εv_3</td>
</tr>
<tr>
<td>v_3</td>
<td>v_1+εv_2</td>
<td>v_2</td>
<td>v_3</td>
<td>v_4+εv_3</td>
<td>v_5</td>
</tr>
<tr>
<td>v_4</td>
<td>ε^2v_1</td>
<td>ε^εv_2</td>
<td>e^{-ε}v_3</td>
<td>v_4</td>
<td>ε^{-2ε}v_5</td>
</tr>
<tr>
<td>v_5</td>
<td>v_1+εv_4+ε^2v_5</td>
<td>v_2+εv_3</td>
<td>v_3</td>
<td>v_4+2εv_5</td>
<td>v_5</td>
</tr>
</tbody>
</table>

There are three cases to be considered: φ(v) > 0, φ(v) < 0 and φ(v) = 0.

— Case 1: If φ(v) > 0, then we put a_1 = a_5 = 0 and a_4 = 1. By adjoint action Ad(e^{εv_3}v), if we take ε = -a_3, then v is equivalent to

\[ v = a_2v_2 + v_4. \]  (3.2)

— Case 2: If φ(v) < 0, then we put a_4 = 0 and a_1 = a_5 = 1. By adjoint action Ad(e^{εv_1}v), if we take ε_1 = a_3, then v is equivalent to

\[ v = v_1 + a_2v_2 + v_5. \]  (3.3)

Another adjoint action Ad(e^{εv_3}v), if we take ε_2 = -a_2, then v is equivalent to

\[ v = v_1 + v_5. \]  (3.3)

— Case 3: If φ(v) = 0, then there are three subcases.

— Subcase 1: We put a_4 = a_5 = 0 and a_1 = 1. By adjoint action Ad(e^{εv_1}v), if we take ε = a_2/a_3, then v is equivalent to

\[ v = v_1 + a_3v_3. \]  (3.4)

— Subcase 2: We put a_4 = a_1 = 0 and a_5 = 1. By adjoint action Ad(e^{εv_5}v), if we take ε = -a_3/a_2, then v is equivalent to

\[ v = a_2v_2 + v_5. \]  (3.5)
Subcase 3: We put \( a_1 = a_4 = a_5 = 0 \). By adjoint action \( \text{Ad}(e^{a_5 v}) \), if we take \( \epsilon = -\frac{a_4}{a_2} \), then \( v \) is equivalent to
\[
v = v_2, \tag{3.6}
\]
or by adjoint action \( \text{Ad}(e^{a_1 v}) \), if we take \( \epsilon = \frac{a_2}{a_3} \), then \( v \) is equivalent to
\[
v = v_3. \tag{3.7}
\]

The optimal system of one-dimensional subalgebras is as follows:
\[
\begin{align*}
& a_2 v_2 + v_4, \\
& v_1 + v_5, \\
& v_1 + a_3 v_3, \\
& a_2 v_2 + v_5, \\
& v_2, \\
& v_3. \tag{3.8}
\end{align*}
\]

We apply these subalgebras to (1.1) and find exact solutions.

4. Group invariant solutions

4.1. Exact solution using the generator \( v_1 + a_3 v_3 \)

In this subsection, we consider the subalgebra \( v_1 + a_3 v_3 \) and take \( a_3 = 1 \). In this case, the invariant surface conditions are given by
\[
\begin{align*}
t u_y + u_t &= -1, \\
t v_y + v_t &= 70. \tag{4.1}
\end{align*}
\]
Solving (4.1), we find
\[
\begin{align*}
u &= -t + g(x, r), \\
v &= h(x, r), \tag{4.2}
\end{align*}
\]
where \( r = t^2 - 2y \). By substituting (4.2) into (1.1), we get
\[
1 - 2gg_r + 4\mu g_{rr} + \lambda hg_x + \lambda \mu g_{xx} = 0, \\
2h_r + g_x = 0. \tag{4.3}
\]
To obtain the travelling wave solutions of (4.3), we suppose that
\[
g = G(z), \quad h = H(z), \quad z = x + ar. \tag{4.4}
\]
Substituting (4.4) into (4.3), we obtain

\[ 1 + (\lambda H - 2aG)G' + (4\mu a^2 + \lambda \mu) G'' = 0, \quad (4.5a) \]
\[ 2aH' + G' = 0. \quad (4.5b) \]

Integrating (4.5b) with respect to \( z \), one finds

\[ H = -\frac{1}{2a} G + c_1. \quad (4.6) \]

Let us substitute (4.6) into (4.5a), to get

\[ 1 - \left( \frac{\lambda}{2a} + 2a \right) GG' + (4\mu a^2 + \lambda \mu) G'' + c_1 \lambda G' = 0. \quad (4.7) \]

By integrating (4.7) with respect to \( z \), we find

\[ z - \left( \frac{\lambda}{2a} + 2a \right) \frac{G^2}{2} + (4\mu a^2 + \lambda \mu) G' + \lambda c_1 G = c_2, \quad (4.8) \]

where \( c_1 \) and \( c_2 \) are integration constants. The solution of (4.8) is given by

\[ G(z) = \frac{C}{2A} - \left( \frac{B}{A^2} \right)^{\frac{1}{3}} \frac{B i'(k)}{B i(k)} + c_3 A i'(k) \quad (4.9) \]

where \( A = \frac{\lambda}{4a} + a, \quad B = 4\mu a^2 + \lambda \mu, \quad C = \lambda c_1, \quad k = \frac{1}{4(AB)^{\frac{1}{3}}} (C^2 - 4Ac_2 + 4Az), \)

\( c_3 \) is the integration constant, \( A i(k) \) and \( B i(k) \) are the Airy functions defined by [29]

\[ A i(k) = \frac{1}{\pi} \int_0^{\infty} \cos \left( \frac{1}{3} t^3 + kt \right) \, dt, \]
\[ B i(k) = \frac{1}{\pi} \int_0^{\infty} \left[ e^{-\frac{1}{3} t^3 + kt} + \sin \left( \frac{1}{3} t^3 + kt \right) \right] \, dt. \quad (4.10) \]

In this case, the solution of (1.1) is given by

\[ u(x, y, t) = -t + \frac{C}{2A} - \left( \frac{B}{A^2} \right)^{\frac{1}{3}} \frac{B i'(k) + c_3 A i'(k)}{B i(k) + c_3 A i(k)}, \]
\[ v(x, y, t) = -\frac{a}{2b} \left( \frac{C}{2A} - \left( \frac{B}{A^2} \right)^{\frac{1}{3}} \frac{B i'(k) + c_3 A i'(k)}{B i(k) + c_3 A i(k)} \right) + c_1. \quad (4.11) \]
4.2. Exact solution using the generator $a_2 v_2 + v_5$

In this subsection, we consider the subalgebra $a_2 v_2 + v_5$ and take $a_2 = 0$. In this case, the invariant surface conditions are given by

$$
\begin{align*}
t x u_x + t y u_y + t^2 u_t &= -(t u + y), \\
t x v_x + t y v_y + t^2 v_t &= -(t v + \frac{x}{\lambda}).
\end{align*}
$$

(4.12)

Solving (4.12), we find

$$
\begin{align*}
u &= -\frac{r}{s} + \frac{1}{x} g(r, s), \\
v &= -\frac{1}{\lambda s} + \frac{1}{x} h(r, s),
\end{align*}
$$

(4.13)

where $r = \frac{y}{x}$, $s = \frac{t}{x}$. Substituting (4.13) into (1.1), we get

$$
\begin{align*}
g(2\lambda \mu - \lambda h + g_r) + \lambda \left[(4\mu - h)(sg_s + rg_r) + \mu \left(s^2 g_{ss} + r^2 g_{rr} + 2rsg_{rs}\right)\right] + \mu g_{rr} &= 0, \\
sg_s + rg_r + g + h_r &= 0.
\end{align*}
$$

(4.14)

To obtain the travelling wave solutions of (4.14), we assume that

$$
h = H(z), \quad g = G(z), \quad z = r + s.
$$

(4.15)

Substituting (4.15) into (4.14), we find

$$
\begin{align*}
G(2\lambda \mu - \lambda H + G') + \lambda \left[(4\mu - H)zG' + \mu z^2 G''\right] + \mu G'' &= 0, \\
zG' + G + H' &= 0,
\end{align*}
$$

(4.16a)

(4.16b)

where $G'$, $G''$, $H'$ and $H''$ are the derivative with respect to $z$. By integrating (4.16b) with respect to $z$ and considering the integration constant to be zero, we obtain

$$
H = -zG.
$$

(4.17)

Let us substitute (4.17) into (4.16a), to obtain

$$
2\lambda \mu \left[G + zG'\right] + \lambda \mu \left[2zG' + z^2 G''\right] + \lambda \left[zG^2 + z^2 GG'\right] + GG' + \mu G'' = 0.
$$

(4.18)

By integrating (4.18) with respect to $z$ and considering the integration constant to be zero, we find

$$
2\lambda \mu zG + \lambda \mu z^2 G' + \frac{\lambda}{2} z^2 G^2 + \frac{1}{2} G^2 + \mu G' = 0.
$$

(4.19)
Equation (4.19) has the following solution:

\[ G(z) = \frac{2\mu\sqrt{\lambda}}{(1 + \lambda z^2) \left(2c_1\mu\sqrt{\lambda} + \tan^{-1}\sqrt{\lambda}z\right)}, \quad (4.20) \]

where \(c_1\) is the integration constant. Finally, the solution of (1.1) is given by

\[
\begin{align*}
  u(x, y, t) &= -\frac{y}{t} + \frac{2\mu\sqrt{\lambda}}{x \left(1 + \lambda z^2\right) \left(2c_1\mu\sqrt{\lambda} + \tan^{-1}\sqrt{\lambda}z\right)}, \\
  v(x, y, t) &= -\frac{x}{\lambda t} - \frac{2\mu\sqrt{\lambda}z}{x \left(1 + \lambda z^2\right) \left(2c_1\mu\sqrt{\lambda} + \tan^{-1}\sqrt{\lambda}z\right)}. \quad (4.21)
\end{align*}
\]

Now, we are interested in the infinite-dimensional subalgebra that is ignored in optimal system calculations.

4.3. Exact solution using the generator \(v_f\)

The invariant surface conditions, in this case, are given by

\[
\begin{align*}
  F(t) u_x &= 0, \\
  F(t) v_x &= -\frac{F'(t)}{\lambda}.
\end{align*}
\]

Solving (4.22), we find

\[
\begin{align*}
  u &= g(y, t), \\
  v &= -\frac{F'(t)}{\lambda F(t)}x + h(y, t).
\end{align*}
\]

Substituting (4.23) into (1.1), we get

\[
\begin{align*}
  g_t &= gg_y + \mu g_{yy}, \quad (4.24a) \\
  h_y &= 0. \quad (4.24b)
\end{align*}
\]

Equation (4.24a) is the one-dimensional Burgers equation. This equation has many famous solutions (see, for example, [28]) and we will not list them here.
4.4. Exact solution using a linear combination of $v_f, v_1, v_2$

In this case, the invariant surface conditions are given by

\[
F(t)u_x + u_y + u_t = 0, \\
F(t)v_x + v_y + v_t = -\frac{F'(t)}{\lambda}.
\] (4.25)

Solving (4.25), we obtain

\[
u = g(r, s), \\
v = -\frac{F(t)}{\lambda} + h(r, s),
\] (4.26)

where $r = -y + t$ and $s = -\int F(t) \, dt + x$. Substituting (4.26) into (1.1), we get

\[
\lambda hg_s + \lambda g_{ss} - (1 + g) g_r + \mu g_{rr} = 0, \\
g_s + h_r = 0.
\] (4.27)

To obtain exact solutions of (4.27), we apply the Lie symmetry analysis to it. In this case, we obtain the following infinitesimal generators:

\[
\Gamma_1 = -(g + 1) \partial_g - h \partial_h + r \partial_r + s \partial_s, \\
\Gamma_2 = \partial_r, \\
\Gamma_3 = \partial_s.
\] (4.28)

We use these generators in the following subsection to obtain exact solutions of (4.27).

4.4.1. The infinitesimal generator $\Gamma_1$ of (4.27)

The solution of the invariant surface conditions, in this case, are given by

\[
g(r, s) = -1 + \frac{1}{r} G(z), \\
h(r, s) = \frac{1}{r} H(z),
\] (4.29)

where $z = \frac{s}{r}$. Substituting (4.29) into (4.27), we find

\[
G(2\mu + G) + (4\mu z + zG + \lambda H)G' + \mu(z^2 + \lambda)G'' = 0, \quad (4.30a) \\
-H + G' - zH' = 0. \quad (4.30b)
\]
Integrate (4.30b) and consider the constant of integration equal \(-\mu\) to get

\[ H = \frac{G + \mu}{z} . \]  

(4.31)

Substituting (4.31) into (4.30a), we obtain

\[
(2\mu z G + \mu z^2 G') + \frac{1}{2} (2z G^2 + 2z^2 G G') + (3\mu z^3 G' + \mu z^2 G'') + (\mu \lambda G' + \mu \lambda G'') + \lambda G G' = 0 . \]  

(4.32)

By integrating (4.32) and considering the integration constant to be zero, we find

\[
\mu z^2 G + \frac{1}{2} z^2 G^2 + \mu z^3 G' + \mu \lambda z G' + \frac{\lambda}{2} G^2 = 0 . \]  

(4.33)

Equation (4.33) has the following solution:

\[
G(z) = \frac{2\mu \sqrt{\lambda}}{(\sqrt{z^2 + \lambda}) \left(2\mu c_1 \sqrt{\lambda} + \ln \frac{z}{\lambda + \sqrt{\lambda z^2 + \lambda^2}}\right)} , \]  

(4.34)

where \(c_1\) is the integration constant. In this case, the solution of (1.1) is given by

\[
u(x, y, t) = -1 + \frac{2\mu \sqrt{\lambda}}{r \left(\sqrt{z^2 + \lambda}\right) \left(2\mu c_1 \sqrt{\lambda} + \ln \frac{z}{\lambda + \sqrt{\lambda z^2 + \lambda^2}}\right)} ,
\]

\[
v(x, y, t) = -\frac{F(t)}{\lambda} \left(\sqrt{z^2 + \lambda}\right) \left(2\mu c_1 \sqrt{\lambda} + \ln \frac{z}{\lambda + \sqrt{\lambda z^2 + \lambda^2}}\right) + \frac{\mu}{s} . \]  

(4.35)

4.4.2. Travelling wave infinitesimal generator of (4.27)

In this subsection, we consider a linear combination of \(\Gamma_2\) and \(\Gamma_3\). In this case, the solution of the invariant surface conditions is given by

\[
 h = H(z) , \quad g = G(z) , \quad z = r + as . \]  

(4.36)

Substituting (4.36) into (4.27), we find

\[
\lambda a H G' + (\lambda a^2 + \mu) G'' - (1 + G) G' = 0 , \]  

(4.37a)

\[
a G' + H' = 0 . \]  

(4.37b)

Integrating (4.37b) with respect to \(z\), we get

\[
H = -a G + c_1 , \]  

(4.38)
where \( c_1 \) is the integration constant. Substituting (4.38) into (4.37a), we obtain
\[
- (\lambda a^2 + 1) GG' + \mu (\lambda a^2 + 1) G'' + (\lambda ac_1 - 1) G' = 0.
\] (4.39)
Let us integrate (4.39) with respect to \( z \), to obtain
\[
G' = A + BG + \frac{1}{2\mu} G^2,
\] (4.40)
where \( A = \frac{c_2}{\mu(\lambda a^2 + 1)} \) and \( B = \frac{1 - \lambda ac_1}{\mu(\lambda a^2 + 1)} \). Equation (4.40) is the Riccati equation that has the following solutions:

— Case 1: the first solution is given by
\[
G(z) = -B\mu - \sqrt{\mu} \sqrt{B^2\mu - 2A} \\
\times \tanh \left( \frac{\sqrt{B^2\mu - 2A}}{2\sqrt{\mu}} z + c_3\sqrt{\mu} \sqrt{B^2\mu - 2A} \right),
\] (4.41)
and the corresponding solution of (1.1) in this case is given by
\[
u(x, y, t) = -\frac{F(t)}{\lambda} + aB\mu + a\sqrt{\mu} \sqrt{B^2\mu - 2A} \\
\times \tanh \left( \frac{\sqrt{B^2\mu - 2A}}{2\sqrt{\mu}} z + c_3\sqrt{\mu} \sqrt{B^2\mu - 2A} \right) + c_1,
\] (4.42)
where \( c_2 \) and \( c_3 \) are arbitrary constants.

— Case 2: The second solution is given by (when \( A = B = 0 \) or \( c_2 = 0 \) and \( c_1 = \frac{1}{a\lambda} \))
\[
G(z) = -\frac{2\mu}{z + c_3},
\] (4.43)
and the exact solution of (1.1) in this case is given by
\[
u(x, y, t) = -\frac{F(t)}{\lambda} + \frac{2a\mu}{z + c_3} + \frac{1}{a\lambda},
\] (4.44)
where \( c_1, c_2 \) and \( c_3 \) are the integration constants.
5. Conclusion

In this paper, by using symmetry analysis method, the Lie algebra of infinitesimal symmetry generators spanned by five-dimensional and infinite-dimensional subalgebra is produced. The optimal system of the five-dimensional subalgebras is computed. These generators are applied to obtain some reduced equations and the exact solutions of the reduced equations are obtained. We get some new exact similarity solutions of Eq. (1.1) in the form of the Airy function (Eq. (4.11)), arctan function (Eq. (4.21)) and logarithmic function (Eq. (4.35)).

REFERENCES