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# SCALAR ONE-LOOP VERTEX INTEGRALS AS MEROMORPHIC FUNCTIONS OF SPACE-TIME DIMENSION $d$ *

Johannes Blümlein[a], Khiem Hong Phan[a,b], Tord Riemann[c,a,†]

[a]Deutsches Elektronen-Synchrotron DESY
Platanenallee 6, 15738 Zeuthen, Germany
[b]University of Science, Vietnam National University
227 Nguyen Van Cu, Dist.5, Ho Chi Minh City, Vietnam
[c]Institute of Physics, University of Silesia
75 Pułku Piechoty 1, 41-500 Chorzów, Poland

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Representations are derived for the basic scalar one-loop vertex Feynman integrals as meromorphic functions of the space-time dimension $d$ in terms of (generalized) hypergeometric functions ${}_2F_1$ and $F_1$. Values at asymptotic or exceptional kinematic points as well as expansions around the singular points at $d = 4+2n$, $n$ being non-negative integers, may be derived from the representations easily. The Feynman integrals studied here may be used as building blocks for the calculation of one-loop and higher-loop scalar and tensor amplitudes. From the recursion relation presented, higher $n$-point functions may be obtained in a straightforward manner.

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## 1. Introduction

The systematic treatment of Feynman integrals is one of the basic ingredients of any perturbative calculation in quantum field theory. In gauge field theories, the Feynman integrals may have both ultraviolet and infrared singularities, and the necessary regularizations are usually performed using a space-time dimension $d = 4 - 2\epsilon$, where $\epsilon$ is the regulator. At one loop, one has to treat two issues concerning dimensionally regularized Feynman integrals: *(i)* the calculation of $n$-point integrals; *(ii)* the calculation of tensor integrals. In a variety of publications, it has been shown that, besides a

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† Corresponding author: `tordriemann@gmail.com`

direct calculation, a general $n$-point tensor Feynman integral may be algebraically reduced to a basis of scalar one- to four-point functions [1], with higher powers $\nu$ of propagators and in higher dimensions $d = 4 + 2n - 2\epsilon$. Using recurrence relations [2–5], one may get representations with all $\nu = 1$, although yet at $d = 4+2n-2\epsilon$ with non-negative integer $n$. Such a representation in higher space-time dimensions may be organized such that it avoids the creation of inverse Gram determinants, which are known to destabilize realistic loop calculations [6, 7].

Having all this in mind, it is evident that the seminal articles by 't Hooft and Veltman on scalar one-loop integrals [8] and by Passarino and Veltman on tensor reduction [9] for one- to four-point functions in 1978 set the stage for decades. They solved the determination of the Laurent expansions in $\epsilon$ for these functions from the leading singular terms upto including the constant terms, at $n = 0$. Later, the leading $\epsilon$ terms were determined in [10], and the general expansion in $\epsilon$ was studied in [11], again at $n = 0$.

Although there are many attempts to determine the scalar one-loop integrals as meromorphic functions in the space-time dimension $d$, a complete solution in terms of special functions has not been given so far. The most important article on the subject is [12], where solutions have been found for scalar one- to four-point integrals in $d$ dimensions by solving iterative difference equations for them. The solutions depend on Gauss' hypergeometric function for two-point functions, additionally on the Appell function $F_1$ (a special case of the Kampé de Fériet function and one of the set of Horn functions) for three-point functions, and additionally on the Lauricella–Saran function $F_{\rm S}$ for four-point functions. In our understanding, the study [12] is not complete because the authors failed to determine sufficiently general expressions for certain boundary terms which they call $b_3$.

In this article, we close the above-mentioned gap left in [12] by applying another technique, starting from Feynman parameter representations for the Feynman integrals, deriving an iterative master integral. For vertices, we solve here the iterative two-dimensional Mellin–Barnes representation. Our version of the boundary term $b_3$ allows to cover the complete physical kinematics in the complex $d$-plane. The most interesting case of four-point functions with a term $b_4$ has also been solved and will be published elsewhere.

## 2. Definitions

The scalar one-loop $n$-point Feynman integrals are defined as

$$J_n \equiv J_n\left(d; \{p_i p_j\}, \{m_i^2\}\right) = \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_n^{\nu_n}} \tag{1}$$

with inverse propagators $D_i = (k+q_i)^2 - m_i^2 + i\varepsilon$. We assume $\nu_i = 1$ as well as momentum conservation and all momenta to be incoming, $\sum_i^n p_i = 0$. The $q_i$ are loop momenta shifts and they will be expressed for applications by the external momenta $p_i$. The $F$ function is independent of a shift of the integration variable $k$ due to the dependence on the differences $q_i - q_j$. Further, the difference of two neighboring momentum shifts $q_i$ equals to an external momentum. We use the Feynman parameter representation for the evaluation of the Feynman integrals (1)

$$J_n = (-1)^n \Gamma\left(n - d/2\right) \int\limits_0^1 \prod_{j=1}^n \mathrm{d}x_j \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{1}{F_n(x)^{n-d/2}} \,. \tag{2}$$

Here, the $F$ function is the second Symanzik polynomial. It is derived from the propagators, $M^2 \equiv x_1 D_1 + \cdots + x_n D_n = k^2 - 2Qk + J$. Using $\delta(1-\sum x_i)$ under the integral in order to transform linear terms in $x$ into bilinear ones, one obtains

$$F_n(x) = -\left(\sum_{i=1}^n x_i\right) \times J + Q^2 \;=\; \frac{1}{2}\sum_{i,j} x_i Y_{ij} x_j - i\varepsilon \,, \tag{3}$$

where the $Y_{ij}$ are elements of the Cayley matrix, introduced for a systematic study of one-loop $n$-point Feynman integrals *e.g.* in [13],

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2 \,. \tag{4}$$

We will discuss the one-loop integrals as functions of two kinematic matrices and determinants, which were introduced by Melrose [13]. The Cayley determinant $\lambda_{12\ldots n}$ is composed of the $Y_{ij}$ introduced in (4), and its determinant is

$$\lambda_n \equiv \lambda_{12\ldots n} = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix} \,. \tag{5}$$

We also define the $(n-1)\times(n-1)$ dimensional Gram determinant

$$G_n \equiv G_{12\cdots n} = -\begin{vmatrix} (q_1-q_n)^2 & \ldots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & \ldots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & \ldots & (q_{n-1}-q_n)^2 \end{vmatrix} \,. \tag{6}$$

Both determinants are independent of a common shifting of the momenta $q_i$. After elimination of one $x$-variable from the $n$-dimensional integral (1), *e.g.* $x_n$, by use of the $\delta$ function in (2), the $F$ function becomes a quadratic form in $x = (x_i)$ with linear terms in $x$ and with an inhomogeneity $R_n$,

$$F_n(x) = (x-y)^T G_n (x-y) + r_n \; - i\varepsilon = \Lambda_n(x) + R_n \,. \tag{7}$$

The following relations are also valid:

$$R_n \equiv r_n - i\varepsilon = -\frac{\lambda_n}{G_n} - i\varepsilon \tag{8}$$

and

$$y_i = \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{G_n}\,\frac{\partial \lambda_n}{\partial m_i^2} \equiv -\frac{\partial_i \lambda_n}{g_n}\,, \qquad i = 1\ldots n\,. \tag{9}$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled. The notations for the $F$ function are finally independent of the choice of the variable which was eliminated by use of the $\delta$ function in the integrand of (2). The inhomogeneity $R_n$ is the only variable carrying the causal $i\varepsilon$ prescription, while *e.g.* $\Lambda(x)$ and the $y_i$ are by definition real.

The simplest case of a one-loop scalar Feynman integral is the one-point function or tadpole

$$J_1\left(d; m^2\right) = \int \frac{\mathrm{d}^d k}{i\pi^{d/2}}\frac{1}{k^2 - m^2 + i\varepsilon} = -\frac{\Gamma(1-d/2)}{(m^2 - i\varepsilon)^{1-d/2}}\,. \tag{10}$$

Finally, we introduce the operator $\boldsymbol{k}^-$, which will reduce an $n$-point Feynman integral $J_n$ to an $(n-1)$-point integral $J_{n-1}$ by shrinking the $k^{\text{th}}$ propagator, $1/D_k$

$$\boldsymbol{k}^- J_n = \boldsymbol{k}^- \int \frac{\mathrm{d}^d k}{i\pi^{d/2}}\frac{1}{\prod_{j=1}^n D_j} = \int \frac{\mathrm{d}^d k}{i\pi^{d/2}}\frac{1}{\prod_{j\neq k, j=1}^n D_j}\,. \tag{11}$$

## 3. The master formula for the Feynman integrals $J_n$

We study the general case with $G_n \neq 0$ and $R_n \neq 0$. Other cases are simply derived from the formulae given here. One may use the well-known Mellin–Barnes representation in order to decompose the integrand of $J_n$ given in (2) as follows:

$$\frac{1}{[\Lambda_n(x) + R_n]^{n-\frac{d}{2}}} = \frac{R_n^{-\left(n-\frac{d}{2}\right)}}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \mathrm{d}s\, \frac{\Gamma(-s)\,\Gamma\left(n-\frac{d}{2}+s\right)}{\Gamma\left(n-\frac{d}{2}\right)}\left[\frac{\Lambda_n(x)}{R_n}\right]^s, \tag{12}$$

for $|\mathrm{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. As a result of (12), the Feynman parameter integral of $J_n$ becomes homogeneous

$$K_n = \prod_{j=1}^{n-1} \int\limits_0^{1-\sum_{i=j+1}^{n-1} x_i} \mathrm{d}x_j \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \int \mathrm{d}S_{n-1} \left[\frac{\Lambda_n(x)}{R_n}\right]^s . \qquad (13)$$

In order to solve this integral, we introduce the differential operator $\hat{P}_n$ [14, 15],

$$\frac{\hat{P}_n}{s}\left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2s}(x_i - y_i)\frac{\partial}{\partial x_i}\left[\frac{\Lambda_n(x)}{R_n}\right]^s = \left[\frac{\Lambda_n(x)}{R_n}\right]^s , \qquad (14)$$

into the integrand of (13)

$$K_n = \frac{1}{s}\int \mathrm{d}S_{n-1}\hat{P}_n\left[\frac{\Lambda_n(x)}{R_n}\right]^s = \frac{1}{2s}\sum_{i=1}^{n-1}\prod_{k=1}^{n-1}\int\limits_0^{u_k} \mathrm{d}x'_k(x_i - y_i)\frac{\partial}{\partial x_i}\left[\frac{\Lambda_n(x)}{R_n}\right]^s . \qquad (15)$$

After a series of manipulations in order to perform one of the $x$-integrations — by partial integration, eating the corresponding differential — and applying a Barnes relation [16] (item 14.53 at page 290 of [17]), one arrives at the following recursion relation:

$$\begin{aligned} J_n\left(d, \{q_i, m_i^2\}\right) \;=\; & \frac{-1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \mathrm{d}s \frac{\Gamma(-s)\Gamma\left(\frac{d-n+1}{2}+s\right)\Gamma(s+1)}{2\Gamma\left(\frac{d-n+1}{2}\right)}\left(\frac{1}{R_n}\right)^s \\ & \times \sum_{k=1}^{n}\left(\frac{1}{R_n}\frac{\partial r_n}{\partial m_k^2}\right)\boldsymbol{k}^- J_n\left(d+2s; \{q_i, m_i^2\}\right) . \qquad (16) \end{aligned}$$

Equation (16) is the master integral for one-loop $n$-point functions in space-time dimension $d$, representing them by $n$ integrals over $(n-1)$-point functions with a shifted dimension $d+2s$. This Mellin–Barnes integral representation is the equivalent to Eq. (19) of [12]. There, an *infinite sum* over a discrete parameter $s$ was derived in order to represent an $n$-point function in space-time dimension $d$ by simpler functions $J_{n-1}$ at dimensions $d+2s$.

## 4. The three-point function

According to master formula (16), we can write the massive 3-point function as a sum of three terms, each of them relying on a two-point function, relying on one-point functions. After analytically performing the two-fold Mellin–Barnes integrals, we arrive at

$$J_3\left(d;\left\{p_i^2\right\},\left\{m_i^2\right\}\right)=J_{123}+J_{231}+J_{312} \tag{17}$$

with

$$\begin{aligned}
J_{123} &= \Gamma\left(2-\frac{d}{2}\right)R_3^{\frac{d}{2}-2}b_{123}\\
&\quad -\frac{\sqrt{\pi}\;\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)}\frac{\partial_3\lambda_3}{\lambda_3}\frac{R_{12}^{\frac{d}{2}-1}}{4\lambda_{12}}\left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}}+\frac{\partial_1\lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}}\right]\\
&\quad\times{}_2F_1\left[\begin{matrix}\frac{d-2}{2},1\,; & \frac{R_{12}}{R_3}\\ \frac{d-1}{2}\,; & \end{matrix}\right]+\frac{2}{d-2}\Gamma\left(2-\frac{d}{2}\right)\frac{\partial_3\lambda_3}{\lambda_3}\\
&\quad\times\left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}}\frac{\left(m_1^2\right)^{\frac{d}{2}-1}}{4\lambda_{12}}F_1\left(\frac{d-2}{2};1,\frac{1}{2};\frac{d}{2};\frac{m_1^2}{R_3},\frac{m_1^2}{R_{12}}\right)+(1\leftrightarrow 2)\right]
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
b_{123} &= -\frac{1}{2G_{12}}\frac{\partial_3\lambda_3}{\lambda_3}\left(\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}}+\frac{\partial_1\lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}}\right){}_2F_1\left[\begin{matrix}1,1\,; & \frac{R_{12}}{R_3}\\ \frac{3}{2}\,; & \end{matrix}\right]\\
&\quad -\frac{\partial_3\lambda_3}{\lambda_3}\left\{\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}}\frac{m_1^2}{4\lambda_{12}}F_1\left(1;1,\frac{1}{2};2;\frac{m_1^2}{R_3},\frac{m_1^2}{R_{12}}\right)+(1\leftrightarrow 2)\right\},
\end{aligned} \tag{19}$$

where $\partial_i\lambda_{j\cdots}$ is defined in (9). Representation (17) is valid for $\mathrm{Re}(\frac{d-2}{2})>0$. The conditions $\left|\frac{m_i^2}{R_{ij}}\right|<1$, $\left|\frac{R_{ij}}{R_3}\right|<1$ had to be met during the derivation. The result may be analytically continued in a straightforward way, however, in the complete complex domain.

## 5. Vertex numerics

In Table I, we show one numerical case, further examples are given in the slides of the presentation at MTTD 2018, see `http://indico.if.us.edu.pl/event/4/`. While we agree completely with the "main" parts of the solutions for the Feynman integrals given in [12], our boundary term has a richer structure and is, contrary to $b_3$ [12], valid for arbitrary kinematics without additional specific considerations.

TABLE I

Numerics for the constant term of a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. This work, (17) to (19) is labelled (TR). Bold input quantities suggest that, according to Eq. (73) in [12] (labelled (OT)), one has to set there $b_3 = 0$. This choice gives a wrong result for $J_3$. If instead, we choose in the numerics for Eq. (75) of [12] that $\sqrt{-G_3} \to \sqrt{-G_3 + \varepsilon I} = +I\sqrt{|G_3|}$, and include the non-vanishing value for $b_3$, the $J_3^{(\mathrm{OT})}$ gets correct. The setting $G_3 - \varepsilon I$ looks counter-intuitive for a "momentum"-like function like $G_3$.

| | |
|---|---|
| $[p_i^2]$, $[m_i^2]$ | $[-100,\ +\mathbf{200},\ -300]$, $[10,\ 20,\ 30]$ |
| $G_3$, $\lambda_3$ | $+\mathbf{480000}$, $-19300000$ |
| $m_i^2/r_3$ | 0.248705, 0.497409, 0.746114 |
| $\sum J$, Eq. (18) | $-0.012307377 - 0.056679689\ I$ |
| $\sum b$, Eq. (19) | $+0.047378343I$ |
| $J_3^{(\mathrm{TR})} = \sum J + \sum b$ | $-0.012307377 - 0.009301346\ I$ |
| $b_3$ [12] | $+0.047378343I$ |
| $b_3 + \sum J$ [12] | $-0.012307377 - 0.009301346\ I$ |
| $J_3^{(\mathrm{OT})} = \sum J$ | $b_3 \to 0$, gets wrong |
| $(-1)\times$FIESTA 3 [18] | $-(0.012307 + 0.009301\ I)$ |
| LoopTools/FF [19] | $-0.012307377 - 0.009301346\ I$ |

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