HIGGS DECAY, Z DECAY AND THE QCD BETA FUNCTION*

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Recent developments in perturbative QCD leading to the beta function in five-loop approximation are presented. In a first step, the two most important decay modes of the Higgs boson are discussed: decays into a pair of gluons and, alternatively, decays into a bottom–antibottom quark pair. Subsequently, the quark mass anomalous dimension is presented which is important for predicting the value of the bottom-quark mass at high scales and, consequently, the Higgs boson decay rate into a pair of massive quarks, in particular into $b\bar{b}$. In the next section, the $\alpha_{\rm s}^4$ corrections to the vector and axial-vector correlators are discussed. These are the essential ingredients for the evaluation of the QCD corrections to the cross section for electron–positron annihilation into hadrons at low and at high energies, to the hadronic decay rate of the τ lepton and for the Z-boson decay rate into hadrons. Finally, we present the prediction for the QCD beta function in five-loop approximation, discuss the analytic structure of the result and compare with experiment at low and at high energies.

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1. Introduction

During the past years, significant progress has been made in the evaluation of higher order QCD corrections to inclusive decay rates. Some of the basic tools of these calculations were formulated already long time ago (see, *e.g.* early reviews [1, 2]).

However, steady progress has been also achieved more recently, pushing *e.g.* the evaluation of QCD corrections to scalar- and vector-current correlators to $\mathcal{O}(\alpha_s^4)$ and, correspondingly, the evaluation of decay rates of scalar and vector particles to the same order [3–5]. Also, along the same line, the evaluation of the QCD beta function has been pushed to the fifth order [6] and, indeed, also this result has been confirmed (and extended to a generic gauge group) by three new, independent calculations [7–9].

2. Dominant Higgs boson decay modes

The two most important decay modes of the Higgs boson are the top quark mediated decay channel into two gluons and the decay into a bottom plus antibottom quark (for a recent review, see [10]). The higher order corrections to these modes have been evaluated up to $\mathcal{O}(\alpha_s^5)$ [11, 12] for the two-gluon channel (very recently even up to $\mathcal{O}(\alpha_s^6)$ [5]) and up to $\mathcal{O}(\alpha_s^4)$ for the $b\bar{b}$ channel [3, 5]. Mixed terms related to the gg and the $b\bar{b}$ mode are treated in [13]. These two modes constitute the dominant Higgs decay channels with branching ratios around 15% for the two-gluon and close to 60% for the $b\bar{b}$ mode.

2.1. Higgs decay into two gluons

Let us start with the two-gluon channel. In the limit $m_t \to \infty$, the part of the effective Lagrangian which determines the coupling of the Higgs boson to gluons is given by

$$\mathcal{L}_{\rm eff} = -2^{1/4} G_{\rm F}^{1/2} H C_1 \left[O_1'\right] . \tag{1}$$

Here, $[O_1']$ is the renormalized counterpart of the bare operator

$$O_1' = G^{0\prime}_{a\mu\nu} G_a^{0\prime\mu\nu} ,$$

with $G_{a\mu\nu}$ standing for the colour field strength. The superscript 0 denotes bare fields, and primed objects refer to the five-flavour effective theory. C_1 stands for the corresponding renormalized coefficient function, which carries all M_t dependence. $\mathcal{L}_{\rm eff}$ thus effectively counts the number of heavy-quark species, which in the Standard Model is restricted to the top quark. In the

Born approximation [14]

$$\Gamma_{\mathrm{Born}}(H \to gg) = \frac{G_{\mathrm{F}} M_H^3}{36\pi\sqrt{2}} \left(\frac{\alpha_{\mathrm{s}}^{(n_l)}(M_H)}{\pi} \right)^2 . \tag{2}$$

The leading order result, being proportional to α_{s}^2, exhibits a strong scale dependence which demonstrates the need for higher order corrections. Over the years subsequently higher orders have been calculated, from NLO through N^{2}LO [15] up to N^{3}LO [11, 12]. Quite recently, even the N^{4}LO corrections became available [5]. (Power-suppressed corrections of the order of $(m_H/M_t)^n$ with $n \leq 5$ were calculated up to N^{2}LO and can be found in the literature [16, 17].)

After a drastic increase of the cross section by about 60% from the NLO corrections, the N^{2}LO terms lead to a further increase of the decay rate by about 20%. This was the motivation for the evaluation of the N^{3}LO terms. Using the optical theorem, the decay rate can be cast into the form of

$$\Gamma(H \to gg) = \frac{\sqrt{2} G_{\mathrm{F}}}{M_H} C_1^2 \mathrm{Im} \Pi^{GG} \left(q^2 = M_H^2 \right) , \tag{3}$$

where

$$\Pi^{GG}\left(q^2\right) = \int e^{iqx} \langle 0 | T \left(\left[O_1' \right](x) \left[O_1' \right](0) \right) | 0 \rangle \mathrm{d}x \,. \tag{4}$$

The combination $[O_1']$ denotes the renormalized counterpart of the bare operator $O_1' = G^{0'}_{a\mu\nu} G^{0'\mu\nu}_a$ and has been introduced above. The normalization C_1 is known to order N^{3}O from massive tadpoles [18].

In total, one finds

$$\Gamma(H \to gg) = \Gamma_{\mathrm{Born}}(H \to gg) \times K \tag{5}$$

with

$$K = 1 + 17.9167\, a_{\mathrm{s}}' + 152.5 \left(a_{\mathrm{s}}' \right)^2 + 381.5 \left(a_{\mathrm{s}}' \right)^3 . \tag{6}$$

Here $a_{\mathrm{s}}' = \alpha_{\mathrm{s}}/\pi$. It is quite remarkable that the residual scale dependence is reduced quite drastically, from $\pm 24\%$ in LO to $\pm 22\%$ in NLO down to $\pm 10\%$ in N^{2}LO and $\pm 3\%$ in N^{3}LO.

2.2. Higgs decay into bottom quarks

The second and, in fact, dominant decay mode of the Higgs boson is the $b\bar{b}$ channel. The decay rate into a quark–antiquark pair, generically denoted by $f\bar{f}$, is given by

$$\Gamma \left(H \to \bar{f} f \right) = \frac{G_{\mathrm{F}}\, M_H}{4\sqrt{2}\pi} m_f^2 \tilde{R} \left(s = M_H^2 \right) , \tag{7}$$

where $\tilde{R}(s) = \mathrm{Im}\tilde{\Pi}(-s-i\epsilon)/(2\pi s)$ stands for the absorptive part of the scalar two-point correlator

$$\widetilde{\Pi}\left(Q^2\right) = (4\pi)^2 i \int \mathrm{d}x e^{iqx} \langle 0|T\left[J_f^{\mathrm{S}}(x) J_f^{\mathrm{S}}(0)\right]|0\rangle \,. \tag{8}$$

This five-loop result has been obtained in [3] and recently confirmed in [5, 8, 9]. Strong cancellations are evident between "kinematical terms", originating from the analytical transition from spacelike to timelike arguments, and "dynamical terms", intrinsic for the calculation in the timelike region. In total, one finds

$$\begin{aligned}\widetilde{R} \;=\; & 1 + 5.667\, a_{\mathrm{s}} + a_{\mathrm{s}}^2\left[51.57 - \underline{15.63} - n_{\mathrm{f}}(1.907 - \underline{0.548})\right] \\ & + a_{\mathrm{s}}^3\left[648.7 - \underline{484.6} - n_{\mathrm{f}}(63.74 - \underline{37.97}) + n_{\mathrm{f}}^2(0.929 - \underline{0.67})\right] \\ & + a_{\mathrm{s}}^4\left[9470.8 - \underline{9431.4} - n_{\mathrm{f}}(1454.3 - \underline{1233.4})\right. \\ & \left. + n_{\mathrm{f}}^2(54.78 - \underline{45.10}) - n_{\mathrm{f}}^3(0.454 - \underline{0.433})\right]\,,\end{aligned} \tag{9}$$

where the underlined terms originate from the analytic continuation from the spacelike to the timelike region. Evidently, the inclusion of the π^2 terms from higher orders alone does not improve the quality of the result. In total, remarkable cancellations are observed between "kinematical" and "dynamical" terms, leading to a nicely "convergent" answer. For $n_{\mathrm{f}} = 5$, the physically relevant result is given by

$$\begin{aligned}\tilde{R}\left(s = M_H^2, \mu = M_H\right) \;&=\; 1 + 5.667\, a_{\mathrm{s}} + 29.147\, a_{\mathrm{s}}^2 + 41.758\, a_{\mathrm{s}}^3 - 825.7\, a_{\mathrm{s}}^4 \\ &=\; 1 + 0.2040 + 0.0378 + 0.0019 - 0.00139\,.\end{aligned} \tag{10}$$

In the last equation, we have substituted $a_{\mathrm{s}}(m_H) = \alpha_{\mathrm{s}}/\pi = 0.0360$, valid for a Higgs mass of 125 GeV and $\alpha_{\mathrm{s}}(M_Z) = 0.118$. The nearly complete compensation between $\mathcal{O}(\alpha_{\mathrm{s}}^3)$ and $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ term may be interpreted as a consequence of an accidentally small coefficient of the α_{s}^3 term.

In total, this leads to a dominant contribution of the $b\bar{b}$ mode with a branching ratio close to 60%. Note that an important ingredient in this context is the mass of the bottom quark at the scale of m_H, which has been taken as [19]

$$m_b(m_H) = 2771 \pm 8|_{m_b} \pm 15|_{\alpha_{\mathrm{s}}}\ \mathrm{MeV}\,. \tag{11}$$

Let us mention in passing that, in order α_{s}^4, there are also interference corrections resulting from mixed terms between $H \to gg$ and $H \to b\bar{b}$ which have been evaluated in [13].

3. Quark mass anomalous dimension

It is well-known that quark masses are conveniently defined to depend on a renormalization scale

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} m|_{g^0,m^0} = m\gamma_m(a_\mathrm{s}) \equiv -m \sum_{i\geq 0} \gamma_i \, a_\mathrm{s}^{i+1} \,, \tag{12}$$

with $a_\mathrm{s} = \alpha_\mathrm{s}/\pi$ and the coefficients γ_i of the quark mass anomalous dimension γ_m are known from γ_0 to γ_4 and thus in five-loop order [20]. (At lower orders, the γ_m was computed in [21–25].) In numerical form and for SU(3), it is given by

$$\begin{aligned}\gamma_m = & -a_\mathrm{s} - a_\mathrm{s}^2\,(4.20833 - 0.138889 n_\mathrm{f}) \\ & -a_\mathrm{s}^3\left(19.5156 - 2.28412 n_\mathrm{f} - 0.0270062 n_\mathrm{f}^2\right) \\ & -a_\mathrm{s}^4\left(98.9434 - 19.1075 n_\mathrm{f} + 0.276163 n_\mathrm{f}^2 + 0.00579322 n_\mathrm{f}^3\right) \\ & -a_\mathrm{s}^5\left(559.7069 - 143.6864\, n_\mathrm{f} + 7.4824\, n_\mathrm{f}^2 + 0.1083\, n_\mathrm{f}^3 - 0.000085359\, n_\mathrm{f}^4\right)\end{aligned} \tag{13}$$

and, thus,

$$\begin{aligned}\gamma_m & \underset{n_\mathrm{f}=3}{=\!=} -a_\mathrm{s} - 3.79167\, a_\mathrm{s}^2 - 12.4202\, a_\mathrm{s}^3 - 44.2629\, a_\mathrm{s}^4 - 198.907\, a_\mathrm{s}^5 \,, \\ \gamma_m & \underset{n_\mathrm{f}=4}{=\!=} -a_\mathrm{s} - 3.65278\, a_\mathrm{s}^2 - 9.94704\, a_\mathrm{s}^3 - 27.3029\, a_\mathrm{s}^4 - 111.59\, a_\mathrm{s}^5 \,, \\ \gamma_m & \underset{n_\mathrm{f}=5}{=\!=} -a_\mathrm{s} - 3.51389\, a_\mathrm{s}^2 - 7.41986\, a_\mathrm{s}^3 - 11.0343\, a_\mathrm{s}^4 - 41.8205\, a_\mathrm{s}^5 \,, \\ \gamma_m & \underset{n_\mathrm{f}=6}{=\!=} -a_\mathrm{s} - 3.37500\, a_\mathrm{s}^2 - 4.83867\, a_\mathrm{s}^3 + 4.50817\, a_\mathrm{s}^4 + 9.76016\, a_\mathrm{s}^5 \,.\end{aligned}$$

Note the significant cancellations between the contributions for n_f^0 and n_f^1 for values of n_f around 4 and 5 which are clearly visible for the four-loop result and persist in five-loop order. This leads to a moderate growth of the series, even for scales as small as 2 GeV, where $a_\mathrm{s} \equiv \alpha_\mathrm{s}/\pi \approx 0.1$. The strong cancellations between different powers of n_f have been anticipated by predictions based on "Asymptotic Padé Approximants" [26–28], the numerical value of the result, however, differs significantly (see Table I).

Let us note in passing that quite recently the result for a general gauge group has been obtained [29, 30].

TABLE I

The exact results for $(\gamma_m)_4$ together with the predictions made with the help of the original APAP method and its two somewhat modified versions.

n_f	3	4	5	6
$(\gamma_m)_4^{\mathrm{exact}}$	198.899	111.579	41.807	−9.777
$(\gamma_m)_4^{\mathrm{APAP}}$ [26]	162.0	67.1	−13.7	−80.0
$(\gamma_m)_4^{\mathrm{APAP}}$ [27]	163.0	75.2	12.6	12.2
$(\gamma_m)_4^{\mathrm{APAP}}$ [28]	164.0	71.6	−4.8	−64.6

4. Z decay in $\mathcal{O}(\alpha_\mathrm{s}^4)$

In view of asymptotic freedom, perturbative QCD can be applied at vastly different energy scales, despite the dramatic variation of the strong coupling between the mass of the τ lepton and, for example, the mass of the Z boson. Starting, for example, at the scale of the τ lepton with

$$\alpha_\mathrm{s}(m_\tau) = 0.332 \pm 0.005|_{\mathrm{exp}} \pm 0.015|_{\mathrm{th}} \tag{14}$$

four loop running and matching at the flavour thresholds lead to the reduction of the strong coupling at the scale of the Z-boson mass [4]

$$\alpha_\mathrm{s}(M_Z) = 0.1202 \pm 0.006|_{\mathrm{exp}} \pm 0.0018|_{\mathrm{th}} \pm 0.0003|_{\mathrm{evol}} \tag{15}$$

by a factor three and a reduction of the uncertainties by nearly a factor ten. In this case, the evolution error receives contributions from uncertainties in the charm- and bottom-quark mass, the variation of the matching scale and the four-loop truncation of the renormalization group equation. The final result is in remarkable agreement with the direct determination of α_s from Z decays which leads to $\alpha_\mathrm{s} = 0.1190 \pm 0.0026|_{\mathrm{exp}}$ and a small theory error. Note that the dominant term in the QCD corrections for Z decays is identical to the correction term for τ decays. However, starting from $\mathcal{O}(\alpha_\mathrm{s}^2)$, one receives additional, new terms in the Z boson case. These arise from the so-called singlet contributions which, in turn, are different for the vector and the axial-vector part.

In total, one finds for the QCD corrected decay rate of the Z boson (neglecting for the moment mass suppressed terms of $\mathrm{O}(m_b^2/M_Z^2)$ and electroweak corrections)

$$R^{\mathrm{nc}} = 3\left[\sum_f v_f^2 r_{\mathrm{NS}}^{\mathrm{V}} + \left(\sum_f v_f\right)^2 r_{\mathrm{S}}^{\mathrm{V}} + \sum_f a_f^2 r_{\mathrm{NS}}^{\mathrm{A}} + r_{\mathrm{S};t,b}^{\mathrm{A}}\right]. \tag{16}$$

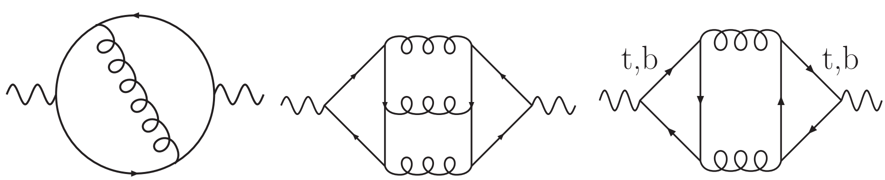
The relative importance of the different terms (Fig. 1) is best seen from the results of the various r-ratios introduced above. In the numerical form [31]

$$\begin{aligned} r^{\mathrm{V}}_{\mathrm{NS}} = r^{\mathrm{A}}_{\mathrm{NS}} &= 1 + a_{\mathrm{s}} + 1.4092\, a_{\mathrm{s}}^2 - 12.7671\, a_{\mathrm{s}}^3 - 79.9806\, a_{\mathrm{s}}^4 \,, \\ r^{\mathrm{V}}_{\mathrm{S}} &= -0.4132\, a_{\mathrm{s}}^3 - 4.9841\, a_{\mathrm{s}}^4 \,, \\ r^{\mathrm{A}}_{\mathrm{S}:t,b} &= (-3.0833 + l_t)\, a_{\mathrm{s}}^2 + \left(-15.9877 + 3.7222\, l_t + 1.9167\, l_t^2\right)\, a_{\mathrm{s}}^3 \\ &\quad + \left(49.0309 - 17.6637\, l_t + 14.6597\, l_t^2 + 3.6736\, l_t^3\right)\, a_{\mathrm{s}}^4 \,, \end{aligned} \tag{17}$$

with $a_{\mathrm{s}} = \alpha_{\mathrm{s}}(M_Z)/\pi$ and $l_t = \ln(M_Z^2/M_t^2)$. Using for the pole mass M_t the value 172 GeV, the axial singlet contribution in numerical form is given by

$$r^{\mathrm{A}}_{\mathrm{S};t,b} = -4.3524\, a_{\mathrm{s}}^2 - 17.6245\, a_{\mathrm{s}}^3 + 87.5520\, a_{\mathrm{s}}^4 \,. \tag{18}$$

The significant decrease of the scale dependence is evident from Fig. 2.



(a) (b) (c)

Fig. 1. Different contributions to r-ratios: (a) non-singlet, (b) vector singlet and (c) axial vector singlet.

Let us recall the basic aspects of these results:

— The non-singlet term dominates all different channels. It starts in the Born approximation and is identical for τ decay, for $\sigma(e^+e^- \to$ hadrons) through the vector current (virtual photon) and for $\Gamma(Z \to$ hadrons) through vector and axial current.

— The singlet axial term starts in order α_{s}^2, is present in $Z \to$ hadrons and depends on $\ln M_Z^2/M_t^2$. Its origin is the strong imbalance between the masses of top and bottom quark [32].

— The singlet vector term is present both in $\gamma^* \to$ hadrons and $Z \to$ hadrons and starts in $\mathcal{O}(\alpha_{\mathrm{s}}^3)$.

— All three terms are known up to order α_{s}^4 and the total rate is remarkably stable under scale variations.

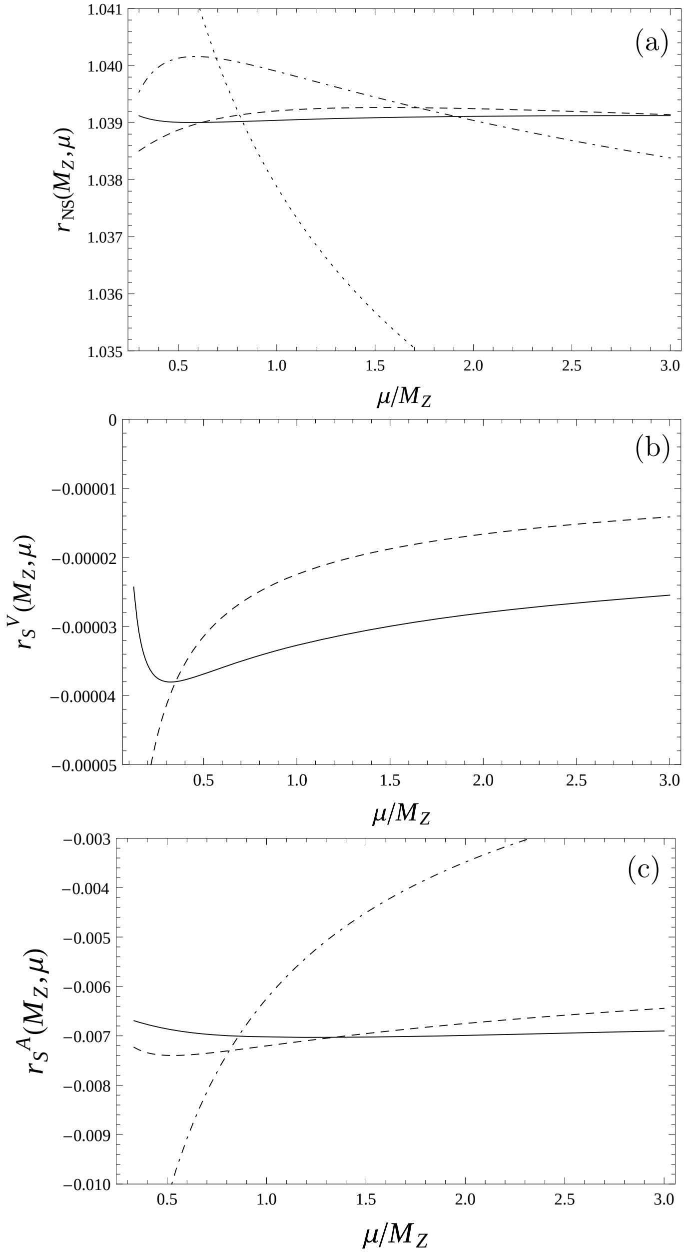


Fig. 2. Scale dependence of (a) non-singlet r_{NS}; (b) vector singlet $r_{\mathrm{S}}^{\mathrm{V}}$ and (c) axial vector singlet $r_{\mathrm{S};t,b}^{\mathrm{A}}$. Dotted, dash-dotted, dashed and solid curves refer to $\mathcal{O}(\alpha_{\mathrm{s}})$ up to $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ predictions. $\alpha_{\mathrm{s}}(M_Z) = 0.1190$ and $n_l = 5$ is adopted in all these curves.

5. Five-loop β function

Asymptotic freedom, manifest by a decreasing coupling with increasing energy, can be considered as the basic prediction of the non-Abelian gauge theories and was crucial for establishing QCD as the theory of strong interactions. The dominant, leading order prediction [33, 34] was quickly followed by the corresponding two- [35, 36] and three-loop [23, 37] results. Subsequently, it took more than 15 years until the four-loop result was evaluated [38] and another seven years until this result was confirmed by an independent calculation [39]. Now, finally, the five-loop result for QCD became available [6], quickly confirmed and generalized to an arbitrary gauge group [7–9].

There are several reasons to push the QCD β function to an order as high as possible. From the practical side, it is important to compare experiment and theory prediction with the best achievable precision. From the theoretical side, one expects that the perturbative series at some point starts to demonstrate its asymptotic divergence, shown by significantly increasing terms. However, as shown below, even up to the fifth order, the series exhibits a remarkably smooth behaviour with continuously decreasing perturbative coefficients. Let us, in a first step, recall the coefficients of the QCD β function defined by

$$\beta(a_{\mathrm{s}}) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} a_{\mathrm{s}}(\mu) = -\sum_{i\geq 0} \beta_i a_{\mathrm{s}}^{i+2} \,. \tag{19}$$

Using the same tools as those discussed in [4, 20], the β function in fifth order is given by

$$\begin{aligned}
\beta_0 &= \frac{1}{4}\left\{11 - \frac{2}{3}n_{\mathrm{f}},\right\}, \qquad \beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_{\mathrm{f}}\right\},\\
\beta_2 &= \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_{\mathrm{f}} + \frac{325}{54}n_{\mathrm{f}}^2\right\},\\
\beta_3 &= \frac{1}{4^4}\left\{\frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right]n_{\mathrm{f}}\right.\\
&\quad \left. + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right]n_{\mathrm{f}}^2 + \frac{1093}{729}n_{\mathrm{f}}^3\right\},\\
\beta_4 &= \frac{1}{4^5}\left\{\frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5\right.\\
&\quad +n_{\mathrm{f}}\left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right]\\
&\quad +n_{\mathrm{f}}^2\left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right]
\end{aligned}$$

$$+n_{\rm f}^3\left[-\frac{630559}{5832}-\frac{48722}{243}\zeta_3+\frac{1618}{27}\zeta_4+\frac{460}{9}\zeta_5\right]$$
$$+n_{\rm f}^4\left[\frac{1205}{2916}-\frac{152}{81}\zeta_3\right]\Bigg\}\,.$$

This result has, in the meantime, been confirmed in [7–9] and even extended to an arbitrary, simple, compact Lie group. The surprising pattern of the delayed appearance of higher transcendentals, already observed in lower orders, repeats itself in the present case: The transcendental numbers ζ_6 and ζ_7 that could be present in β_4 in principle, are evidently absent, similarly to the absence of ζ_4 and ζ_5 in the result for β_3.

Let us reemphasize the surprising smallness of the perturbative coefficients, characterized by the small deviations from the leading order result. Consider the ratio $\bar\beta\equiv\frac{\beta}{-\beta_0 a_{\rm s}^2}=1+\sum_{i\ge1}\bar\beta_i a_{\rm s}^i$ for two characteristic values of $n_{\rm f}$:

$$\begin{aligned}\bar\beta(n_{\rm f}=4) &= 1+1.54\,a_{\rm s}+3.05\,a_{\rm s}^2+15.07\,a_{\rm s}^3+27.33\,a_{\rm s}^4\,,\\ \bar\beta(n_{\rm f}=5) &= 1+1.26\,a_{\rm s}+1.47\,a_{\rm s}^2+9.83\,a_{\rm s}^3+7.88\,a_{\rm s}^4\,.\end{aligned}$$

Indeed, an extremely modest growth of the perturbative coefficients is observed. Remarkably enough, the rough pattern of the coefficients is indeed in qualitative agreement with the expectations for the $n_{\rm f}$ dependence of β_4 based on the method of "Asymptotic Padé Approximant" [26] (the boxed term was used as an input):

$$\begin{aligned}\beta_4^{\rm APAP} &\approx 740-213\,n_{\rm f}+20\,n_{\rm f}^2-0.0486\,n_{\rm f}^3-\boxed{0.001799\,n_{\rm f}^4}\,,\\ \beta_4^{\rm exact} &\approx 524.56-181.8\,n_{\rm f}+17.16\,n_{\rm f}^2-0.22586\,n_{\rm f}^3-0.001799\,n_{\rm f}^4\,.\end{aligned}$$

However, large cancellations occur for $n_{\rm f}=3,4,5$, leading to drastic disagreement for the final predicitions for the corresponding values of β_4.

As stated before, the smallness of the higher order coefficients, in particular for the $n_{\rm f}$ values of interest, leads to a remarkable stabilization of the results. The excellent agreement between $\alpha_{\rm s}$ values from vastly different energy scales indeed persists in higher orders. Let us, as a typical example, recall the comparison between the strong coupling at the scales of m_τ and M_Z. Starting with the value $\alpha_{\rm s}(m_\tau)=0.33\pm0.014$, one arrives, after running and matching at the charm and bottom threshold at the value $\alpha_{\rm s}^{(5)}=0.1198\pm0.0015$. From the direct measurement of Z-boson decays combined in the electroweak precision data, on the other hand, one obtains the result $\alpha_{\rm s}^{(5)}=0.1197\pm0.0028$ in remarkable agreement with the previous value.

6. Summary

A sizable number of four- and five-loop QCD results has been evaluated during the past years. $\mathcal{O}(\alpha_s^4)$ corrections of Higgs boson decays to fermions, of τ-lepton decays to hadrons, Z decays to hadrons and of corrections to the familiar R ratio (with $R \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$) are among the most prominent examples. These calculations have been complemented by the most recent result along the same lines, the five-loop QCD β function. No sign of an onset of the asymptotically expected divergence of the series is observed. Excellent agreement between theory and experiment for a large number of predictions is observed. For the moment, the precision of the theoretical prediction is significantly ahead of the experimental results.

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