HADAMARD'S PROBLEM OF DIFFUSION OF WAVES*

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The status of Hadamard's problem of diffusion of waves for second order hyperbolic equations of normal hyperbolic type in four independent variables is reviewed wherein the contributions of Myron Mathisson are highlighted. A new family of non-trivial, non-self-adjoint wave equations which satisfy Huygens' principle in the strict sense is given.

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1. Introduction

In a seminal 1939 paper Myron Mathisson [37] made a fundamental contribution to the solution of Hadamard's problem of diffusion of waves first posed in his 1923 Yale Lectures [31]. This problem arises in the study of Cauchy's problem for second order, linear, homogeneous, partial differential equations of normal hyperbolic type in n independent variables. Such an equation may be written in coordinate invariant form as

$$F(u) \equiv g^{ij}\nabla_i\nabla_j u + A^i\nabla_i u + Cu = 0\,, \tag{1.1}$$

where g^{ij} are the contravariant components of the metric tensor $\boldsymbol{g}$ of a Lorentzian space $(M,\boldsymbol{g})$ of signature $2-n$ and ∇_i denotes the covariant derivative with respect to the Lorentzian connection.

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Cauchy's problem for the equation (1.1) is the problem of determining a solution which assumes given values for u and its normal derivative on a given space-like $(n-1)$-dimensional submanifold S. These given values are called the *Cauchy data*. The first general solution to Cauchy's problem for (1.1) was given by Hadamard [31]. Alternate solutions have been presented by Mathisson [36], Sobolev [46], Bruhat [10], and Douglis [25].

The question of how the value of the solution u at a point $x_0 \in M$ depends on the Cauchy data is of considerable interest. Hadamard shows that in general $u(x_0)$ depends on the data on and in the interior of the intersection of the retrograde characteristic conoid $C^-(x_0)$ with the initial surface S. If the solution depends only on the data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$ for every Cauchy problem and for every x_0, one says that the equation satisfies *Huygens' principle* or is a *Huygens equation* or in the terminology of Mathisson an *equation of pure waves.* Examples of such equations are the ordinary wave equations

$$\frac{\partial^2 u}{\partial x^{12}} - \sum_{i=2}^{2m} \frac{\partial^2 u}{\partial x^{i2}} = 0\,, \tag{1.2}$$

in an even number of variables $n = 2m \geqslant 4$.

Hadamard asked the fundamental question: for which equations is Huygens' principle true. This is called *Hadamard's problem* in the literature. He showed that in order for Huygens' principle to be valid it is necessary that n be even and $\geqslant 4$. He further showed that a necessary and sufficient condition for its validity is that the elementary solution contain no logarithmic term. He recognized the limitation of his condition stating ([31], p.236):

> We have said that give *an* answer and not *the* answer, to our question: for it is clear that we can wish it "plus résolu" than it has been in the above. We have enunciated the necessary and sufficient condition, but we do not know how equations satisfying it can be found, or even whether any exist except (e_{2m-1}) (our (1.2)) (and, of course, those that are deduced from (e_{2m-1}) by trivial transformations).

The problem is quite difficult since the condition involves the coefficients of (1.1) in a very indirect and complicated manner. Since none other than (1.2) were known, he suggested that as a first step one should attempt to prove that every Huygens equation is equivalent to some equation of the form (1.2). This suggestion has been called *Hadamard's conjecture* in the literature (see Courant and Hilbert [18], p. 765).

Recall that two equations of the form (1.1) are said to be *equivalent* if they are related by one of the following transformations called *trivial transformations* that preserve the Huygens' property of the equation:

(a) a transformation of coordinates,

(b) multiplication of the equation by a non-vanishing factor $e^{-2\phi}$, where ϕ is a function on M (this transformation induces a conformal transformation of the metric),

(c) replacement of the unknown function u by λu, where λ is a non-vanishing function on M.

An equation (1.1) which is equivalent to an equation (1.2) is said to be *trivial*.

The first significant progress towards the solution of Hadamard's problem was made by Mathisson [37, 38]. Studying the case $n = 4$, he proved Hadamard's conjecture under the assumption that the Lorentzian metric g is flat. He also claimed to have proved the conjecture in the general case $n = 4$ [37]. However, only the proof for the case when g is assumed flat was published before his untimely death in 1940. Mathisson's proof is divided into two parts. He had apparently concluded that the solution to the problem could not be obtained from Hadamard's condition stating in [38] that

> La condition de M. Hadamard ne nous en dit rien; et en générale, il semble fort malaisé d'en tirer des indications plus précises sur la *forme* des équations à ondes pures.

His first step then was to derive an alternate necessary and sufficient condition based on a solution to Cauchy's problem for (1.1) in the case n even that had been given an in earlier paper [36]. In this approach the necessary and sufficient condition for a Huygens equation is that the approximate elementary solution (parametrix) v which he introduces and which reduces the solution of Cauchy's problem to that of an integral equation of Volterra type, is in fact an exact solution; that is it satisfies

$$G(v) = 0\,, \tag{1.3}$$

where

$$G(v) \equiv g^{ij}\nabla_i\nabla_j v - \nabla_i(A^i v) + Cv\,, \tag{1.4}$$

is the adjoint differential operator. The second part of the proof consists in finding the equations (up to equivalence) where this condition is satisfied. It is this part that has had a lasting impact on subsequent research. Mathisson's strategy for the solution of (1.3) is the following:

1. Expand in the neighbourhood of a point x_0 the function v in powers of the spherical polar coordinate r.

2. Calculate, using this expansion, the analogous expansion of $G(v)$.

3. Set to zero the consecutive terms of the expansion of $G(v)$ and utilize each equation thus obtained to simplify the following ones.

4. Profit from the circumstance that the choice of the point x_0 is arbitrary.

He states that it seems reasonable to hope that the first few necessary conditions obtained in this way would be sufficient to completely characterize the coefficients of all equations of pure waves (Huygens equations). Mathisson in addition uses the trivial transformations (a) and (c) to simplify the calculations. Following this plan for the case when

$$g_{ij} = \mathrm{diag}(1, -1, -1, -1)\,, \tag{1.5}$$

he obtains the following conditions from the first three terms in the expansion:

$$C - \frac{1}{2}\frac{\partial A^i}{\partial x^i} - \frac{1}{4}A_i A^i \;=\; 0\,, \tag{1.6}$$

$$\frac{\partial H_i{}^j}{\partial x^j} \;=\; 0\,, \tag{1.7}$$

$$H_{ij} \equiv \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) \;=\; 0\,. \tag{1.8}$$

These conditions are necessarily invariant under the transformations (a) and (c). The equation (1.8) implies that there exists a function λ such that

$$A_i = -2\frac{\partial \log \lambda}{\partial x^i}\,. \tag{1.9}$$

Now the substition (c) transforms (1.1) into an equation of the same form, namely

$$F[u] := g^{ij}\nabla_i\nabla_j u + \overline{A}^i\nabla_i u + \overline{C}u = 0\,, \tag{1.10}$$

where

$$\overline{A}_i = A_i + 2\frac{\partial \log \lambda}{\partial x^i}\,, \tag{1.11}$$

$$\overline{C} = C + \lambda^{-1}\Box\lambda + A^i\frac{\partial \log \lambda}{\partial x^i}\,, \tag{1.12}$$

where $\Box \equiv g^{ij}\nabla_i\nabla_j$ denotes the wave operator. It follows from (1.9), (1.11), and (1.12) that $\overline{A}^i = 0$ and $\overline{C} = 0$, which implies that (1.1) where g is given by (1.5) is equivalent to the ordinary wave equation (1.2) in four dimensions. This completes Mathisson's proof of Hadamard's conjecture in this case.

Hadamard re-visited Mathisson's proof in a 1942 paper [32] dedicated to his memory. He adopts the general approach followed by Mathisson but bases his proof instead on the necessary and sufficient condition that he gives in [31]. Hadamard justifies his choice in the following comment on Mathisson's proof:

> Now, it seemed to me that this roundabout use of an integral equation by introduction of a special parametrix could be avoided, since the exact elementary solution can actually be constructed; and indeed, it seems that the proof becomes simpler this way.

Hadamard's more direct approach leads to a concise proof for the conjecture that has had important consequences for subsquent work on the problem.

Hadamard's conjecture is now known not to be true in general. The first counter-examples were given by Stellmacher [47] in 1953 for the case $n = 6$. In a later paper [48] he gives examples for all even dimensions $n \geqslant 6$. These examples are given by the equation

$$\frac{\partial^2 u}{\partial x^{12}} - \sum_{i=2}^{2m} \frac{\partial^2 u}{\partial x^{i2}} + \left(\frac{\lambda_1}{(x^1)^2} - \sum_{i=2}^{2m} \frac{\lambda_i}{(x^i)^2} \right) u = 0\,, \tag{1.13}$$

where

$$-\lambda_i = \nu_i(\nu_i + 1)\,, \qquad \nu_i = 0, 1, 2, \ldots \tag{1.14}$$

$$\sum_{i=2}^{2m} \nu_i \leqslant m - 2\,. \tag{1.15}$$

For example, when $m = 3$, one possibility is

$$\frac{\partial^2 u}{\partial x^{12}} - \sum_{i=2}^{2m} \frac{\partial^2 u}{\partial x^{i2}} - \frac{2u}{(x^1)^2} = 0\,, \tag{1.16}$$

which is one of the first examples given by Stellmacher. In order to see that the equation (1.13) is not equivalent to the wave equation (1.2) one notes that necessary and sufficient conditions for equivalence are [17, 28]

$$C_{ijkl} \;=\; 0\,, \tag{1.17}$$

$$H_{ij} \;=\; 0\,, \tag{1.18}$$

$$\mathcal{C} := C - \frac{1}{2} A^i{}_{;i} - \frac{1}{4} A_i A^i - \frac{n-2}{4(n-1)} R = 0\,, \tag{1.19}$$

where C_{ijkl} denotes the Weyl conformal curvature tensor, R the curvature scalar associated to the metric g_{ij}, and $;i$ denotes the covariant derivative. The definitions of these quantities are as follows:

$$C_{ijkl} \equiv R_{ijkl} - 2 g_{[i[l} L_{j]k]}\,, \tag{1.20}$$

where R_{ijkl} is the Riemann curvature tensor, $R_{jk} \equiv g^{il} R_{ijkl}$ is the Ricci tensor, $R \equiv g^{jk} R_{jk}$ is the curvature scalar, and $L_{ij} \equiv -R_{ij} + \frac{1}{6} g_{ij} R$. The result then follows, since the conditions (1.17) and (1.18) hold for (1.13), while (1.19) does not since $C \neq 0$. We shall shortly see why a counter-example of the form (1.13) cannot exist for the case $n = 4$.

Counter-examples for $n = 4$ were given by Günther [29] in 1965. These examples are given by the wave equation

$$\Box u = 0 \,, \tag{1.21}$$

on the Lorentzian spaces with metric

$$ds^2 = 2dx^1 dx^2 - a_{\alpha\beta} dx^\alpha dx^\beta \,, \qquad (\alpha, \beta = 3, 4) \,, \tag{1.22}$$

where the symmetric matrix $(a_{\alpha\beta})$ is positive definite with elements that are functions only of x^1. The above metric may interpreted in the framework of general relativity as an exact plane wave solution of the vacuum or Einstein–Maxwell field equations. It has been studied in this context by Ehlers and Kundt [26] in a different coordinate system where it has the form

$$ds^2 = 2dv[du + (Dz^2 + \overline{D}\overline{z}^2 + ez\overline{z})dv] - 2dzd\overline{z} \,, \tag{1.23}$$

where D and $e = \overline{e}$ are functions only of v.

The counter-example of Günther shows that Mathisson's claim [37] that Hadamard's conjecture is true in general for $n = 4$, is false. However, it took forty-six years to determine the status of his claim. At the time of writing, in spite of the considerable progress which has been made, Hadamard's problem for $n = 4$ remains unsolved. For the cases $n = 2m$, $m = 3, 4, 5, \ldots$, Berest [6, 7, 9] has obtained many important results. However, the problem for these cases also remains unresolved. For a review of the status of the problem for $n = 2m$, see Berest and Vasilov [8]. Hadamard's problem has been generalized to systems of equations of type (1.1) including Maxwell's equations and higher spin wave equations by Günther and Wünsch. See Günther [30] and Belger, Schimming and Wünsch [5] for reviews.

The purpose of the present paper is to describe the current status of the problem for $n = 4$, and to present an apparently new non-trivial, non-self-adjoint Huygens equation. We shall follow Mathisson's general strategy for attacking the problem. However, we shall base our calculations on Hadamard's necessary and sufficient condition which is re-written in terms of the theory of distributions. The required theory of elementary solutions in this language and exact form of the necessary and sufficient condition used is given in Section 2. The first six necessary conditions together with a sketch of their derivation is given in Section 3. Consequences of the necessary conditions in the form of five theorems are presented in Section 4. Section 5 contains an apparently new example of a non-trivial, non-self-adjoint Huygens equation. A summary of the status of Hadamard's problem and a suggestion for the direction of future research is given in Section 6. It should be noted that all considerations in this paper are purely local.

2. Elementary solutions

To proceed with the discussion we need to examine Hadamard's necessary and sufficient condition for the validity of Huygens' principle. This condition may be expressed in terms of the *elementary solutions* of (1.1) which are distributions $E^{\pm}_{x_0}(x)$ which satisfy the equation

$$G(E^{\pm}_{x_0}(x)) = \delta_{x_0}(x)\,, \tag{2.1}$$

where $\delta_{x_0}(x)$ is the Dirac delta distribution (see Friedlander [27]). Lichnerowicz [35] has shown that these elementary solutions exist and are unique for C^∞ equations. Furthermore for $n = 4$, they decompose as follows:

$$E^{\pm}_{x_0}(x) = V(x_0,x)\delta^{\pm}(\varGamma(x_0,x)) + \mathcal{V}^{\pm}(x_0,x)\Delta^{\pm}(x_0,x)\,, \tag{2.2}$$

where x and x_0 belong to some simple convex set $\varOmega$ of M. The function V in (2.2) is defined by

$$V(x_0,x) = \frac{1}{2\pi}\exp\left\{-\frac{1}{4}\int\limits_0^{s(x)}(g^{ij}\varGamma_{;ij} - 8 - A^i\varGamma_{,i})\frac{dt}{t}\right\}\,, \tag{2.3}$$

where the integration is along the geodesic joining x_0 and x, $\varGamma(x_0,x)$ is, up to a sign, the square of the geodesic distance between x_0 and x, and s is an affine parameter. The functions $\mathcal{V}^{\pm}$ are defined on the closures of the sets $D^{\pm}(x_0)$ which denote the respective interiors of the future and past pointing characteristic conoids $C^{\pm}(x_0)$, as follows:

$$G(\mathcal{V}^{\pm})(x_0,x) = 0\,, \qquad x \in D^{\pm}(x_0)\,, \tag{2.4}$$

$$\mathcal{V}^{\pm} = \frac{V(x_0,x)}{s(x)}\int\limits_0^{s(x)}\frac{G(V)}{V}dt, x \in C^{\pm}(x_0)\,. \tag{2.5}$$

Finally, we have

$$\delta^{\pm}(\varGamma(x_0,x)) = \begin{cases}\delta(\varGamma(x_0,x))\,, & x \in C^{\pm}(x_0)\,,\\ 0\,, & x \in C^{\mp}(x_0)\,,\end{cases} \tag{2.6}$$

while $\Delta^{\pm}$ denote the characteristic functions on $D^{\pm}(x_0)$.

In terms of the functions involved in the definition of the elementary solutions, Hadamard's necessary and sufficient condition takes the form

$$\mathcal{V}^{\pm}(x_0, x) = 0, \forall x_0 \in M, \forall x \in D^{\pm}(x_0)\,. \tag{2.7}$$

From (2.2) we see that (2.7) is equivalent to the elementary solutions having support only on the characteristic semi-conoids $C^{\pm}(x_0)$. For purposes of calculations a more useful form of the condition (2.7), first given by Hadamard [32], is

$$[G(V)(x_0, x)] = 0\,, \qquad \forall x_0 \in M\,, \tag{2.8}$$

where the brackets $[\ldots]$ signify the restriction of the enclosed function to the set

$$C(x_0) = C^{+}(x_0) \cup C^{-}(x_0)\,. \tag{2.9}$$

The convenience of the condition (2.8) results in part from the fact that the function V defined by (2.3) may be expressed as

$$V(x_0, x) = \frac{1}{2\pi}(\rho(x_0, x))^{-\frac{1}{2}} \exp\left\{\frac{1}{4}\int\limits_0^{s(x)} A^i \Gamma_{,i}\frac{dt}{t}\right\}\,, \tag{2.10}$$

where

$$\rho(x_0, x) \equiv 8(g(x)g(x_0))^{\frac{1}{2}}\left[\det\left(\frac{\partial^2 \Gamma}{\partial x^i \partial x_0^j}\right)\right]^{-1}\,, \tag{2.11}$$

is the so called discriminant function and $g(x) = \det(g_{ij}(x))$.

3. Necessary conditions

Following Mathisson's strategy we now proceed to determine necessary conditions on the coefficients g^{ij}, A^i, and C of (1.1) in order that Huygens' principle be satisfied. For this purpose we use Hadamard's necessary and sufficient condition (2.8). In view of the complexity of the condition the calculations are lengthy and quite involved. We shall now give a brief description of how the conditions are derived, referring the reader to the articles [1, 39, 40] for the details of the calculations.

To begin we shall need the transformation laws for the coefficients of (1.1) under the trivial transformations (b) and (bc) which is a combination of (b) and (c), introduced by Hadamard [32], defined as follows:

(bc) replacement of the function u in (1.1) by λu ($\lambda \neq 0$) and simultaneous multiplication of the equation by λ^{-1}.

(The transformation (bc) has the property of leaving invariant the Lorentzian metric g^{ij}.) The transformations (b) and (bc) transform the differential operator $F(u)$ into an operator $\overline{F}(u)$ of the same form but with different coefficients $\tilde{g}^{ij}$, $\overline{A}^i$, and $\overline{C}$. Explicitly

$$\overline{F}(u) \equiv \tilde{g}^{ij} u_{;ij} + \overline{A}^i u_{,i} + \overline{C} u\,, \tag{3.1}$$

where

$$\tilde{g}^{ij} = e^{-2\phi} g^{ij}\,, \qquad \tilde{g}_{ij} = e^{2\phi} g_{ij}\,, \tag{3.2}$$

$$\overline{A}_i = A_i + 2(\log)_{,i} - (n-2)\phi_{,i}\,, \qquad \overline{A}^i = \tilde{g}^{ij}\overline{A}_i\,, \tag{3.3}$$

$$\overline{C} = e^{-2\phi}\left(C + \lambda^{-1}\Box\lambda + A^i(\log\lambda)_i\right)\,. \tag{3.4}$$

The above transformations induce the following transformation on tensors introduced earlier:

$$\begin{aligned} \tilde{C}^i{}_{jkl} &= C^i{}_{jkl}\,, && (3.5)\\ \overline{H}_{ij} &= H_{ij}\,, && (3.6)\\ \overline{\mathcal{C}} &= e^{-2\phi}\mathcal{C}\,. && (3.7) \end{aligned}$$

The transformation laws for the adjoint differential operator and the elementary solutions are respectively [40]

$$\begin{aligned} \overline{G}(v) &= \lambda e^{-n\phi} G(\lambda^{-1} e^{(n-2)} v)\,, && (3.8)\\ \overline{E}^{\pm}_{x_0}(x) &= \lambda\lambda_0^{-1} e^{(2-n)\phi} E^{\pm}_{x_0}(x)\,, && (3.9) \end{aligned}$$

where $\lambda_0 \equiv \lambda(x_0)$.

In particular when $n = 4$,

$$\overline{E}^{\pm}_{x_0}(x) = \lambda\lambda_0^{-1} e^{-2\phi} E^{\pm}_{x_0}(x)\,. \tag{3.10}$$

It follows from (3.10) that the transformation laws for V and $\mathcal{V}$ are given by

$$\begin{aligned} [\overline{V}] &= \lambda_0^{-1} a_1 [\lambda e^{-2\phi} V]\,, && (3.11)\\ \overline{\mathcal{V}}^{\pm} &= \lambda_0^{-1}\lambda e^{-2\phi}\mathcal{V}^{\pm}\,, && (3.12) \end{aligned}$$

where

$$a_1 = \frac{1}{s}\int_0^{s(x)} e^{2\phi} dt\,. \tag{3.13}$$

Finally we have
$$[G(\overline{V})] = \lambda_0^{-1} a_1 [\lambda e^{-4\phi} G(V)]\,, \tag{3.14}$$
from which it follows that the condition (2.8) is invariant under trivial transformations (as it must be).

If one considers only the transformation (bc), then it follows from (2.3) and (3.3) that
$$\overline{V} = \lambda \lambda_0^{-1} V\,. \tag{3.15}$$
In contrast to (3.12) the above transformation holds at every point in some normal neighbourhood of x_0, not only on $C(x_0)$.

We are now in a position to describe how the necessary conditions are derived. One begins by choosing an arbitrary point $x_0 \in M$. Then a trivial transformation (b) is made such that

$$\overset{o}{\tilde{L}}_{ij} = \overset{o}{\tilde{L}}_{(ij;k)} = \overset{o}{\tilde{L}}_{(ij;kl)} = \ldots = 0\,, \tag{3.16}$$

where the small o over each tensor indicates evaluation at x_0. Next, following Hadamard [32] we choose the trivial transformation (bc) by setting

$$\lambda(x) = \exp\left\{ -\frac{1}{4} \int_0^{s(x)} A^i \Gamma_{,i} \frac{dt}{t} \right\}\,, \tag{3.17}$$

where the tildes have been dropped. It follows that $\lambda_0 = 1$ and that

$$\overline{V}(x_0, x) = \frac{1}{2\pi} \rho^{-\frac{1}{2}}\,. \tag{3.18}$$

The choice (3.17) is equivalent to the requirement that

$$\overline{A}^i \Gamma_{,i} = 0\,. \tag{3.19}$$

Finally the trivial transformation (a) is specified by choosing a system of normal coordinates (x^i) with origin x_0. Recall that these coordinates are defined by the condition [45]

$$g_{ij} x^j = \overset{o}{g}_{ij}\, x^j\,. \tag{3.20}$$

In view of the above choices for the trivial transformations the function V has the particularly simple form

$$V \overset{*}{=} \frac{1}{2\pi} \left(\frac{\overset{o}{g}}{g} \right)^{\frac{1}{4}}\,, \tag{3.21}$$

where the bar has been dropped and $\stackrel{*}{=}$ signifies that the equation holds in normal coordinates. It follows that the condition (2.8) may be expressed as

$$[\sigma(x_0, x)] \stackrel{*}{=} 0\,, \tag{3.22}$$

where (dropping the tildes and bars)

$$\sigma \stackrel{*}{=} \gamma + A^i g^{jk} g_{jk,i} + 4A^i{}_{,i} - 4C\,, \tag{3.23}$$

and where

$$\gamma \stackrel{*}{=} (g^{ij} g^{kl} g_{kl,i})_{,j} + \frac{1}{4} g^{ij} g_{ij,k} g^{kl} g^{mn} g_{mn,l}\,. \tag{3.24}$$

The condition (3.19) now takes the form

$$A_i x^i \stackrel{*}{=} 0\,, \tag{3.25}$$

which is the same as that obtained by Hadamard in the flat space case. Since this condition must hold for all x in some normal neighbourhood, one has at x_0

$$\overset{o}{A}_i = \overset{o}{A}_{(i,j)} = \overset{o}{A}_{(i,jk)} = \ldots = 0\,. \tag{3.26}$$

These are the relations obtained by Mathisson [38] and Günther [28] from (3.3) by a suitable choice of the derivatives of $\log\lambda$ at x_0. However, their true origin seems to a consequence of the choice (3.17). (Günther [28] uses Mathisson's form of the necessary and sufficient condition. In this work he seems to have been unaware of Hadamard's 1942 paper [32].)

Since σ must vanish on $C(x_0)$, the following conditions must be satisfied by σ and its derivatives at x_0:

$$\overset{o}{\sigma} = \overset{o}{\sigma}_{,i} = \mathrm{TS}(\overset{o}{\sigma}_{;ij}) = \mathrm{TS}(\overset{o}{\sigma}_{;ijk}) = \mathrm{TS}(\overset{o}{\sigma}_{;ijkl}) = \ldots = 0\,, \tag{3.27}$$

where TS$(\ldots)$ denotes the trace-free symmetric part of the enclosed tensor. The derivatives of σ at x_0 are calculated in a systematic way from Taylor expansions about x_0 of the tensors g_{ij}, g^{ij}, A^i, and the function C. This has been carried out to fifth order by the second author [1, 39, 40] and to sixth order (in the case $A^i = 0$) by Rinke and Wünsch [44] using the methods of Herglotz [33] and Günther [28].

For the purposes of illustration we shall give these expansions only to second order which is sufficient to enable us to derive the first necessary condition. One has

$$g_{ij} \stackrel{*}{=} \overset{o}{g}_{ij} + \frac{1}{3}\, \overset{o}{R}_{iklj}\, x^{kl}\,, \tag{3.28}$$

$$g^{ij} \stackrel{*}{=} \overset{o}{g}{}^{ij} - \frac{1}{3}\,\overset{o}{R}{}^{i}{}_{kl}{}^{j} x^{kl}\,, \tag{3.29}$$

$$A_i \stackrel{*}{=} \overset{o}{H}_{ij}\, x^j + \frac{2}{3}\,\overset{o}{H}_{ij;k}\, x^{jk}\,, \tag{3.30}$$

$$C \stackrel{*}{=} \overset{o}{C} + \overset{o}{C}_{;i}\, x^i + \overset{o}{C}_{;ij}\, x^{ij}\,, \tag{3.31}$$

where $x^{ij} \equiv x^i x^j$. It follows from the above and (3.23) and (3.24) that

$$\overset{o}{\sigma} \stackrel{*}{=} 4\,\overset{o}{C}\ . \tag{3.32}$$

Thus, on account of (3.27) the first necessary condition has the form

$$\overset{o}{C} = 0\,, \tag{3.33}$$

under our special choice of trivial transformations. In order to express this condition in a form invariant under trivial transformations, it is necessary to find an invariant which reduces to $\overset{o}{C}$ when our special choice of trivial transformations is made. Such a quantity is the Cotton invariant defined by (1.19) which obeys the transformation law (3.7). Thus the general form of (3.33) at x_0 is

$$\overset{o}{\mathcal{C}} \equiv \overset{o}{C} - \frac{1}{2}\,\overset{o}{A}{}^{i}{}_{;i} - \frac{1}{4}\,\overset{o}{A}_i \overset{o}{A}{}^{i} + \frac{1}{6}\,\overset{o}{R} = 0\,.$$

Since x_0 was chosen arbitrarily we must have at every point of M

$$C - \frac{1}{2}A^{i}{}_{;i} - \frac{1}{4}A_i A^i - \frac{1}{6}R = 0\,, \tag{3.34}$$

which is our first necessary condition for a Huygens equation. The subsequent conditions are obtained by similar procedures (see [40] for details) and, following Mathisson's strategy [38], by using the preceding conditions to simplify the following ones.

Each necessary condition must be expressed by the vanishing of a tensor (necessarily trace-free and symmetric) which is invariant under the trivial transformations [40]. In the case of the self-adjoint equation ($A^i = 0$), this involves the study of conformally invariant tensors which are constructed from the metric tensor and its partial derivatives up to a certain order. A theory of these tensors has been developed by Wünsch [50]. This theory and its application to Huygens' principle is described in Günther [30].

We list the first six necessary conditions (I–VI) below.

The history of these conditions is now described. Hölder [34] found Condition I in the case $A^i = C = 0$; Mathisson [37] found it in general. Mathisson [38], Hadamard [32], and Asgeirson [4] obtained the Conditions I, II, and III in the case g^{ij} constant. The Conditions I to IV, for

$$\text{I} \qquad C - \tfrac{1}{2}A^i{}_{;i} - \tfrac{1}{4}A_iA^i - \tfrac{1}{6}R = 0$$

$$\text{II} \qquad H_{ij;}{}^j = 0$$

$$\text{III} \qquad S_{ijk;}{}^k - \tfrac{1}{2}C^k{}_{ij}{}^l L_{kl} + 5\left(H_{ik}H_j{}^k - \tfrac{1}{4}g_{ij}H_{kl}H^{kl}\right) = 0$$

$$\text{IV} \qquad \mathrm{TS}\left(S_{ijl}H_k{}^l + C^l{}_{ij}{}^m H_{kl;m}\right) = 0$$

$$\begin{aligned}\text{V} \qquad \mathrm{TS}(&3C_{mijn;o}C^m{}_{kl}{}^n{}_;{}^o + 8C^m{}_{ij}{}^n{}_{;k}S_{mnl} + 40S_{ij}{}^m S_{klm} \\ &- 8C^m{}_{ij}{}^n S_{mn;kl} - 24C^m{}_{ij}{}^n S_{kl;mn} + 4C^m{}_{ij}{}^n C_n{}^o{}_{km}L_{lo} \\ &+ 12C^m{}_{ij}{}^n C^o{}_{kln}L_{mn} + 12H_{mi;jk}H^m{}_l - 16H_{mi;j}H^m{}_{k;l} \\ &- 84H^m{}_i C_{mjkn}H^n{}_l - 18H_{mi}H^m{}_j L_{kl}) = 0\end{aligned}$$

$$\begin{aligned}\text{VI} \qquad \mathrm{TS}(&36C^n{}_{ij}{}^o C_{oklp;n}H^p{}_m - 6C^n{}_{ij}{}^o{}_{;k}C_{olm}{}^p H_{np} - 138S_{ij}{}^n C_{nklo}H^o{}_m \\ &+ 6S_{ijn}H^n{}_{k;lm} + 6C^n{}_{ij}{}^o{}_{;k}H_{nl;om} - 24S_{ijn;k}H^n{}_{l;m} \\ &+ 12C^n{}_{ij}{}^o L_{nk}H_{ol;m} - 9C^n{}_{ij}{}^o{}_{;k}L_{nl}H_{om} - 9S_{ijn}L_{kl}H^n{}_m) = 0\end{aligned}$$

the general case, were given by Günther [28] and independently by Chevalier [15] for $A^i = C = 0$. McLenaghan [39] obtained Condition V when $R_{ij} = 0$. Subsequently Wünsch [49] gave it when $A^i = 0$. Condition V, in the general case, was found by McLenaghan [40]. Condition VI was obtained by Anderson and McLenaghan [1]. Condition VII for the self-adjoint equation has been given by Rinke and Wünsch [44]. However, it is too lengthy to be given here.

4. Consequences of the necessary conditions

If $R_{ijkl} = 0$, (Minkowski space) and if $A^i = 0$, Condition I implies that $C = 0$. For this reason no non-trivial Huygens equations of the form $\Box u + Cu = 0$, can be constructed in Minkowski space. This fact may explain why it took considerably longer to find a counter-example to Hadamard's conjecture for $n = 4$ than for $n = 6, 8, \ldots$.

If $R_{ijkl} = 0$, the Conditions I to III imply that a Huygens equation is equivalent to the wave equation (1.2) with $m = 2$. This is the result of Mathisson already described in Section 1. The proof depends on the following lemma [28]:

Lemma 4.1. *If*

$$H_{ik}H_j{}^k - \frac{1}{4}g_{ij}H_{kl}H^{kl} = 0\,, \tag{4.1}$$

then (1.1) is equivalent by a transformation (bc) to the conformally invariant equation

$$\Box u + \frac{1}{6}Ru = 0\,. \tag{4.2}$$

Proof. The left hand side of (4.1) may be interpreted as the energy momentum tensor of the "Maxwell field" H_{ij}. It is known that the energy momentum tensor vanishes if and only if the Maxwell field vanishes. (See Mathisson [38] for a proof.) Thus it follows from (1.8) that the one-form $\boldsymbol{A} \equiv A_i dx^i$ is closed and locally exact. Thus there exists a function α such that $\boldsymbol{A} = \mathrm{d}\alpha$. It follows that for the transformation (bc) defined by $\lambda = \exp(-\alpha/2)$, one has $\overline{A}_i = 0$, and hence from Condition I that $\overline{C} = R/6$. □

Hadamard's problem is solved in the case that $(M, \boldsymbol{g})$ is conformally related to an empty space ($R_{ij} = 0$) (vacuum or Ricci flat) by the following result [39]:

Theorem 4.1. *The equation (1.1) on a conformally empty space $(M, \boldsymbol{g})$ is a Huygens equation if and only if it is equivalent to the equation*

$$\Box u = 0\,, \tag{4.3}$$

on a plane wave space with metric given by (1.23).

Remark 4.1. *It should be noted that the curvature scalar R vanishes identically for all plane wave spaces. Furthermore, the plane wave metrics (1.22) or (1.23) are conformally related to empty plane wave metrics which for (1.23) satisfy $e = 0$. In particular Theorem 4.1 holds when $R_{ij} = 0$. This result has important consequences in General Relativity.*

Progress on removing the restriction on the Ricci tensor required by Theorem 4.1 has been difficult. The theorem suggests a revision of Hadamard's conjecture that would state that *any equation (1.1) is a Huygens equation if and only if it is equivalent to the wave equation $\Box u = 0$, on a plane wave space with metric (1.23).* Such a result has been proved in several important cases which will be described in the sequel. However, the existence of an apparently new non-self-adjoint Huygens equation not equivalent to (4.3) on a plane wave space, and which will be presented at the end of this paper, shows that the revised conjecture, like the original one, is not true in general.

The approach used to prove these results is to consider separately each of the five possible Petrov types [24,43] of the Weyl conformal curvature tensor C_{ijkl} of the background space. This is a natural approach since Petrov type is invariant under a general conformal transformation. The possible Petrov types and their characterization are given in the table below.

The vectors l and n in the table below, called *principal null vectors (pnv)*, satisfy

$$g_{ij} l^i l^j = g_{ij} n^i n^j = 0\,. \tag{4.4}$$

Petrov type	Equations satisfied by C_{ijkl}
I	$l_{[i}C_{j]mn[k}l_{l]}l^m l^n = 0$
II	$l_{[i}C_{j]mnk}l^m l^n = 0$
D	$l_{[i}C_{j]mnk}l^m l^n = n_{[i}C_{j]mnk}n^m n^n = 0$
III	$l_{[i}C_{j]mkl}l^m = 0$
N	$C_{ijkl}l^l = 0$
—	$C_{ijkl} = 0$

For type I there are four such vectors which satisfy the equation (each null vector is said to be *simple* in this case), in type II three (one repeated and two simple), in type III two (one repeated and one simple), in type D two repeated, and in type N one repeated null vector. It is worth noting that the Weyl tensor corresponding to the plane wave metric (1.23) is Petrov type N or —.

We also need an invariant classification for the Maxwell tensor H_{ij}. It is said to be *singular* if there exists a null vector l_i such that $H_{ij}l^j = 0$, and *non-singular* if $l_{[i}H_{j]k}l^k = 0$. Such a null vector is called a principal null vector of the Maxwell tensor. A pnv of the Maxwell tensor whose direction coincides with the direction of a pnv of the Weyl tensor is said to be *aligned*.

The conjecture has been proved for type N (the most degenerate) and type III. It has been proved for type D for the conformally invariant equation (4.2). A partial result for the non-self-adjoint equation on type D background spaces has been found. Some preliminary results have been obtained for type II for the conformally invariant equation. The details of these results are given in the following theorems. However, a non-self-adjoint Huygens equation has been found on a type D background space which is not equivalent to (4.3) on a plane wave space. Thus the revised conjecture is not true in general.

Theorem 4.2. *An equation (1.1) on a Petrov type N background space is a Huygens equation if and only if it is equivalent to the wave equation (4.3) on a plane wave space with metric (1.23).*

This theorem is due to Carminati and McLenaghan [11] for the case of the conformally invariant equation (4.2) and to McLenaghan and Walton [41] for the non-self-adjoint equation (1.1).

Theorem 4.3. *There exist no Petrov type III background spaces for which the equation (1.1) is a Huygens equation.*

This theorem is due to Carminati and McLenaghan [13] and Czapor, McLenaghan and Sasse [23] for the case of the conformally invariant equation (4.2) and to Anderson, McLenaghan and Walton [3] and Anderson, McLenaghan and Sasse [2] for the non-self-adjoint equation (1.1).

Theorem 4.4. *There exists no Petrov type D space on which the conformally invariant wave equation (4.2) is a Huygens equation.*

This theorem is due to Carminati and McLenaghan [12], Wünsch [51], and McLenaghan and Williams [42].

Theorem 4.5. *Let (1.1) be any non-self-adjoint equation on a Petrov type D background space. If the Maxwell tensor corresponding to A^i is singular and its principal null direction is aligned with that of one of the repeated principal null directions of the Weyl tensor, then (1.1) is not a Huygens equation.*

This theorem is due to Chu, Czapor and McLenaghan [16].

Theorem 4.6. *The validity of Huygens' principle for the conformally invariant scalar wave equation (1.1) on a Petrov Type II space implies that the repeated principal null congruence of the Weyl tensor defined by the null vector field l^i is geodesic, shear free and hypersurface orthogonal; that is*

$$l_{i;j}l^j = fl_i\,, \qquad l_{(i;j)}l^{i;j} = \frac{1}{2}(l_{;i}l^i)^2\,, \qquad l_{[i;j}l_{k]} = 0\,.$$

This theorem is due to Carminati, Czapor, McLenaghan and Williams [14].

We now give an outline of the proofs of the above theorems.

1. Select a null frame $(l, n, m, \overline{m})$ such that l (and n) is (are) repeated principal null vector field(s) of the Weyl tensor.

2. With the help of the NPspinor package [22] in the computer algebra system Maple, express the Conditions II to VII in terms of the NP spin coefficients and frame components of the tensors H_{ij}, C_{ijkl}, L_{ij}, S_{ijk}, $\ldots$ using a two-component spinor calculus.

3. Use the remaining freedom in the choice of the null frame, the conformal freedom, and the trivial transformation (bc) to simplify as much as possible the system of equations obtained by Step 2.

4. Obtain the integrability conditions for the system of equations resulting from Step 3, the NP field equations, and the NP Bianchi identities with the help of the NP package [21] in Maple.

5. If necessary use Maple's Groebner package [19, 20] to analyze the system of polynomial equations resulting from Steps 4 and 5. Maple's debever package is used if integration of Cartan's first structural equations is required.

The details of the proofs may be found in the cited papers.

5. Example of new non-self-adjoint Huygens equation

In this section we exhibit an apparently new example of a non-self-adjoint Huygens equation that is non-trivial and not equivalent to wave equation (4.3) on a plane wave space. Consider a background space with metric

$$ds^2 = \frac{2dudv}{\left[1 - \frac{1}{8}(R+\beta)uv\right]^2} - \frac{2dzd\overline{z}}{\left[1 + \frac{1}{8}(R-\beta)z\overline{z}\right]^2}\,, \tag{5.1}$$

where R and β are any real constants. Consider also the one-form

$$\begin{aligned}\boldsymbol{A} = {} & \frac{1}{2}(H_3 + \overline{H}_3)(1-\alpha uv)^{-1}(v\mathrm{d}u - u\mathrm{d}v) \\ & + \frac{1}{2}(H_3 - \overline{H}_3)(1+\delta z\overline{z})^{-1}(\overline{z}\mathrm{d}z - z\mathrm{d}\overline{z})\,,\end{aligned} \tag{5.2}$$

where

$$|H_3| = \left(\frac{\beta R}{60}\right)^{\frac{1}{2}}, \qquad \alpha = \frac{1}{8}(R+\beta)\,, \qquad \delta = \frac{1}{8}(R-\beta)\,. \tag{5.3}$$

Then the equation

$$\Box u + A^i \nabla_i u + Cu = 0\,, \tag{5.4}$$

where

$$C = \frac{1}{2}A^i{}_{;i} + \frac{1}{4}A^i A_i + \frac{1}{6}R\,, \tag{5.5}$$

satisfies Huygens' principle if

$$R/\beta = 3/5\,. \tag{5.6}$$

The Huygens' property of (5.4) may be verified by showing that Hadamard's necessary and sufficient condition (2.8) is satisfied. The fact that (5.4) is not a trivial equation follows from the property that the Weyl tensor corresponding to the metric (5.1) vanishes if and only if the Ricci scalar $R = 0$. However, the condition (5.6) shows that $R\beta \neq 0$ which also implies that $d\boldsymbol{A} \neq 0$. Further, (5.4) is not equivalent to (4.3) on a plane wave space, since the Weyl tensor of (5.1) is Petrov type D since $R \neq 0$, while the Weyl tensor of the plane wave metric (1.23) is Petrov type N. Recall that Petrov type is a conformally invariant property. *We thus claim that the equation (5.4) on a background space with metric (5.1), vector field (5.2), and C given by (5.5) is a non-trivial Huygens equation if (5.6) is satisfied. Moreover, it is not equivalent to the equation (4.3) on a plane wave space.* The details of the proof of our claim will be published elsewhere.

6. Conclusion

We have illustrated how Mathisson's strategy together with the Petrov classification has been used to solve Hadamard's problem for the equation (1.1) for $n = 4$. The problem has been completely solved for the degenerate Petrov types III and N. It has also been completely solved for the conformally invariant wave equation (4.2) for Petrov type D. However, the problem remains open for (1.1) on type D background spaces. The existence of a non-trivial, non-self-adjoint Huygens equation which is not equivalent to the wave equation on a plane wave space, shows that the solution may be richer than previously thought. Some preliminary results have been obtained for the conformally invariant equation for type II. However, the complete solution of the equations for type II and those arising from the generic case of type I appear intractable even with the use of powerful computer algebra packages that helped solve the more degenerate cases.

It thus seems that Mathisson's strategy, conceived almost seventy years ago, has been exploited to its fullest possible extent. Further progress on the solution of Hadamard's problem will depend on a new approach. Such an approach might involve a deeper analysis of Hadamard's necessary and sufficient condition on the entire null conoid rather the study of the (infinite) sequence of necessary conditions that arise from it at the vertex which is the essence of Mathisson's approach.

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