

CRITICAL LOOK AT THE TIME–ENERGY UNCERTAINTY RELATIONS*

K. URBANOWSKI

University of Zielona Góra, Institute of Physics
Prof. Z. Szafrana 4a, 65-516 Zielona Góra, Poland

K.Urbanowski@if.uz.zgora.pl
k.a.urbanowski@gmail.com

(Received October 31, 2019)

We analyze the time–energy uncertainty relation postulated by Heisenberg and the derivation of the Mandelstam–Tamm time–energy uncertainty relation. The conclusion is that these relations cannot be considered as universally valid.

DOI:10.5506/APhysPolB.51.71

1. Introduction

Before the emergence of quantum mechanics, physicists were convinced that two different physical quantities can always be measured at the same time with any accuracy. Heisenberg analyzing such quantities as a position and a momentum of the moving electron found that such a belief is wrong on the quantum level, that is, in all cases when a particle manifests its quantum properties [1, 2]. Results of this Heisenberg’s analysis is known as the uncertainty relations. These uncertainty relations describe connections between uncertainties of the position and momentum and also between uncertainties of time and energy [1]. We have a mathematically rigorous derivation of the position–momentum uncertainty relation but so far within the Schrödinger and von Neumann quantum mechanics, there does not exist a rigorous derivation of the time–energy uncertainty relation. Thus, the time-uncertainty relation still requires its analysis and checking whether it is correct and well-motivated by postulates of quantum mechanics. We present here an analysis of the Heisenberg and Mandelstam–Tamm time–energy uncertainty relations and show that they cannot be considered as universally valid. In Sections 2 and 3, the reader finds theory and calculations. Discussion is presented in Section 4. Section 5 contains conclusions.

* Presented at the 3rd Jagiellonian Symposium on Fundamental and Applied Subatomic Physics, Kraków, Poland, June 23–28, 2019.

2. Preliminaries: Uncertainty principle

The most known form of the uncertainty principle is the Heisenberg uncertainty principle for the position and momentum [1, 2]

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad (1)$$

where Δx and Δp_x are Heisenberg's “*uncertainties*”. Unfortunately, there is no precise definition of these “*uncertainties*” in [1]. The rigorous definition of uncertainties was proposed in [3, 4]. Following [3, 4], the uncertainty relation can be written as (see *e.g.* [5])

$$\Delta_\phi x \Delta_\phi p_x \geq \frac{\hbar}{2}, \quad (2)$$

where $\Delta_\phi x$ and $\Delta_\phi p_x$ are the standard (root-mean-square) deviations: In the general case for an observable F , the standard deviation is defined as follows: $\Delta_\phi F = \|\delta F|\phi\rangle\|$, where $\delta F \stackrel{\text{def}}{=} F - \langle F \rangle_\phi \mathbb{I}$, and $\langle F \rangle_\phi \stackrel{\text{def}}{=} \langle \phi|F|\phi \rangle$ is the average (or expected) value of an observable F in a system whose state is represented by the normalized vector $|\phi\rangle \in \mathcal{H}$, provided that $|\langle \phi|F|\phi \rangle| < \infty$. Equivalently: $\Delta_\phi F \equiv \sqrt{\langle F^2 \rangle_\phi - \langle F \rangle_\phi^2}$. (Here, F stands for position and momentum operators x and p_x as well as for their squares.) The observable F is represented by a Hermitian operator F acting in a Hilbert space \mathcal{H} of states $|\phi\rangle$. In general, relation (2) results from basic assumptions of the quantum theory and from the geometry of Hilbert space [6]. Similar relations hold for any two observables, say A and B , represented by noncommuting Hermitian operators A and B acting in the Hilbert space of states (see [3, 4]), such that $[A, B]$ exists and $|\phi\rangle \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, ($\mathcal{D}(\mathcal{O})$ denotes the domain of an operator \mathcal{O} or of a product of operators)

$$\Delta_\phi A \Delta_\phi B \geq \frac{1}{2} \left| \langle [A, B] \rangle_\phi \right|. \quad (3)$$

Inequality (3) is rigorous and its derivation is simple. Indeed, let us consider two observables represented by noncommuting operators A and B . Then, if to apply the definition of δF to operators A and B , respectively, one finds that $[A, B] \equiv [\delta A, \delta B] \neq 0$. Hence, for all $|\phi\rangle \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$,

$$\begin{aligned} |\langle \phi|[A, B]|\phi \rangle|^2 &\equiv |\langle \phi|[\delta A, \delta B]|\phi \rangle|^2 = |\langle \phi|\delta A \delta B|\phi \rangle - (\langle \phi|\delta A \delta B|\phi \rangle)^*|^2 \\ &= 4 |\operatorname{Im} [\langle \phi|\delta A \delta B|\phi \rangle]|^2 \leq 4 |\langle \phi|\delta A \delta B|\phi \rangle|^2 \\ &\leq 4 \|\delta A|\phi\rangle\|^2 \|\delta B|\phi\rangle\|^2 \equiv 4 (\Delta_\phi A)^2 (\Delta_\phi B)^2, \end{aligned} \quad (4)$$

which reproduces inequality (3).

3. Analysis of the Heisenberg and Mandelstam–Tamm time–energy uncertainty relations

Heisenberg in [1] postulated also the validity of a relation analogous to (1) for the time and energy (see also [7]). This relation was a result of his heuristic considerations and it is usually written as follows:

$$\Delta_{\phi}t \Delta_{\phi}E \geq \frac{\hbar}{2}. \quad (5)$$

Similar relation was derived by Mandelstam and Tamm [8]. Their derivation is reproduced in [5] and goes as follows: The operator B in (3) is replaced by the selfadjoint nondepending on time Hamiltonian H of the system considered and $\Delta_{\phi}B$ is replaced by $\Delta_{\phi}H$ and then identifying the standard deviation $\Delta_{\phi}H$ with $\Delta_{\phi}E$, one finds that

$$\Delta_{\phi}A \Delta_{\phi}E \geq \frac{1}{2} \left| \langle [A, H] \rangle_{\phi} \right|, \quad (6)$$

where it is assumed that A does not depend upon the time t explicitly, $|\phi\rangle \in \mathcal{D}(HA) \cap \mathcal{D}(AH)$, and $[A, H]$ exists. Next, using the Heisenberg representation and corresponding equation of motion, one obtains $\langle [A, H] \rangle_{\phi} \equiv i\hbar \frac{d}{dt} \langle A \rangle_{\phi}$. This relation allows one to replace inequality (6) by the following one:

$$\Delta_{\phi}A \Delta_{\phi}E \geq \frac{\hbar}{2} \left| \frac{d}{dt} \langle A \rangle_{\phi} \right|. \quad (7)$$

(Relations (6), (7) are rigorous.) Next, authors of [5, 8] and many others divide both sides of inequality (7) by the term $|\frac{d}{dt} \langle A \rangle_{\phi}|$, which leads to the following relation:

$$\tau_A \Delta_{\phi}E \geq \frac{\hbar}{2}, \quad (8)$$

where $\tau_A \stackrel{\text{def}}{=} \frac{\Delta_{\phi}A}{|\frac{d}{dt} \langle A \rangle_{\phi}|}$. Relation (8) is known as the Mandelstam–Tamm time–energy uncertainty relation. Relation (8) and the above-described derivation of this relation is accepted by many authors analyzing this problem or applying this relation (see, *e.g.* [9–12] and many other papers). On the other hand, there are some formal controversies regarding the role and importance of Δt in (5) or τ_A in (8). These controversies are caused by the fact that in the quantum mechanics, the time t is a parameter. Simply, it cannot be described by the Hermitian operator, say T , acting in the Hilbert space of states (that is time cannot be an observable) such that $[H, T] = i\hbar \mathbb{I}$ if the Hamiltonian H is bounded from below (see [13] and also, *e.g.* [9, 14]).

Therefore, the status of relation (5) and relations (2), (3) is not the same regarding the basic principles of the quantum theory (see also discussion, *e.g.*, in [15–18]).

The Mandelstam–Tamm uncertainty relation (8) is also not free of controversies. People applying and using the above-described derivation of (8) made use (consciously or not) of a hidden assumption that right-hand sides of Eqs. (6), (7) are non-zero, which means that there should not exist any vector $|\phi_\beta\rangle \in \mathcal{H}$ such that $\langle [A, H] \rangle_{\phi_\beta} = 0$, or $\frac{d}{dt} \langle A \rangle_{\phi_\beta} = 0$.

Basic principles of mathematics require that before the dividing the both sides of Eq. (7) by $|\frac{d}{dt} \langle A \rangle_\phi|$, one should check whether $\frac{d}{dt} \langle A \rangle_\phi$ is different from zero or not. Let us do this now: Let $\Sigma_H \subset \mathcal{H}$ be a set of eigenvectors $|\phi_\beta\rangle$ of H for the eigenvalues E_β and let Σ_A denote the set of eigenvectors $|\phi_\alpha\rangle$ for A . We have $H|\phi_\beta\rangle = E_\beta|\phi_\beta\rangle$ for all $|\phi_\beta\rangle \in \Sigma_H$ and, therefore, for all $|\phi_\beta\rangle \in \Sigma_H \cap \mathcal{D}(A)$, $\langle [A, H] \rangle_{\phi_\beta} = i\hbar \frac{d}{dt} \langle A \rangle_{\phi_\beta} \equiv 0$. Similarly, $\Delta_{\phi_\beta} H = \sqrt{\langle H^2 \rangle_{\phi_\beta} - (\langle H \rangle_{\phi_\beta})^2} \stackrel{\text{def}}{=} \Delta_{\phi_\beta} E \equiv 0$, for all $|\phi_\beta\rangle \in \Sigma_H$. This means that in all such cases the non-strict inequality (7) takes the form of the following equality $\Delta_\phi A 0 = \frac{\hbar}{2} 0$. In other words, one cannot divide the both sides of inequality (7) by $|\frac{d}{dt} \langle A \rangle_\phi| \equiv 0$ for all $|\phi_\beta\rangle \in \Sigma_H$, because in all such cases the result is an undefined number and such mathematical operations are unacceptable.

Similar picture one meets when $|\phi\rangle = |\phi_\alpha\rangle$ is an eigenvector for A . (This case was also noticed in [19].) Then also for any $|\phi_\alpha\rangle \in \Sigma_A \cap \mathcal{D}(H)$, $|\frac{d}{dt} \langle A \rangle_\phi| \equiv 0$ and $\Delta_\phi A \equiv 0$. Thus, one finds that $0 \Delta_\phi H = \frac{\hbar}{2} 0$, and once again dividing both sides of this equation by zero has no mathematical sense. Now note that relations (2), (3) are always satisfied for all $|\phi\rangle \in \mathcal{H}$ fulfilling the conditions specified before Eq. (3). In contrast to this property, results $\Delta_\phi A 0 = \frac{\hbar}{2} 0$ and $0 \Delta_\phi H = \frac{\hbar}{2} 0$ mean that we have proved that the Mandelstam–Tamm relation (8) cannot be true not only on the set $\Sigma_H \subset \mathcal{H}$, whose span is usually dense in \mathcal{H} , but also on the set $\Sigma_A \subset \mathcal{H}$.

Hence, the conclusion that such relations as (8) cannot be considered as correct and rigorous seems to be justified: The relations of type (8) cannot hold on the above-described linearly dense sets in the state space \mathcal{H} and, therefore, such relations cannot be considered as universally valid.

4. Discussion

Conclusions presented in the previous section agree with the intuitive understanding of stationary states: If the system is in a stationary state, say $|\phi_\beta\rangle$, then one knows that at any instant of time $\langle E \rangle_\phi \equiv \langle H \rangle_\phi = E_\beta$ and $\Delta_{\phi_\beta} E = \Delta_{\phi_\beta} H = 0$ and thus relations (5) and (8) cannot hold in such a case (see [20]).

In addition to the doubts discussed above and relating to validity of the time–energy uncertainty relations, a thorough analysis of relation (5) suggests one more interpretative ambiguity. Analyzing the ideas expressed in [1, 2], it can be seen that Heisenberg was sure that the time–energy uncertainty relation is a completely general relation and applies in the quantum world without any exceptions. This means that according to Heisenberg’s ideas, this relation should be also valid in the case of photons. Then, let us invoke a much older relation, namely the Planck–Einstein relation: $E_\phi = h\nu_\phi$, (where h is the Planck’s constant and ν_ϕ is the frequency), which constituted one of the foundations enabling the emergence of quantum mechanics. Next, using the $\nu_\phi = \frac{1}{\mathcal{T}_\phi}$, (where \mathcal{T}_ϕ is the period), one can rewrite the Planck–Einstein relation as follows:

$$\mathcal{T}_\phi E_\phi = h > \frac{\hbar}{2}. \quad (9)$$

Note that the inequality $\mathcal{T}_\phi E_\phi > \frac{\hbar}{2}$ coincides with the strong case of the Heisenberg inequality (5). The problem is that relation (9) (and $E_\phi = h\nu_\phi$) combines exact values of time $t = \mathcal{T}_\phi$ and energy E_ϕ (or E_ϕ and ν_ϕ) with each other, while inequality (5) combines uncertainties of time t and energy E . In the light of this analysis, the standard interpretation of the Heisenberg relation (5) may not be obvious and correct. Simply: Equation (9) says that if one finds that the exact value of the energy of the photon in the state $|\phi\rangle$ is E_ϕ , then one is sure that the period is exactly \mathcal{T}_ϕ (or that the frequency is $\nu_\phi = 1/\mathcal{T}_\phi$) and, of course, because the value of E is exact, then in this case, there must be $\Delta_\phi E = 0$. At the same time, the minimal uncertainty version of inequality (5) states that if the value of E is exact and thus $\Delta_\phi E = 0$ then, simultaneously, there must be $\Delta_\phi t = \infty$, which means that it should be impossible to determine the exact value of the period \mathcal{T}_ϕ or frequency ν_ϕ .

As it was mentioned in [20], it is possible to apply relation (8) to unstable states modeled by wave-packets of eigenvectors of H corresponding to the continuous part of the spectrum of H (see also *e.g.* [8, 21–23]), but this is quite another situation than that described by the standard relations (2), (3) — for details, see [20].

5. Conclusions

The analysis and the discussion of relations (5) and (8) in previous sections show that these time–energy uncertainty relations are not well-founded and cannot be considered as universally valid. Thus, using them as the basis for predictions of the properties and of a behavior of some systems in physics or astrophysics (including cosmology — see *e.g.* [12]), one should be very careful in interpreting and applying the obtained results.

REFERENCES

- [1] W. Heisenberg, *Z. Phys.* **43**, 172 (1927); English translation in: *Quantum Theory and Measurements*, J.A. Wheeler, W.H. Zurek (Eds.), Princeton University Press, 1983.
- [2] W. Heisenberg, *The Physical Principles of the Quantum Theory*, Dover Publications Inc., 1930.
- [3] H.P. Robertson, *Phys. Rev.* **34**, 163 (1929).
- [4] E. Schrödinger, *Sitzungsber. Preuss. Akad. Wiss. Berlin* **19**, 296 (1930); English translation: *Bulg. J. Phys.* **26**, 193 (1999) [[arXiv:quant-ph/9903100](#)].
- [5] A. Messiah, *Quantum Mechanics*, vol. I, North Holland Publ. Co., Amsterdam 1962.
- [6] G. Teschl, *Mathematical Methods in Quantum Mechanics (With Applications to Schrödinger Operators)*, *Graduate Studies in Mathematics*, Vol. 99, American Mathematical Society, Providence, Rhode Island 2009.
- [7] M. Jammer, *The Philosophy of Quantum Mechanics*, Wiley, New York 1974.
- [8] L. Mandelstam, I.G. Tamm, *J. Phys. USSR* **9**, 249 (1945); *I.E. Tamm Selected Papers*, B.M. Bolotovskii, V.Ya. Frenkel (Eds.), Springer, Berlin 1991.
- [9] J. Butterfield, *On Time in Quantum Physics*, in: *The Blackwell Companion to the Philosophy of Time*, A. Bardou, H. Dyke (Eds.), Wiley-Blackwell, 2013, pp. 220–241 [[arXiv:1406.4745](#) [physics.hist-ph]].
- [10] M. Bauer, P.A. Mello, *Proc. Nat. Acad. Sci. USA* **73**, 283 (1976).
- [11] E.A. Gislason, N.H. Sabelli, *Phys. Rev. A* **31**, 2078 (1985).
- [12] S. Kalyana Rama, *Phys. Lett. B* **519**, 103 (2001).
- [13] W. Pauli, *General Principles of Quantum Mechanics*, Springer-Verlag, New York 1980, p. 63, footnote 2.
- [14] P. Busch, *The Time–Energy Uncertainty Relation*, in: *Time in Quantum Mechanics*, G. Muga, R. Sala Mayato, I.L. Egusquiza (Eds.), 2nd ed., Springer-Verlag Berlin, Heidelberg 2008.
- [15] Yu.I. Vorontsov, *Sov. Phys. Usp.* **24**, 150 (1981).
- [16] J. Hilgevoord, *Am. J. Phys.* **64**, 1451 (1996).
- [17] J. Hilgevoord, *Am. J. Phys.* **66**, 396 (1998).
- [18] S. Briggs, *J. Phys.: Conf. Ser.* **99**, 012002 (2008).
- [19] J.E. Gray, A. Vogt, *J. Math. Phys.* **46**, 052108 (2005).
- [20] K. Urbanowski, [arXiv:1810.11462](#) [quant-ph]; [arXiv:1908.05273](#) [quant-ph].
- [21] V. Fock, N. Krylov, *JETP* **17**, 93 (1947) [*J. Phys. USSR* **11**, 112 (1947)].
- [22] K. Bhattacharyya, *J. Phys. A: Math. Gen.* **16**, 2993 (1983).
- [23] T.B. Boykin *et al.*, *Eur. J. Phys.* **28**, 673 (2007).