CLASSIFICATION OF THE TRACELESS RICCI TENSOR IN 4-DIMENSIONAL PSEUDO-RIEMANNIAN SPACES OF NEUTRAL SIGNATURE

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The traceless Ricci tensor $C_{ab}$ in 4-dimensional pseudo-Riemannian spaces equipped with the metric of the neutral signature is analyzed. Its algebraic classification is given. This classification uses the properties of $C_{ab}$ treated as a matrix. The Petrov–Penrose types of Plebański spinors associated with the traceless Ricci tensor are given. Finally, the classification is compared with a similar classification in the complex case.

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1. Introduction

The algebraic structure of Ricci tensor in general relativity was investigated by many authors (see, e.g., Petrov [1], Plebański [2], Penrose [3], Hall [4], Plebański and Stachel [5]). In particular, in 1964, an algebraic classification of the traceless Ricci tensor $C_{ab}$ in real 4-dimensional Lorentzian manifolds was given by Plebański in his distinguished work in Acta Physica Polonica [2]. Investigation of this problem was motivated by the obvious relation between traceless Ricci tensor and the tensor of matter $T_{ab}$. Plebański proved that there were exactly 15 different types of the tensor of matter. In [2], the algebraic structure of $C_{ab}$ is investigated from several points of view. First, $C_{ab}$ is considered as a matrix. Then the structure of the so-called Plebański spinors has been investigated. It appeared that any tensor of matter can be represented as a superposition of three energy-momentum tensors of the electromagnetic type. Careful analysis of this fact was the third line of studies on the properties of $C_{ab}$ presented in [2].
In the seventies, a great deal of interest was devoted to the complex 4-dimensional spaces. It appeared that the Plebański algebraic classification of the traceless Ricci tensor could be easily carried over to the complex spacetimes [6]. Since it does not make any sense to distinguish between spacelike and timelike vectors in complex spaces, one could expect that the structure of $C_{ab}$ in complex spaces should be less complicated than the analogous structure in the Lorentzian case. Surprisingly, there appeared 17 different types of the traceless Ricci tensor in complex spacetime. Some of these types do not have their counterparts in real Lorentzian case. Results from [6] allowed one to understand better the complex relativity and the differences between complex and real manifolds.

It is worth to note that in both papers [2, 6], the spinorial formalism has been intensively used [7–9]. It helped to simplify the calculations and allowed to define spinorial objects (like the Plebański spinors), which appeared to be essential in further analysis.

Recently, the real 4-dimensional spaces equipped with the metric of the neutral (ultrahyperbolic) signature $(+ + --)$ has attracted the great deal of interest. The Walker and Osserman spaces, integrable systems, self-dual and anti-self-dual structures, para-Hermite and para-Kähler structures — these all concepts are related to the real 4-dimensional, neutral spaces. Especially interesting are recently discovered relations between real 4-dimensional, neutral Einstein spaces equipped with the para-Kähler structure and the 5-dimensional spaces equipped with the $(2,3,5)$-distributions [10, 11]. Thus, it seems that the 4-dimensional pseudo-Riemannian spaces with neutral signature will play more and more important role in the theoretical physics.

Our paper is devoted to such spaces. We investigate the algebraic structure of traceless Ricci tensor $C_{ab}$ in the real 4-dimensional, neutral spaces. To classify $C_{ab}$, we follow the works by Plebański and Przanowski using the same techniques. Our approach uses discrete classification (the number and type of eigenvectors of $C_{ab}$) and the continuous classification (the number and type of eigenvalues of the characteristic polynomial of $C_{ab}$). Moreover, we distinguish spacelike, timelike and null eigenvectors. The Plebański spinors have the same structure as a self-dual or anti-self-dual Weyl spinors and in neutral signature case, they can be divided into 10 different Petrov–Penrose types. This way, we obtain another criteria helpful in classification of the traceless Ricci tensor. Finally, we arrive at 33 different types of $C_{ab}$. We realize that the structure of the traceless Ricci tensor is much richer that we could suspect.

It is well-known that real analytic spaces can be obtained from the complex spaces as the real slices. In many cases, real analytic spaces with the metric of the neutral signature can be obtained from the complex ones par-
ticularly simple. It is enough to replace complex variables by real ones and holomorphic functions by real analytic ones. However, in classification of the $C_{ab}$, there appear subtle differences between complex spaces and real neutral spaces. Single complex type of $C_{ab}$ splits in a few subtypes in the real case. It is related to the existence of the spacelike and timelike vectors in real spaces. In Section 3, we point all these differences by listing generic complex types and real types into which these complex types split.

We believe that our work fills the gap left by the works of Plebański and Przanowski published in *Acta Physica Polonica B* and will be helpful in analysis of non-Einsteinian para-Hermite and para-Kähler spaces. Some applications of ideas presented here have been already used in our work [12].

The paper is organized as follows. In Section 2, a portion of basic facts about the null and orthonormal tetrad in both complex and real neutral spaces is presented. Then we discuss the different types of the roots of the 4$\textsuperscript{th}$ order polynomial and the criteria which allow to distinguish these types. The polynomials with the complex and real coefficients are both discussed. The essential difference between Petrov–Penrose classification of the 4-index, dotted and undotted totally symmetric spinors in complex and real neutral spaces are also sketched. Finally, the new symbol of the type of traceless Ricci tensor is introduced (2.17). At the first glance, this symbol is more complicated than the symbols used by Plebański and Przanowski in [2, 6]. We believe however, that the great number of different types of $C_{ab}$ in real neutral spaces and the complexity of the degeneration schemes (like Scheme 2) justify using such a symbol.

Section 3 is devoted to the detailed classification of the traceless Ricci tensor. We present the canonical forms of $C_{ab}$ and we discuss its possible degenerations. Also, the Petrov–Penrose type of the Plebański spinors is analyzed. The results are gathered in the tables and also the graphs of possible degenerations are presented. Concluding remarks end the paper.

2. Preliminaries

2.1. Formalism

In this section, we present the foundations of the formalism used in this paper. For more detailed treatment, see [7–9].

We consider 4-dimensional manifold $\mathcal{M}$ equipped with the metric tensor $ds^2$. $\mathcal{M}$ could be complex analytic differentiable manifold endowed with a holomorphic metric $ds^2$ or a real 4-dimensional smooth differentiable manifold endowed with a real smooth metric $ds^2$ of the neutral signature $(++--)$. Thus, one deals with *complex relativity* (CR) or with *real ultra-hyperbolic (neutral) relativity* (UR).
The metric of $\mathcal{M}$ in null tetrad $(e^1, e^2, e^3, e^4)$ reads

$$ds^2 = g_{ab} e^a e^b = 2e^1 e^2 + 2e^3 e^4,$$

$$(g_{ab}) := \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}. \quad (2.1)$$

In orthonormal tetrad $(E^1', E^2', E^3', E^4')$, the metric takes the form of

$$ds^2 = g_{a'b'} E^a' E^{b'} = E^1' E^1' + E^2' E^2' - E^3' E^3' - E^4' E^4',$$

$$(g_{a'b'}) := \text{diag}(1, 1, -1, -1). \quad (2.2)$$

The relation between null and orthonormal tetrad is

$$\begin{cases}
\sqrt{2} E^1' = e^1 + e^2 \\
\sqrt{2} E^2' = e^3 + e^4 \\
\sqrt{2} E^3' = e^1 - e^2 \\
\sqrt{2} E^4' = e^3 - e^4
\end{cases} \iff \begin{cases}
\sqrt{2} e^1 = E^1' + E^3' \\
\sqrt{2} e^2 = E^1' - E^3' \\
\sqrt{2} e^3 = E^2' + E^4' \\
\sqrt{2} e^4 = E^2' - E^4'.
\end{cases} \quad (2.3)$$

In the spinorial formalism, the metric reads

$$ds^2 = -\frac{1}{2} g_{AB} g^{AB}, \quad A = 1, 2, \quad B = \hat{1}, \hat{2}, \quad (2.4)$$

where $g_{AB}$ are given by

$$\left( g^{AB} \right) := \sqrt{2} \begin{bmatrix}
e^4 & e^2 \\
e^1 & -e^3
\end{bmatrix}. \quad (2.5)$$

Let us consider now the pair of normalized undotted and dotted spinors, $(k_A, l_B)$, $(\tilde{k}_{\hat{A}}, \tilde{l}_{\hat{B}})$ $k^A l_A = 1$ and $\tilde{k}^\hat{A} \tilde{l}_\hat{A} = 1$. They generate the new null tetrad $(\tilde{e}^1, \tilde{e}^2, \tilde{e}^3, \tilde{e}^4)$ according to the formulas:

$$\sqrt{2} \tilde{e}^1 := k_A l_B g^{AB}, \quad \sqrt{2} \tilde{e}^2 := l_A k_B g^{AB}, \quad -\sqrt{2} \tilde{e}^3 := k_A \tilde{k}_{\hat{B}} g^{AB}, \quad \sqrt{2} \tilde{e}^4 := \tilde{l}_A l_{\hat{B}} g^{AB}. \quad (2.6)$$

Define the matrix $g_{a\hat{A}\hat{B}}$ by the relation $g^{a\hat{B}} = g_{a\hat{A}} e^a$. The following identities hold

$$g_{a\hat{A}\hat{B}} g^{b\hat{A}\hat{B}} = -2\delta^b_a, \quad g_{a\hat{A}\hat{B}} g^{a\hat{C}\hat{D}} = -2\delta^C_A \delta_{\hat{B}}^{\hat{D}}. \quad (2.7)$$
Thus, we have
\[
\begin{align*}
\sqrt{2}\tilde{e}_1^a &= k_A l_B g_{aAB} \\
\sqrt{2}\tilde{e}_2^a &= l_A k_B g_{aAB} \\
-\sqrt{2}\tilde{e}_3^a &= k_A k_B g_{aAB} \\
\sqrt{2}\tilde{e}_4^a &= l_A l_B g_{aAB}
\end{align*}
\]
\[\iff
\begin{align*}
\tilde{e}_{2a} g_{AB}^a &= -\sqrt{2} k_A l_B \\
\tilde{e}_{1a} g_{AB}^a &= -\sqrt{2} l_A k_B \\
\tilde{e}_{4a} g_{AB}^a &= \sqrt{2} k_A k_B \\
\tilde{e}_{3a} g_{AB}^a &= -\sqrt{2} l_A l_B
\end{align*}
\] (2.8)

2.2. The Petrov–Penrose classification of totally symmetric 4-index spinors

Algebraic classification of totally symmetric 4-index spinors has been presented in classical paper [7], see also [2]. It can be applied for SD (or ASD) part of the Weyl spinor $C_{ABCD}$ ($C_{\dot{A}\dot{B}\dot{C}\dot{D}}$, respectively). We use these results to classify the Plebański spinors (2.11). First, we consider complex undotted Plebański spinor $V_{ABCD}$ and its contraction with the arbitrary 1-index spinor $\xi^A$:
\[
\Omega := V_{ABCD} \xi^A \xi^B \xi^C \xi^D.
\]
Clearly, $\Omega$ has the form of $\Omega = (\xi^2)^4 V(z)$, where $V(z)$ is a 4th order polynomial in $z := \xi^1 / \xi^2$. Due to the fundamental theorem of algebra, $\Omega$ can be always brought to the factorized form $\Omega = (\alpha_A \xi^A)(\beta_B \xi^B)(\gamma_C \xi^C)(\delta_D \xi^D)$. Because of the arbitrariness of $\xi^A$, we find
\[
V_{ABCD} = \alpha_{(A}\beta_{B}\gamma_{C}\delta_{D)}.
\] (2.9)

In general, 1-index spinors $\alpha_A$, $\beta_A$, $\gamma_A$ and $\delta_A$ are mutually linearly independent. Such case corresponds to the case when the polynomial $V(z)$ has four different roots. The possible coincidences between spinors $\alpha_A$, $\beta_A$, $\gamma_A$ and $\delta_A$ brought us to the well-known Petrov–Penrose diagram (Scheme 1).

\[
\begin{align*}
\alpha_{(A}\beta_{B}\gamma_{C}\delta_{D)} & \quad \downarrow \\
\alpha_{(A}\alpha_{B}\beta_{C}\delta_{D)} & \quad \downarrow \\
\alpha_{(A}\alpha_{B}\beta_{C}\gamma_{D)} & \quad \downarrow \\
0 & \quad \alpha_{A}\alpha_{B}\alpha_{C}\alpha_{D} \quad \alpha_{(A}\beta_{B}\beta_{C}\beta_{D)}
\end{align*}
\]

Scheme 1: The Petrov–Penrose diagram.

In the complex case, there are 6 different Petrov–Penrose types of the spinor $V_{ABCD}$. On the other hand, if we consider real totally symmetric 4-index spinor $V_{ABCD}$, then the scheme of the roots of $V(z)$ is more complicated. There appear 10 different Petrov–Penrose types. The symbols which are usually used as abbreviations of the corresponding Petrov–Penrose types of spinor $V_{ABCD}$ and the scheme of the roots of the polynomial $V(z)$ are gathered in Table I. (In Table I, $Z$ means that the root is complex, while $R$
stands for the real root, the power denotes the multiplicity of corresponding root, spinors $\alpha_A, \beta_A, \gamma_A$ and $\delta_A$ are complex, spinors $\mu_A, \nu_A, \xi_A$ and $\zeta_A$ are real, bar stands for the complex conjugation).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Type & $V_{ABCD} =$ & Roots of $\mathcal{V}(z)$ \\
\hline
[I] & $\alpha(A_B \gamma_C \delta_D)$ & $Z_1 Z_2 Z_3 Z_4$ \\
\hline
[I] & $\alpha(A_B \gamma_C \gamma_D)$ & $Z_1^2 Z_2^2 Z_3^2$ \\
\hline
[D] & $\alpha(A_B \gamma_C \gamma_D)$ & $Z_1^2 Z_2^2 Z_3^2$ \\
\hline
[N] & $\alpha(A_B \gamma_C \gamma_D)$ & $Z^4$ \\
\hline
\hline
\end{tabular}
\end{table}

It is clear that the Petrov–Penrose types of both real and complex totally symmetric 4-index spinors $V_{ABCD}$ are related to the nature of roots of the corresponding polynomial $\mathcal{V}(z)$. It is well-known that such a 4th order polynomial can be always brought to the canonical form. The criteria which allow to distinguish the scheme of roots of the 4th order polynomial in the canonical form are discussed in the next subsection.

Of course, similar classification can be applied for the dotted 4-index spinors $V_{\dot{A}\dot{B}\dot{C}\dot{D}}$ and for the “dotted” polynomial $\dot{\mathcal{V}}(\dot{z})$.

2.3. Traceless Ricci tensor

The relation between traceless Ricci tensor $C_{ab}$ and its spinorial image $C_{ABCD}$ reads

$$C_{ab} = g_a^A g_b^B g_{CD} C_{ABCD} \iff C_{ABCD} = \frac{1}{4} C_{ab} g_{AC}^a g_{BD}^b$$

(2.10) (compare (2.7)). Using spinorial image $C_{ABCD}$ of the traceless Ricci tensor $C_{ab}$, one defines the undotted and dotted Plebański spinors by the relations [2, 8]

$$V_{ABCD} := 4 C_{AB}^{\hat{M} \hat{N}} C_{AC} \hat{M} \hat{N}, \quad V_{\dot{A}\dot{B}\dot{C}\dot{D}} = 4 C_{MN} (\dot{A}\dot{B} C^{\hat{M} \hat{N}} \dot{C} \dot{D}) .$$

(2.11)
The dotted and undotted Plebański spinors are totally symmetric $V_{ABCD} = V_{(ABCD)}$ and $V_{\dot{A}\dot{B}\dot{C}\dot{D}} = V_{(\dot{A}\dot{B}\dot{C}\dot{D})}$.

The characteristic polynomial of the matrix $C_a^b$ of the traceless Ricci tensor reads

$$W(x) := \det (C_a^b - x\delta_a^b) = \sum_{i=0}^{4} (-1)^i \left. C \right|^{i}_{[0]} x^4 - \left. C \right|^{i}_{[1]} x^3 + \left. C \right|^{i}_{[2]} x^2 - \left. C \right|^{i}_{[3]} x + \left. C \right|^{i}_{[4]},$$

where the coefficients $\left. C \right|^{i}$ are given by

$$\left. C \right|^{0} := 1, \quad \left. C \right|^{k}_{[a_1...a_k]} := C_a^{a_1}...C_a^{a_k}, \quad k = 1, 2, 3, 4. \quad (2.13)$$

Since the matrix $C_a^b$ is traceless, we find that $\left. C \right|^{1} := C_a^{a} = 0$ so, finally, the characteristic polynomial $W(x)$ takes the form

$$W(x) = x^4 + \left. C \right|^{2}_{[2]} x^2 - \left. C \right|^{3}_{[3]} x + \left. C \right|^{4}_{[4]}. \quad (2.14)$$

In UR coefficients, $\left. C \right|^{i} \in \mathbb{R}$. Criteria which allow us to distinguish the properties of the roots of $W(x)$ have been widely discussed in [8, 13–15]. Define

$$-8J := \frac{1}{2} \left. C \right|^{3}_{[3]} - \frac{4}{3} \left. C \right|^{2}_{[2][4]} + \frac{1}{27} \left. C \right|^{3}_{[2]}, \quad I := \left. C \right|^{4}_{[4]} + \frac{1}{12} \left. C \right|^{2}_{[2]}, \quad K := \frac{1}{4} \left. C \right|^{3}_{[3]}$$

$$L := \frac{1}{6} \left. C \right|^{2}_{[2]}, \quad N := \frac{1}{4} \left. C \right|^{2}_{[2]} - \left. C \right|^{3}_{[4]}, \quad P := -9 \left. C \right|^{2}_{[2]} - 2 \left. C \right|^{3}_{[3]} \left( \left. C \right|^{2}_{[2]} - 4 \left. C \right|^{3}_{[4]} \right).$$

Then, the discriminant of polynomial (2.14) reads

$$\Delta = 256 \left( I^3 - 27J^2 \right). \quad (2.16)$$

As it was mentioned in the previous subsection, there are exactly 9 cases which should be distinguished using the criteria from Table II.

In CR, the coefficients $\left. C \right|^{i} \in \mathbb{C}$ are complex. There are only 5 distinct cases (see Table III).
TABLE II

Roots of the quartic equation with real coefficients.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta &lt; 0 )</td>
<td>( R_1R_2ZZ )</td>
</tr>
<tr>
<td>( \Delta &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>( L &lt; 0 ) and ( N &gt; 0 )</td>
<td>( R_1R_2R_3R_4 )</td>
</tr>
<tr>
<td>( L \geq 0 ) or ( N &lt; 0 )</td>
<td></td>
</tr>
<tr>
<td>( \Delta = 0 )</td>
<td></td>
</tr>
<tr>
<td>( I \neq 0 ), ( J \neq 0 )</td>
<td>( P &gt; 0 ) ( R_1R_2R_3^2 )</td>
</tr>
<tr>
<td>( K \neq 0 ) or ( N \neq 0 )</td>
<td>( P &lt; 0 ) ( R^2ZZ )</td>
</tr>
<tr>
<td>( I \neq 0 ), ( J \neq 0 )</td>
<td>( J &lt; 0 ) ( R_1^2R_2^2 )</td>
</tr>
<tr>
<td>( K = N = 0 )</td>
<td>( J &gt; 0 ) ( Z^2Z^2 )</td>
</tr>
<tr>
<td>( I = J = 0 )</td>
<td>( N \neq 0 ) ( K \neq 0 ) ( R_1R_2^3 )</td>
</tr>
<tr>
<td>( N = K = 0 )</td>
<td>( R^4 )</td>
</tr>
</tbody>
</table>

TABLE III

Roots of the quartic equation with complex coefficients.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \neq 0 )</td>
<td>( Z_1Z_2Z_3Z_4 )</td>
</tr>
<tr>
<td>( \Delta = 0 )</td>
<td></td>
</tr>
<tr>
<td>( I \neq 0 ), ( J \neq 0 )</td>
<td>( P \neq 0 ) ( Z_1Z_2Z_3^2 )</td>
</tr>
<tr>
<td>( P = 0 )</td>
<td>( Z_1^2Z_2^2 )</td>
</tr>
<tr>
<td>( I = J = 0 )</td>
<td>( L \neq 0 ) ( Z_1Z_2^3 )</td>
</tr>
<tr>
<td>( L = 0 )</td>
<td>( Z_4 )</td>
</tr>
</tbody>
</table>

2.4. Terminology and symbols

To classify traceless Ricci tensor in \( \textbf{UR} \), we use the notation similar to Plebański’s notation from [2] and the Plebański–Przanowski notation from [6]. The number of eigenvectors are considered as a main criterion, while the properties of the eigenvalues and the form of the minimal polynomial serve as subcriteria.

The complete information about the type of the matrix, \( (C^a_b) \), is gathered in the symbol

\[
[A]_j \otimes [B]_k [n_1E_1 - n_2E_2 - \ldots]^{v}_{(q_1q_2\ldots)}. \tag{2.17}
\]

Inside the square bracket, all different eigenvalues \( E_i, i = 1, 2, \ldots, N_0 \) to-
gethertogetherwiththeirmultiplicities\(n_i\)arelisted. Of course,
\[
\begin{align*}
  n_1 + n_2 + \cdots + n_{N_0} &= 4, \\
  n_1 E_1 + n_2 E_2 + \cdots + n_{N_0} E_{N_0} &= 0.
\end{align*}
\]
(2.18)
The last equality follows from the fact that the matrix \((C^a_b)\) is traceless.

The characteristic polynomial takes the form of
\[
W(x) = \prod_{i=1}^{N_0} (x - E_i)^{n_i}.
\]
(2.19)

A complex eigenvalue is denoted by \(Z\) and a real one by \(R\). Real eigenvalues have additional superscript which denotes the type of the corresponding eigenvector. \(R^s\) means that the eigenvector which corresponds to the eigenvalue \(R\) is spacelike, \(R^t\) — timelike, \(R^n\) — null, \(R^{ns}\) — null or spacelike, \(R^{nt}\) — null or timelike and, finally, \(R^{nst}\) means that the eigenvector can be of the arbitrary type. [With respect to the orthonormal tetrad (2.2), the definitions of spacelike, timelike and null vectors are as follows: \(V^a V_a > 0\) means that \(V^a\) is spacelike, \(V^a V_a < 0\) stands for a timelike vector and, finally, \(V^a V_a = 0\) means that the vector is null].

Superscript \(v\) denotes the number of eigenvectors. Numbers \(q_i\) in the round bracket determine the form of the minimal polynomial, \(i.e.,\) the polynomial of the lowest possible order with the leading term equal to 1 such that \(W_{\text{min}}(C^a_b) = 0\). Namely, the minimal polynomial of the matrix \((C^a_b)\) has the form of
\[
W_{\text{min}}(x) = \prod_{i=1}^{N_0} (x - E_i)^{q_i}.
\]
(2.20)

Finally, the symbol \([A]_j \otimes [B]_k\) defines the Petrov–Penrose types of the Plebański spinors, \(V_{ABCD}\) and \(\tilde{V}_{\dot{A}\dot{B}\dot{C}\dot{D}}\), respectively (2.11). For example, \([\text{III}]_r \otimes [\text{N}]_r\) means that \(V_{ABCD}\) is of the type \([\text{III}]_r\), while \(\tilde{V}_{\dot{A}\dot{B}\dot{C}\dot{D}}\) is of the type \([\text{N}]_r\).

### 3. Classification of the traceless Ricci tensor in UR

#### 3.1. Parent Types

The eigenvalue criteria (Table II), the number and the type of eigenvectors and the Petrov–Penrose type of the undotted and dotted Plebański spinors allow to distinguish exactly 33 types of the traceless Ricci tensor. They appear as the degenerations of 9 parent Types (according to Plebański’s terminology, “Types” by capital “T”). Each of these parent Types has the
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The minimal equation being exactly the Hamilton–Cayley equation. The symbols of the Types are quite similar to the symbols of Petrov–Penrose types of the Plebański spinors. To distinguish them, we do not put the symbol of the Types into the square bracket (as we do in the case of the Petrov–Penrose types of the Plebański spinors). Types I and II have subscripts “r” (all eigenvectors real), “c” (all eigenvectors complex) or “rc” (two eigenvectors complex, one or two eigenvectors real). Types III and IV have only real eigenvectors. However, Type III admits two null eigenvectors (subscript “n”), one null eigenvector and one timelike (subscript “t”) or one null eigenvector and one spacelike (subscript “s”). We use the symbols of the parent Types in the CR like in [6] (I, II, III\textsubscript{C}, III\textsubscript{N} and IV).

### TABLE IV

Parent Types of $C_{ab}$.

<table>
<thead>
<tr>
<th>Type</th>
<th>Symbols of the parent Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>I\textsubscript{c}</td>
<td>$[I_{c} \otimes I_{c}]\left[Z_{1} - \bar{Z}<em>{1} - Z</em>{2} - \bar{Z}<em>{2}\right]^{4}</em>{(1111)}$, $[I_{c} \otimes I_{c}]\left[Z_{1} - \bar{Z}<em>{1} - Z</em>{2} - \bar{Z}<em>{2}\right]^{4}</em>{(1111)}$</td>
</tr>
<tr>
<td>I\textsubscript{rc}</td>
<td>$[I_{rc} \otimes I_{rc}]\left[Z - \bar{Z} - R_{1}^{s} - R_{2}^{s}\right]^{4}_{(1111)}$</td>
</tr>
<tr>
<td>I\textsubscript{r}</td>
<td>$[I_{r} \otimes I_{r}]\left[R_{1}^{s} - R_{2}^{s} - R_{3}^{t} - R_{4}^{t}\right]^{4}<em>{(1111)}$, $[I</em>{r} \otimes I_{r}]\left[R_{1}^{s} - R_{2}^{s} - R_{3}^{t} - R_{4}^{t}\right]^{4}_{(1111)}$</td>
</tr>
<tr>
<td>II\textsubscript{rc}</td>
<td>$[II_{rc} \otimes II_{rc}]\left[Z - \bar{Z} - 2R_{n}^{n}\right]^{3}<em>{(112)}$, $[II</em>{rc} \otimes II_{rc}]\left[Z - \bar{Z} - 2R_{n}^{n}\right]^{3}_{(112)}$</td>
</tr>
<tr>
<td>II\textsubscript{r}</td>
<td>$[II_{r} \otimes II_{r}]\left[R_{1}^{s} - R_{2}^{s} - 2R_{3}^{n}(112)\right]$, $[II_{r} \otimes II_{r}]\left[R_{1}^{s} - R_{2}^{s} - 2R_{3}^{n}(112)\right]$</td>
</tr>
<tr>
<td>III\textsubscript{n}</td>
<td>$[D_{r} \otimes III_{r}]\left[2R_{1}^{n} - 2R_{2}^{n}\right]^{2}<em>{(22)}$, $[II</em>{r} \otimes D_{r}]\left[2R_{1}^{n} - 2R_{2}^{n}\right]^{2}_{(22)}$</td>
</tr>
<tr>
<td>III\textsubscript{s}</td>
<td>$[III_{s} \otimes III_{s}]\left[R_{1}^{s} - 3R_{2}^{n}\right]^{2}_{(13)}$</td>
</tr>
<tr>
<td>III\textsubscript{t}</td>
<td>$[III_{t} \otimes III_{t}]\left[R_{1}^{t} - 3R_{2}^{n}\right]^{2}_{(13)}$</td>
</tr>
<tr>
<td>IV</td>
<td>$[N_{r} \otimes III_{r}]\left[4R_{1}^{n}\right]^{1}<em>{(4)}$, $[III</em>{r} \otimes N_{r}]\left[4R_{1}^{n}\right]^{1}_{(4)}$</td>
</tr>
</tbody>
</table>

In the next section, we present the canonical forms of the parent Types and Tables of possible degenerations together with the continuous characteristics of the matrix $(C_{ab})$. For the canonical forms, we use both null and orthonormal tetrads. The classification of the traceless Ricci tensor in complex spaces can be treated as a “generic” classification for the UR. This is why we list the Plebański–Przanowski types described in details in [6] (we keep the original symbols of types used in [6]). For the Plebański–Przanowski classification, we use the abbreviation PP classification.
3.2. Type I (4 eigenvectors)

3.2.1. Type I\(_r\) (4 real eigenvectors; 2 spacelike and 2 timelike eigenvectors)

The canonical form of \(C_{ab}\) for the parent Type I\(_r\) reads

\[
C_{ab} = R_1^s E_{1'a}E_{1'b} + R_2^s E_{2'a}E_{2'b} - R_3^t E_{3'a}E_{3'b} - R_4^t E_{4'a}E_{4'b}
\]

\[
= \frac{1}{2} \left( R_1^s - R_3^t \right) (e_{1a}e_{1b} + e_{2a}e_{2b}) + \frac{1}{2} \left( R_1^s + R_3^t \right) (e_{1a}e_{2b} + e_{2a}e_{1b})
\]

\[
+ \frac{1}{2} \left( R_2^s - R_4^t \right) (e_{3a}e_{3b} + e_{4a}e_{4b}) + \frac{1}{2} \left( R_2^s + R_4^t \right) (e_{3a}e_{4b} + e_{4a}e_{3b}) .
\]

(3.1)

The eigenvectors and corresponding eigenvalues are:

\[ E_{1'} \leftrightarrow R_1^s , \quad E_{2'} \leftrightarrow R_2^s , \quad E_{3'} \leftrightarrow R_3^t , \quad E_{4'} \leftrightarrow R_4^t . \]

The eigenvalues have to satisfy

\[ R_1^s + R_2^s + R_3^t + R_4^t = 0 . \]

Using (2.8), one finds the form of the Plebański spinors

\[
V_{ABCD} = \frac{1}{2} \left( R_1^s - R_3^t \right) \left( R_2^s - R_4^t \right) \left( k_A k_B k_C k_D + l_A l_B l_C l_D \right)
\]

\[
+ \frac{1}{2} \left( (R_2^s - R_4^t)^2 - (3R_1^s + R_3^t) \left( R_1^s + 3R_3^t \right) \right) k_{(A} k_{B} l_{C} l_{D}) ,
\]

\[
V_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \frac{1}{2} \left( R_1^s - R_3^t \right) \left( R_2^s - R_4^t \right) \left( \tilde{k}_{\tilde{A}} \tilde{k}_{\tilde{B}} \tilde{k}_{\tilde{C}} \tilde{k}_{\tilde{D}} + \tilde{l}_{\tilde{A}} \tilde{l}_{\tilde{B}} \tilde{l}_{\tilde{C}} \tilde{l}_{\tilde{D}} \right)
\]

\[
+ \frac{1}{2} \left( (R_2^s - R_4^t)^2 - (3R_1^s + R_3^t) \left( R_1^s + 3R_3^t \right) \right) k_{(\tilde{A}} \tilde{k}_{B} l_{\tilde{C}} l_{\tilde{D}}) .
\]

(3.2)

Investigation of the polynomials \(\mathcal{V}(z)\) and \(\mathcal{\tilde{V}}(z)\) (defined in Subsection 2.2) proves that both Plebański spinors are, in general, of the Petrov–Penrose types \([I]_r\) or \([I]_c\). Define the quantity \(\sigma_1\) by the formula

\[
\sigma_1 := \left( R_3^t - R_1^s \right) \left( R_3^t - R_2^s \right) \left( R_3^t + R_1^s + 2R_2^s \right) \left( R_3^t + 2R_1^s + R_2^s \right) .
\]

(3.3)

Then we get the criterion

\[
\sigma_1 < 0 \iff \text{both } V_{ABCD} \text{ and } V_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} \text{ are of the type } [I]_r ,
\]

\[
\sigma_1 > 0 \iff \text{both } V_{ABCD} \text{ and } V_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} \text{ are of the type } [I]_c .
\]

(3.4)
TABLE V

<table>
<thead>
<tr>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1Z_2Z_3Z_4$</td>
<td>$[C_1 - C_2 - C_3 - C_4]_4$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_1Z_2Z_3^2$</td>
<td>$[C_1 - C_2 - 2N]_3$</td>
</tr>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_1^2Z_2^2$</td>
<td>$[2N_1 - 2N]_2$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_1Z_2^3$</td>
<td>$[C_1 - 3N]_2$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z^4$</td>
<td>$[4N]_1$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Scheme 2: Degeneration scheme of the Type $I_r$. 
3.2.2. Type $I_c$ (4 complex eigenvectors)

The canonical form of the $C_{ab}$ for the parent Type $I_c$ has the form of

$$C_{ab} = \frac{1}{2} (Z_1 + \bar{Z}_1) (E_{2a}E_{2b} - E_{4a}E_{4b}) + \frac{i}{2} (Z_1 - \bar{Z}_1) (E_{2a}E_{4b} + E_{4a}E_{2b}) + \frac{1}{2} (Z_2 + \bar{Z}_2) (E_{1a}E_{1b} - E_{3a}E_{3b}) + \frac{i}{2} (Z_2 - \bar{Z}_2) (E_{1a}E_{3b} + E_{3a}E_{1b})$$

$$= \frac{1}{2} (Z_1 + \bar{Z}_1) (e_{3a}e_{4b} + e_{4a}e_{3b}) + \frac{i}{2} (Z_1 - \bar{Z}_1) (e_{3a}e_{3b} - e_{4a}e_{4b}) + \frac{1}{2} (Z_2 + \bar{Z}_2) (e_{1a}e_{2b} + e_{2a}e_{1b}) + \frac{i}{2} (Z_2 - \bar{Z}_2) (e_{1a}e_{1b} - e_{2a}e_{2b}). \quad (3.5)$$

The eigenvectors and corresponding eigenvalues are:

$$\frac{1}{\sqrt{2}} (E_{2'} + iE_{4'}) \longleftrightarrow Z_1, \quad \frac{1}{\sqrt{2}} (E_{2'} - iE_{4'}) \longleftrightarrow \bar{Z}_1,$$

$$\frac{1}{\sqrt{2}} (E_{1'} + iE_{3'}) \longleftrightarrow Z_2, \quad \frac{1}{\sqrt{2}} (E_{1'} - iE_{3'}) \longleftrightarrow \bar{Z}_2.$$

The constraints for the eigenvalues read

$$\text{Im}(Z_1) \neq 0, \quad \text{Im}(Z_2) \neq 0, \quad \text{Re}(Z_1) + \text{Re}(Z_2) = 0.$$

The Plebanski spinors are

$$V_{ABCD} = 2\text{Im}(Z_1)\text{Im}(Z_2) (k_Ak_Bk_Ck_D + l_Al_Bl_Cl_D)$$

$$- (2(\text{Im}(Z_1))^2 + 2(\text{Im}(Z_2))^2 + 8(\text{Re}(Z_1))^2) \; k_{(A}k_{B}l_{C}l_{D}),$$

$$V_{\bar{A}\bar{B}\bar{C}\bar{D}} = -2\text{Im}(Z_1)\text{Im}(Z_2) (k_{\bar{A}}k_{\bar{B}}k_{\bar{C}}k_{\bar{D}} + l_{\bar{A}}l_{\bar{B}}l_{\bar{C}}l_{\bar{D}})$$

$$- (2(\text{Im}(Z_1))^2 + 2(\text{Im}(Z_2))^2 + 8(\text{Re}(Z_1))^2) \; k_{(\bar{A}\bar{B}}l_{\bar{C}}l_{\bar{D})} \quad (3.6)$$

and they both are, in general, of the type $[I]_r$ and $[I]_c$. To distinguish these two types, we have the following criterion

$$\text{Im}(Z_1)\text{Im}(Z_2) < 0 \iff V_{ABCD} \text{ is of the type } [I]_c$$

and $V_{\bar{A}\bar{B}\bar{C}\bar{D}}$ is of the type $[I]_r$,

$$\text{Im}(Z_1)\text{Im}(Z_2) > 0 \iff V_{ABCD} \text{ is of the type } [I]_r$$

and $V_{\bar{A}\bar{B}\bar{C}\bar{D}}$ is of the type $[I]_c$. \quad (3.7)

It is interesting to note that only the subtype $[I]_r \otimes [I]_c [Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]_{1111}^4$ allows the degeneration into the type $[D]_r \otimes [D]_c [2Z - 2\bar{Z}]_{1111}^4$. 


TABLE VI
Subtypes of the Type $I_c$.

<table>
<thead>
<tr>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1Z_2Z_3Z_4$</td>
<td>$[C_1 - C_2 - C_3 - C_4]_4$</td>
</tr>
<tr>
<td>$Z_3^2Z_2^2$</td>
<td>$[2N_1 - 2N]_2$</td>
</tr>
<tr>
<td>$Z_1^2Z_2^2$</td>
<td>$Z^2\bar{Z}^2$</td>
</tr>
</tbody>
</table>

$[I]_c \otimes [I]_r [Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]_4^{(1111)}$  $[I]_r \otimes [I]_c [Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]_4^{(1111)}$

$[D]_r \otimes [D]_c [2Z - 2\bar{Z}]_4^{(11)}$

$[[I]_c \otimes [I]_r [Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]_4^{(1111)}]$

$[[I]_r \otimes [I]_c [Z_1 - \bar{Z}_1 - Z_2 - \bar{Z}_2]_4^{(1111)}]$

Scheme 3: Degeneration scheme of the Type $I_c$.

3.2.3. Type $I_{rc}$ (4 eigenvectors; two complex, one timelike and one spacelike eigenvectors)

The canonical form of the $C_{ab}$ for the parent Type $I_{rc}$ is:

$$C_{ab} = R_1^s E_{1'a} E_{1'b} - R_2^t E_{3'a} E_{3'b}$$
$$+ \frac{1}{2} (Z + \bar{Z}) (E_{2'a} E_{2'b} - E_{4'a} E_{4'b}) + \frac{i}{2} (Z - \bar{Z}) (E_{2'a} E_{4'b} + E_{4'a} E_{2'b})$$
$$= \frac{1}{2} (R_1^s - R_2^t) (e_{1a} e_{1b} + e_{2a} e_{2b}) + \frac{1}{2} (R_1^s + R_2^t) (e_{1a} e_{2b} + e_{2a} e_{1b})$$
$$+ \frac{1}{2} (Z + \bar{Z}) (e_{3a} e_{4b} + e_{4a} e_{3b}) + \frac{i}{2} (Z - \bar{Z}) (e_{3a} e_{3b} - e_{4a} e_{4b}) . \quad (3.8)$$

The eigenvectors and corresponding eigenvalues are:

$$E_{1'} \longleftrightarrow R_1^s , \quad E_{3'} \longleftrightarrow R_2^t ,$$

$$\frac{1}{\sqrt{2}} (E_{2'} + i E_{4'}) \longleftrightarrow Z , \quad \frac{1}{\sqrt{2}} (E_{2'} - i E_{4'}) \longleftrightarrow \bar{Z} .$$

The relations between eigenvalues read

$$\text{Im}(Z) \neq 0 , \quad R_1^s + R_2^t + 2 \text{Re}(Z) = 0 .$$
The Plebański spinors have the following form:

\[
V_{ABCD} = (R_1^s - R_2^t) \text{Im}(Z)(k_Ak_Bk_Ck_D - l_Al_Bl_Cl_D) + \frac{1}{2} \left( (R_1^s - R_2^t)^2 - 4(\text{Im}(Z))^2 - 16(\text{Re}(Z))^2 \right) k_{(A}k_{B}\bar{l}_{C}l_{D)}.
\]

\[
V_{\dot{A}\dot{B}\dot{C}\dot{D}} = (R_1^s - R_2^t) \text{Im}(Z)(k_{\dot{A}}k_{\dot{B}}k_{\dot{C}}k_{\dot{D}} - l_{A}\bar{l}_{B}l_{C}l_{D}) + \frac{1}{2} \left( (R_1^s - R_2^t)^2 - 4(\text{Im}(Z))^2 - 16(\text{Re}(Z))^2 \right) k_{(\dot{A}\dot{B}\bar{l}_{C}l_{D)}}.
\]

(3.9)

Both Plebański spinors for the nondegenerate Type $I_{rc}$ are of the Petrov–Penrose type $[I]_{rc}$.

**TABLE VII**

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1Z_2Z_3Z_4$</td>
<td>$[C_1 - C_2 - C_3 - C_4]_4$</td>
<td>$[I]<em>{rc} \otimes [I]</em>{rc} [Z - \bar{Z} - R_1^s - R_2^t]_4^{(1111)}$</td>
</tr>
<tr>
<td>$Z_1Z_2Z_3^2$</td>
<td>$[C_1 - C_2 - 2N]_2$</td>
<td>$[D]_r \otimes [D]_r [Z - \bar{Z} - 2R^{nst}]_4^{(111)}$</td>
</tr>
</tbody>
</table>

\[
[Z - \bar{Z} - R_1^s - R_2^t]_4^{(1111)} \rightarrow [D]_r \otimes [D]_r [Z - \bar{Z} - 2R^{nst}]_4^{(111)}
\]

Scheme 4: Degeneration scheme of the Type $I_{rc}$.

### 3.3. Type II (3 eigenvectors)

#### 3.3.1. Type II$_r$ (3 eigenvectors; one timelike, one spacelike and one null eigenvectors)

The canonical form of the $C_{ab}$ for the parent Type II$_r$ is:

\[
C_{ab} = R_1^s E_{1'a}E_{1'b} - R_2^t E_{2'a}E_{2'b} + R_3^n (E_{2'a}E_{2'b} - E_{4'a}E_{4'b}) + \frac{1}{2} (E_{2'a}E_{2'b} + E_{4'a}E_{4'b} - E_{2'a}E_{4'b} - E_{4'a}E_{2'b})
\]

\[
= \frac{1}{2}(R_1^s - R_2^t) (e_{1a}e_{1b} + e_{2a}e_{2b}) + \frac{1}{2} (R_1^s + R_2^t) (e_{1a}e_{2b} + e_{2a}e_{1b}) + R_3^n (e_{3a}e_{4b} + e_{4a}e_{3b}) + e_{4a}e_{4b}.
\]

(3.10)
The eigenvectors and corresponding eigenvalues are:

\[ E_1' \leftrightarrow R_1^s, \quad E_3' \leftrightarrow R_2^t, \quad \frac{1}{\sqrt{2}}(E_2' - E_4') \leftrightarrow R_3^n. \]

The eigenvalues have to satisfy the relation

\[ R_1^s + R_2^t + 2R_3^n = 0. \]

The Plebański spinors for the Type II\(_r\) can be brought to the form

\[
\begin{align*}
V_{ABCD} &= \frac{1}{2} \left( 2(R_1^s - R_2^t) k_{(A}k_{B)} - (3R_1^s + R_2^t) (R_1^s + 3R_2^t) l_{(A}l_{B)} k_{C}k_{D)} , \\
V_{\dot{A}\dot{B}\dot{C}\dot{D}} &= \frac{1}{2} \left( 2(R_1^s - R_2^t) k_{(\dot{A}}k_{\dot{B})} - (3R_1^s + R_2^t) (R_1^s + 3R_2^t) l_{(\dot{A})\dot{B})} k_{\dot{C}}k_{\dot{D)}}. 
\end{align*}
\]

Consider the quantity \( \sigma_2 \)

\[ \sigma_2 := (R_1^s - R_2^t) (3R_1^s + R_2^t) (R_1^s + 3R_2^t) . \]

Then we find the following criterion

\[ \sigma_2 > 0 \iff V_{ABCD} \text{ and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ are of the type } [\text{II}]_r , \]
\[ \sigma_2 < 0 \iff V_{ABCD} \text{ and } V_{\dot{A}\dot{B}\dot{C}\dot{D}} \text{ are of the type } [\text{II}]_{rc} . \]

**TABLE VIII**

Subtypes of the Type II\(_r\).

<table>
<thead>
<tr>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td>Type II</td>
</tr>
<tr>
<td>( Z_1Z_2Z_3^2 )</td>
<td>([C_1 - C_2 - 2N]_4)</td>
</tr>
<tr>
<td>( Z_1^2Z_2^2 )</td>
<td>([2N_1 - 2N]_{(1-2)})</td>
</tr>
<tr>
<td>( Z_1Z_2^3 )</td>
<td>([C_1 - 3N]_3)</td>
</tr>
<tr>
<td>( Z^4 )</td>
<td>(^{\text{(3)}}[4N]_2)</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3.3.2. Type II$_{rc}$ (3 eigenvectors; two complex and one null eigenvectors)

The canonical form of the $C_{ab}$ for the parent Type II$_{rc}$ has the form of

$$
C_{ab} = \frac{1}{2} (Z + \bar{Z}) (E'_{1a} E'_{1b} - E'_{3' a} E'_{3'b}) + \frac{i}{2} (Z - \bar{Z}) (E'_{1'a} E'_{3'b} + E'_{3' a} E'_{1'b}) \\
+ R^n(E'_{2'a} E'_{2'b} - E'_{4' a} E'_{4'b}) + \frac{1}{2} (E'_{2'a} E'_{2'b} + E'_{4'a} E'_{4'b} - E'_{2'a} E'_{4'b} - E'_{4'a} E'_{2'b}) \\
= \frac{1}{2} (Z + \bar{Z}) (e_{1a} e_{2b} + e_{2a} e_{1b}) + \frac{i}{2} (Z - \bar{Z}) (e_{1a} e_{1b} - e_{2a} e_{2b}) \\
+ R^n(e_{3a} e_{4b} + e_{4a} e_{3b}) + e_{4a} e_{4b}.
$$

(3.14)

The eigenvectors and corresponding eigenvalues are given by

$$
\frac{1}{\sqrt{2}} (E'_{1} + iE'_{3}) \longleftrightarrow Z , \quad \frac{1}{\sqrt{2}} (E'_{1} - iE'_{3}) \longleftrightarrow \bar{Z} , \\
\frac{1}{\sqrt{2}} (E'_{2} - E'_{4}) \longleftrightarrow R^n.
$$

The conditions for eigenvalues are:

$$
\text{Im}(Z) \neq 0 , \quad R^n + \text{Re}(Z) = 0 .
$$

The Plebański spinors read

$$
V_{ABCD} = 2 \left( \text{Im}(Z) k_{(A} k_{B} - (\text{Im}(Z))^2 + 4(\text{Re}(Z))^2 \right) l_{(A'l_B)} k_{C} k_{D}) , \\
V_{\bar{A}\bar{B}\bar{C}\bar{D}} = 2 \left( -\text{Im}(Z) k_{(\bar{A}} k_{\bar{B}} - (\text{Im}(Z))^2 + 4(\text{Re}(Z))^2 \right) l_{(\bar{A}'l_{\bar{B}})} k_{\bar{C}} k_{\bar{D}}) .
$$

(3.15)
This time, we find the following criterion:

\[
\text{Im}(Z) > 0 \iff V_{ABCD} \text{ is of the type } [II]_r \\
\quad \quad \text{and } V_{\bar{A}\bar{B}\bar{C}\bar{D}} \text{ is of the type } [II]_{rc}, \\
\text{Im}(Z) < 0 \iff V_{ABCD} \text{ is of the type } [II]_{rc} \\
\quad \quad \text{and } V_{\bar{A}\bar{B}\bar{C}\bar{D}} \text{ is of the type } [II]_r. 
\]

(3.16)

**TABLE IX**

Subtypes of the Type $II_{rc}$.

<table>
<thead>
<tr>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td>Type $II$</td>
</tr>
<tr>
<td>$Z_1Z_2Z_3^2$</td>
<td>$[C_1 - C_2 - 2N]_4$</td>
</tr>
<tr>
<td>$Z\bar{Z}R^2$</td>
<td>$[II]<em>r \otimes [II]</em>{rc} [Z - \bar{Z} - 2R^n]_3^{(112)}$</td>
</tr>
<tr>
<td>$[II]_{rc} \otimes [II]_r [Z - \bar{Z} - 2R^n]_3^{(112)}$</td>
<td></td>
</tr>
</tbody>
</table>

3.4. Type III (2 eigenvectors)

3.4.1. Types $III_s$ and $III_t$ (2 eigenvectors; one null and one spacelike or timelike eigenvectors)

The canonical form of the $C_{ab}$ for the parent Type $III_t$ is:

\[
C_{ab} = -R_1^t E_{3'a} E_{3'b} + R_2^n (E_{1'a} E_{1'b} + E_{2'a} E_{2'b} - E_{4'a} E_{4'b}) \\
+ E_{1'a} E_{2'b} + E_{2'a} E_{1'b} - E_{1'a} E_{4'b} - E_{4'a} E_{1'b} \\
= \frac{1}{2} (R_2^n + R_1^t) (e_{1a}e_{2b} + e_{2a}e_{1b}) + \frac{1}{2} (R_2^n - R_1^t) (e_{1a}e_{1b} + e_{2a}e_{2b}) \\
+ R_2^n (e_{3a}e_{4b} + e_{4a}e_{3b}) + e_{1a}e_{4b} + e_{4a}e_{1b} + e_{2a}e_{4b} + e_{4a}e_{2b}. 
\]

(3.17)

The eigenvectors and corresponding eigenvalues are

\[
E_{3'} \longleftrightarrow R_1^t, \quad e_4 \longleftrightarrow R_2^n. 
\]

The eigenvalues have to satisfy the condition

\[
R_1^t + 3R_2^n = 0. 
\]

The Plebański spinors have the following form:

\[
V_{ABCD} = -2 \left( k_{(A} + 8R_2^n l_{(A)} \right) k_B k_C k_D), \\
V_{\bar{A}\bar{B}\bar{C}\bar{D}} = -2 \left( k_{(\bar{A}} + 8R_2^n l_{(\bar{A})} \right) k_{\bar{B}} k_{\bar{C}} k_{\bar{D}}). 
\]

(3.18)
For the nondegenerate Type IIIₜ, these spinors are both of the Petrov–Penrose type $[III]_r$.

The canonical form of the $C_{ab}$ for the parent Type IIIₛ reads

$$C_{ab} = R^s_1 E_1 a E_1 b + R^n_2 (E_2 a E_2 b - E_3 a E_3 b - E_4 a E_4 b)$$
$$+ E_3 a E_2 b + E_2 a E_3 b - E_4 a E_3 b - E_3 a E_4 b$$
$$= \frac{1}{2} (R^s_1 - R^n_2) (e_1 a e_1 b + e_2 a e_2 b) + \frac{1}{2} (R^s_1 + R^n_2) (e_1 a e_2 b + e_2 a e_1 b)$$
$$+ R^n_2 (e_3 a e_4 b + e_4 a e_3 b) + e_1 a e_4 b + e_4 a e_1 b - e_2 a e_4 b - e_4 a e_2 b.$$  \hspace{1cm} (3.19)

The eigenvectors and corresponding eigenvalues are:

$$E_1 \leftrightarrow R^s_1, \quad e_4 \leftrightarrow R^n_2.$$  

The eigenvalues satisfy the relation

$$R^s_1 + 3 R^n_2 = 0.$$  

The Plebański spinors read

$$V_{ABCD} = -2 \left( k_{(A} - 8 R^n_2 l_{(A} \right) k_{B} k_C k_{D)}$$
$$V_{\dot{A}\dot{B}\dot{C}\dot{D}} = -2 \left( \dot{k}_{(\dot{A}} + 8 R^n_2 l_{(\dot{A}} \right) k_{\dot{B}} k_{\dot{C}} k_{\dot{D)}}$$  \hspace{1cm} (3.20)

and they represent the Petrov–Penrose type $[III]_r$.

**TABLE X**

Subtypes of the Types IIIₜ and IIIₛ.

<table>
<thead>
<tr>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td>Type IIIₜ</td>
</tr>
<tr>
<td>$Z_1 Z_2^3$</td>
<td>$[C_1 - 3 N]_4$</td>
</tr>
<tr>
<td>$Z^4$</td>
<td>$[4 N]_3$</td>
</tr>
</tbody>
</table>

$$[III]_r \otimes [III]_r \{ [R^s_1 - 3 R^n_2]_1^{(13)} \}$$
$$\quad \uparrow$$
$$[N]_r \otimes [N]_r \{ [4 R^{nt}_2]_1^{(3)} \}$$

Scheme 6: Degeneration scheme of the Types IIIₜ and IIIₛ.
3.4.2. Type III\(_n\) (2 eigenvectors; both null)

We find here two subtypes. The canonical form of the \(C_{ab}\) reads

\[
C_{ab} = e_{1a}e_{1b} + e_{4a}e_{4b} + R_1^n(e_{3a}e_{4b} + e_{4a}e_{3b}) + R_2^n(e_{1a}e_{2b} + e_{2a}e_{1b}).
\]

The eigenvectors and corresponding eigenvalues are:

\[e_4 \leftrightarrow R^n_1, \quad e_1 \leftrightarrow R^n_2.\]

For the first subtype, the Plebański spinors read

\[
V_{ABCD} = -8(R^n_1)^2 k_{(A}k_{B}l_{C}l_{D)} ,
\]
\[
V_{\hat{A}\hat{B}\hat{C}\hat{D}} = 2(k_{(\hat{A}} + 2R^n_1 l_{(\hat{A})})(k_{\hat{B}} - 2R^n_1 l_{\hat{B})}k_{\hat{C}}k_{\hat{D})}.
\]

Undotted Plebański spinor for the first subtype of the Type III\(_n\) is of the type [D]_{\text{r}} and the dotted one is of the type [II]_{\text{r}}.

The second possibility is

\[
C_{ab} = e_{2a}e_{2b} + e_{4a}e_{4b} + R_1^n(e_{3a}e_{4b} + e_{4a}e_{3b}) + R_2^n(e_{1a}e_{2b} + e_{2a}e_{1b}).
\]

The eigenvectors and corresponding eigenvalues are given by

\[e_4 \leftrightarrow R^n_1, \quad e_2 \leftrightarrow R^n_2.\]

The second subtype is characterized by the following Plebański spinors

\[
V_{ABCD} = 2(k_{(A} + 2R^n_1 l_{(A}))(k_{B} - 2R^n_1 l_{B})k_{C}k_{D}),
\]
\[
V_{\hat{A}\hat{B}\hat{C}\hat{D}} = -8(R^n_1)^2 k_{(\hat{A}k_{\hat{B}l_{\hat{C}l_{\hat{D})}}}
\]

and they are of the Petrov–Penrose types [II]_{\text{r}} and [D]_{\text{r}}, respectively.

The eigenvalues in both subtypes have to satisfy the relation

\[R_1^n + R_2^n = 0.\]
Subtypes of the Type $\text{III}_n$.

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Type $\text{III}_N$</th>
<th>Eigenvalues</th>
<th>Type $\text{III}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1^2 Z_2^2$</td>
<td>$[2N_1 - 2N]^a_4$</td>
<td>$R_1^2 R_2^2$</td>
<td>$[\text{D}] \otimes [\text{II}]_r, [2R_n^1 - 2R_n^2]^{(22)}$</td>
</tr>
<tr>
<td></td>
<td>$[2N_1 - 2N]^b_4$</td>
<td></td>
<td>$[\text{II}]_r \otimes [\text{D}]_r, [2R_n^1 - 2R_n^2]^{(22)}$</td>
</tr>
<tr>
<td>$Z^4$</td>
<td>$(2)[4N]^a_2$</td>
<td>$R^4$</td>
<td>$[-] \otimes [\text{N}]_r, [4R_n^1]^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$(2)[4N]^b_2$</td>
<td></td>
<td>$[\text{N}]_r \otimes [-], [4R_n^1]^{(2)}$</td>
</tr>
</tbody>
</table>

3.5. Type IV (1 null eigenvector)

Finally, for the Type IV, we find two subtypes with canonical forms given by

\[ C_{ab} = e_1 a e_1 b + e_2 a e_4 b + e_4 a e_2 b \]  \hspace{1cm} (3.25)

or

\[ C_{ab} = e_2 a e_2 b + e_1 a e_4 b + e_4 a e_1 b . \]  \hspace{1cm} (3.26)

The eigenvectors and corresponding eigenvalues are:

\[ e_4 \leftrightarrow R^n, \quad R^n = 0. \]

The Plebański spinors for the first subtype read

\[ V_{ABCD} = -2 k_A k_B k_C k_D, \]
\[ V_{\dot{A}\dot{B}\dot{C}\dot{D}} = -4 k_{(\dot{A}\dot{B}\dot{C}\dot{D})}. \]  \hspace{1cm} (3.27)

The Petrov–Penrose types of these spinors are $[\text{N}]_r$ and $[\text{III}]_r$, respectively. For the second subtype, we find

\[ V_{ABCD} = -4 k_{(A} k_B k_C l_{D)} , \]
\[ V_{\dot{A}\dot{B}\dot{C}\dot{D}} = -2 k_{\dot{A}} k_{\dot{B}} k_{\dot{C}} k_{\dot{D}} \]  \hspace{1cm} (3.28)

and the Petrov–Penrose types of Plebański spinors are $[\text{III}]_r$ and $[\text{N}]_r$. 
Subtypes of the Type IV.

<table>
<thead>
<tr>
<th>PP classification</th>
<th>Neutral signature classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td>Type IV</td>
</tr>
<tr>
<td>$Z^4$</td>
<td>$[4N]^a_4$</td>
</tr>
<tr>
<td></td>
<td>$[4N]^b_4$</td>
</tr>
</tbody>
</table>

4. Concluding remarks

In this paper, we analyzed the algebraic structure of the traceless Ricci tensor in 4-dimensional spaces equipped with the metric of the neutral signature. Detailed considerations brought us to the conclusion that there are 33 essentially different types of $C_{ab}$ in such spaces. Our classification is purely algebraic. The alternate way of classification of traceless Ricci tensor in the Lorentzian spaces was given by Penrose [3]. It is an interesting question how the Penrose approach can be used in our case. We are going to study this problem soon.

In our work [12], some of the types of $C_{ab}$ have been related to the existence of the so-called, congruences of the SD null strings. Another way of further investigations is to find a more detailed classification of the congruences of SD null strings and relate such a classification with the types of $C_{ab}$ presented here. This question will be investigated elsewhere.

As mentioned in Introduction, we hope that our present work fills the gap left by two papers by Plebański and Przanowski [2, 6] in Acta Physica Polonica B.

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REFERENCES