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CONGRUENCES OF NULL STRINGS AND THEIR RELATIONS WITH WEYL TENSOR AND TRACELESS RICCI TENSOR*

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4-dimensional spaces equipped with 2-dimensional completely integrable distributions are considered. The integral manifolds of such distributions are totally null and totally geodesics 2-dimensional surfaces (*the null strings*). The relations between congruences (foliations) of the null strings and SD Weyl spinor and traceless Ricci tensor is analyzed. Finally, some explicit Einstein metrics of the spaces which admit the existence of the congruences of the null strings are presented.

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1. Introduction

We analyze 4-dimensional complex manifolds equipped with the holomorphic metric (*complex relativity* abbreviated by **CR**) or 4-dimensional real manifolds equipped with the smooth metric of the neutral signature $(++--)$ (*real ultrahyperbolic relativity* abbreviated by **UR**). Additionally, manifolds are equipped with a 2-dimensional completely integrable distribution. The integral manifolds of such distribution are 2-dimensional, totally null and totally geodesic surfaces, called *the null strings*. The family of such surfaces constitute *the congruence of the null strings*. In what follows, we consider SD congruences (abbreviate by *cns*). The existence of such structures appeared to be very important in the theory of exact solutions of the vacuum Einstein field equations in complex and real para-Hermite and para-Kähler spaces [1]. Moreover, there is a strong influence of the cns on the algebraic structure of the traceless Ricci tensor [2, 7]. In what follows, we use the spinorial formalism (see *e.g.* [5]). All considerations are purely local.

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2. Congruences of the null strings

2.1. Definition and properties

Definition 2.1. Congruence (foliation) of SD null strings in a complex (real neutral) manifold $\mathcal{M}$ is a family of totally null and totally geodesics 2-dimensional holomorphic (real smooth) surfaces, such that for every point $p \in \mathcal{M}$, there exists only one surface of this family such that p belongs to this surface.

It has been proved in [6] that a manifold $\mathcal{M}$ admits a cns if and only if there exists a nowhere vanishing undotted 1-index spinor field m_A such that

$$m^A m^B \nabla_{A\dot{M}} m_B = 0\,. \tag{2.1}$$

Equations (2.1) are called *the SD null string equations*. The cns can be associated with the 2-form Σ, namely $\Sigma = m_A m_B S^{AB}$, where the symmetric spinors S^{AB} constitute the basis of SD 2-forms. In what follows, we abbreviate the cns defined by the 2-form Σ by *Σ-congruence*. From (2.1), we find

$$\nabla_{A\dot{M}} m_B = Z_{A\dot{M}} m_B + \in_{AB} M_{\dot{M}}\,, \tag{2.2}$$

where $Z_{A\dot{M}}$ is the *Sommers vector* and the spinor $M_{\dot{M}}$ is the *expansion of the cns* [6]. With the fixed Riemannian structure, the expansion describes the most important and invariant property of the cns. If $M_{\dot{M}} = 0$, then the 2-dimensional distribution $\mathcal{D}_{m^A} := \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{B}} b^{\dot{B}} \neq 0$ is parallely propagated. It means that $\nabla_X V \in \mathcal{D}_{m^A}$ for every vector field $V \in \mathcal{D}_{m^A}$ and for arbitrary vector field X. Such cns are called *nonexpanding* or *plane*. If $M_{\dot{M}} \neq 0$, then we deal with *expanding* (or *deviating*) cns.

2.2. Relation between cns and SD Weyl spinor and traceless Ricci tensor

From the integrability conditions of Eqs. (2.2), one finds two Theorems [2]:

Theorem 2.2. *If a spinor m_A generates a congruence of SD null strings, then it is a Penrose spinor.* ■

Theorem 2.3. *If a spinor m_A generates a nonexpanding congruence of SD null strings then (i) m_A is a multiple Penrose spinor and (ii) the SD Weyl spinor is of the types [II,D] iff the curvature scalar $R \neq 0$ and of the types [III,N,−] iff $R = 0$.* ■

The spinorial image $C_{AB\dot{M}\dot{N}}$ of the traceless Ricci tensor C_{ab} reads

$$C_{AB\dot{M}\dot{N}} = m_A m_B A_{\dot{M}\dot{N}} + 2m_{(A}\mu_{B)} B_{\dot{M}\dot{N}} + \mu_A \mu_B C_{\dot{M}\dot{N}}\,, \tag{2.3}$$

where m_A and μ_A constitute the base of undotted 1-index spinors, $\mu^A m_A = 1$. From the integrability conditions of Eqs. (2.2), we find

$$2B_{\dot{A}\dot{B}} = \mu_N \left(Z^N{}_{(\dot{A}} M_{\dot{B})} - \nabla^N{}_{(\dot{A}} M_{\dot{B})} \right) + \nabla_{N(\dot{A}} Z^N{}_{\dot{B})} \, , \tag{2.4a}$$

$$2C_{\dot{A}\dot{B}} = m_N \left(\nabla^N{}_{(\dot{A}} M_{\dot{B})} - Z^N{}_{(\dot{A}} M_{\dot{B})} \right) . \tag{2.4b}$$

The spinor $A_{\dot{A}\dot{B}}$ is not determined. Let us assume now that the congruence of SD null strings is nonexpanding what implies $C_{\dot{A}\dot{B}} = 0$. Then we find the characteristic polynomial of the C_{ab} in the form of

$$\mathcal{W}(x) := \det(C^a{}_b - x\delta^a{}_b) = \left(x^2 + 2b\right)^2 \, , \qquad \text{where} \quad b := B_{\dot{A}\dot{B}} B^{\dot{A}\dot{B}} \, . \tag{2.5}$$

From (2.5), one finds the following Theorem:

Theorem 2.4 (Przanowski, [7]). *In complex spaces, the existence of nonexpanding congruence of SD null strings implies that the characteristic polynomial of traceless Ricci tensor has two double or one quadruple eigenvalue, namely $\lambda = \pm\sqrt{-2b}$.* ■

Theorem 2.4 holds true in **UR**. Moreover, there is a significant relation between the existence of the nonexpanding congruences of SD null strings and the existence of the null eigenvectors of the traceless Ricci tensor. Namely, one finds [2]:

Theorem 2.5. *Let $\mu_A r_{\dot{B}}$ be a null eigenvector of the traceless Ricci tensor and let m_A be any spinor such that $m_A \mu^A \neq 0$. Then $m_A r_{\dot{B}}$ is also a null eigenvector of the traceless Ricci tensor iff $C_{\dot{A}\dot{B}} r^{\dot{B}} = 0$, where $C_{\dot{A}\dot{B}}$ is defined by m_A and μ_A according to (2.3).* ■

Obviously, Theorem 2.5 holds true for the nonexpanding congruences of SD null strings. Using the discrete and continuous characteristic of the traceless Ricci tensor in **CR** [8] and in **UR** [3], one arrives at Table I. In Table I, we define $a := A_{\dot{A}\dot{B}} A^{\dot{A}\dot{B}}$ and $r := A_{\dot{A}\dot{B}} B^{\dot{A}\dot{B}}$, and we put the information about the number of the null eigenvectors of the traceless Ricci tensor. If there exists only one null eigenvector, then it must be tangent to the null string. For the two null eigenvectors, we find that they can be both tangent to the null string (2^{ss}) or only one of them is tangent to the null string (2^{sn}). If there are three or four null eigenvectors, then exactly two of them are tangent to the null string.

TABLE I

Possible types of the traceless Ricci tensor admitted in the spaces equipped with exactly one nonexpanding congruence of the SD null strings.

Criteria			Types in **UR**		Types in **CR**	
$b \neq 0$	$ab \neq r^2$		$^{[\mathrm{II}]_r \otimes [\mathrm{D}]_r}[2R_1^n - 2R_2^n]^2_{(22)}$	2^{ss}	$[2N_1 - 2N]^b_4$	2^{ss}
	$ab = r^2\,,\ b > 0$		$^{[\mathrm{D}]_r \otimes [\mathrm{D}]_c}\left[2Z - 2\bar{Z}\right]^4_{(11)}$	0	$[2N_1 - 2N]_2$	4
	$ab = r^2\,,\ b < 0$		$^{[\mathrm{D}]_c \otimes [\mathrm{D}]_c}[2R_1^s - 2R_2^t]^4_{(11)}$	0	$[2N_1 - 2N]_2$	4
			$^{[\mathrm{D}]_r \otimes [\mathrm{D}]_r}[2R_1^{nst} - 2R_2^{nst}]^4_{(11)}$	4		
			$^{[\mathrm{D}]_r \otimes [\mathrm{D}]_r}[2R_1^{nst} - 2R_2^n]^3_{(12)}$	3	$[2N_1 - 2N]_{(1-2)}$	3
$b = 0$	$B_{\dot{A}\dot{B}} \neq 0$	$r \neq 0$	$^{[\mathrm{III}]_r \otimes [\mathrm{N}]_r}[4R^n]^1_{(4)}$	1	$[4N]^b_4$	1
		$r = 0\,,\ a \neq 0$	$^{[\mathrm{N}]_r \otimes [\mathrm{N}]_r}[4R^{nt}]^2_{(3)}$	1	$[4N]_3$	1
			$^{[\mathrm{N}]_r \otimes [\mathrm{N}]_r}[4R^{ns}]^2_{(3)}$	1		
		$r = a = 0$	$^{[-] \otimes [\mathrm{N}]_r}[4R^n]^2_{(2)}$	2^{sn}	$^{(2)}[4N]^a_2$	2^{sn}
	$B_{\dot{A}\dot{B}} = 0$	$r = 0\,,\ a \neq 0$	$^{[\mathrm{N}]_r \otimes [-]}[4R^n]^2_{(2)}$	2^{ss}	$^{(2)}[4N]^b_2$	2^{ss}
		$r = a = 0$	$^{[-] \otimes [-]}[4R^{nst}]^4_{(1)}$	4	$[4N]_1$	4
			$^{[-] \otimes [-]}[4R^{nst}]^3_{(2)}$	3	$^{(3)}[4N]_2$	3

2.3. Two complementary congruences of the SD null strings

Now we assume the existence of two complementary cns. Two cns Σ and $\widetilde{\Sigma}$ are *complementary (transversal)*, if $\Sigma \wedge \widetilde{\Sigma} \neq 0$. In this case, SD Weyl spinor cannot be of the Petrov–Penrose type [N] [2]. If both congruences are nonexpanding, then we arrive at the following theorem:

Theorem 2.6. *If the complex space admits two complementary congruences of SD null strings and both of them are nonexpanding, then SD Weyl spinor is of the type* [D] *iff* $R \neq 0$ *and of the type* $[-]$ *iff* $R = 0$. ■

Interesting fact is that the existence of two congruences of SD null strings completely determines the form of the traceless Ricci tensor. Except (2.4a) and (2.4b), we find that expression for $A_{\dot{B}\dot{C}}$ ($\widetilde{M}_{\dot{B}}$ is the expansion of the $\widetilde{\Sigma}$-congruence)

$$2A_{\dot{A}\dot{B}} = \mu_N \nabla^N{}_{(\dot{A}} \widetilde{M}_{\dot{B})} + \mu_N Z^N{}_{(\dot{A}} \widetilde{M}_{\dot{B})} + \widetilde{M}_{(\dot{A}} \widetilde{M}_{\dot{B})}\,. \tag{2.6}$$

If C_{ab} is generated by the two nonexpanding cns, we have $A_{\dot{A}\dot{B}} = 0 \Longrightarrow a = r = 0$ and the following theorem holds true [2]:

Theorem 2.7. *Let us assume, that the traceless Ricci tensor generated by two complementary, nonexpanding congruences of SD null strings admits a null eigenvector. Then there are two linearly-independent null eigenvectors, the first is tangent to the Σ-congruence, the second to the $\widetilde{\Sigma}$-congruence or there are four linearly-independent null eigenvectors, the first pair is tangent to the Σ-congruence, the second pair to the $\widetilde{\Sigma}$-congruence.* ■

It is easy to extract from Table I possible types of the C_{ab} generated by the two nonexpanding cns. There are 5 such types in **UR** and 3 in **CR**.

3. Examples

In this section, we present some examples of the metrics which admit the existence of cns. As a first example, we consider the space of the types [D] ⊗ [II] with nonzero cosmological constant Λ admitting two nonexpanding congruences of SD null strings and one nonexpanding congruence of ASD null strings. This metric belongs to the para-Kähler class and it reads

$$\mathrm{d}s^2 = 2(\Omega_p \mathrm{d}p + \Omega_q \mathrm{d}q)\mathrm{d}x + 2(\Sigma_x \mathrm{d}x + \Sigma_y \mathrm{d}y)\mathrm{d}p\,, \tag{3.1}$$

where (x,y,q,p) are local coordinates and $\Omega = \Omega(x,p,q)$ and $\Sigma = \Sigma(x,y,p)$ are arbitrary functions, $\Omega_p := \partial\Omega/\partial p$, *etc.* In the Einstein case ($C_{ab} = 0$, $R = -4\Lambda$), metric (3.1) takes the form of

$$\begin{aligned}\mathrm{d}s^2 &= \frac{2}{\Lambda}\Bigg(\partial_x\partial_p \ln\frac{(Q+X)^2(M+N)^2}{X_x N_p}\,\mathrm{d}p\,\mathrm{d}x + \partial_x\partial_q \ln\frac{(Q+X)^2}{X_x}\,\mathrm{d}q\,\mathrm{d}x \\ &\quad + \partial_p\partial_y \ln\frac{(M+N)^2}{N_p}\,\mathrm{d}p\,\mathrm{d}y\Bigg)\,,\end{aligned} \tag{3.2}$$

where $Q = Q(q,p)$, $X = X(x,p)$, $M = M(y,x)$ and $N = N(p,x)$ are arbitrary functions.

The second example is the space of the type [D] ⊗ [D] with nonzero cosmological constant Λ equipped with two SD and two ASD nonexpanding congruences of null strings. The metric reads now

$$\mathrm{d}s^2 = 2e^F \mathrm{d}y\,\mathrm{d}q + 2e^G \mathrm{d}x\,\mathrm{d}p\,, \tag{3.3}$$

where (x,y,q,p) are local coordinates and $F = F(y,q)$ and $G = G(x,p)$ are arbitrary functions. The curvature and traceless Ricci tensor read

$$\begin{aligned} C^{(3)} &= \dot{C}^{(3)} = \frac{R}{6}\,, \qquad R = 2F_{yq}\,e^{-F} + 2G_{xp}\,e^{-G}\,, \\ 2C_{12} &= -2C_{34} = F_{yq}\,e^{-F} - G_{xp}\,e^{-G}\,. \end{aligned} \tag{3.4}$$

The only possible types of C_{ab} admitted by metric (3.3) are ${}^{[\mathrm{D}]_c\otimes[\mathrm{D}]_c}[2R_1^s - 2R_2^t]^4_{(11)}$, ${}^{[\mathrm{D}]_r\otimes[\mathrm{D}]_r}[2R_1^{nst} - 2R_2^{nst}]^4_{(11)}$ and ${}^{[-]\otimes[-]}[4R^{nst}]^4_{(1)}$ in **UR** and the types $[2N_1 - 2N]_2$ and $[4N]_1$ in **CR**. In the Einstein case, metric (3.3) reads

$$\mathrm{d}s^2 = \frac{2\mathrm{d}q\,\mathrm{d}y}{\left(1+\frac{\Lambda}{2}qy\right)^2} + \frac{2\mathrm{d}p\,\mathrm{d}x}{\left(1+\frac{\Lambda}{2}px\right)^2} \tag{3.5}$$

and the curvature coefficients are

$$C^{(3)} = \dot{C}^{(3)} = -\frac{2\Lambda}{3}\,. \tag{3.6}$$

Metric (3.5) is the most general metric of the homogeneous Einstein para-Kähler spaces [4]. All metrics presented here are the general solutions of the considered problems. More examples can be found in [1].

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