ALPHA DECAY OF DEFORMED EVEN–EVEN NUCLEI*

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(Received January 17, 2017)

Using the decay theory of Goldberger and Watson, we analyzed the α decay of even–even axially symmetric nuclei. The decay is regarded as a transition, caused by any residual interaction, between the bound state of the parent nucleus and the continuous spectrum. In addition, the corrections to nuclear and Coulomb interactions due to deformation are also treated as a perturbation. Basis wave functions of the continuous spectrum, calculated in quasi-classical approximation, are shown to have inside the nucleus the amplitude of the order of square root from the transmission coefficient through the Coulomb barrier. General formula is derived for the α-decay rate, which correlates with standard result in the case of transitions between the ground states of even–even nuclei.

DOI:10.5506/APhysPolBSupp.10.69

1. Introduction

Alpha decay is usually described (see, *e.g.*, [1, 2]) in the framework of Gamov's model [3], in which Gamov assumed that at the initial moment $t = 0$ inside the nucleus, there is an α particle, described by the wave packet with wave vectors, which satisfy the Bohr–Sommefeld condition for a quasi-bound state in the nuclear potential well. Respectively, the α-decay rate is determined by the oversimplified formula

$$\lambda = p\nu e^{-2S}\,, \tag{1}$$

where p means the formation probability for the α particle, ν the assault frequency, and e^{-2S} the transmission probability through the Coulomb barrier.

However, the most direct description of any decay process, including α or cluster decay, is provided by the decay theory, given in the book [4]. Applying such an approach, we consider the α decay as a transition between

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* Presented at the XXIII Nuclear Physics Workshop "Marie and Pierre Curie", Kazimierz Dolny, Poland, September 27–October 2, 2016.

the initial bound state φ_a of the parent nucleus and the final states of continuous spectrum φ_b. In the initial moment $t = 0$, there is only the parent nucleus, described by the wave function $\Psi(0) = \varphi_a$, without any hint on the α particle. Afterwards, the wave function of the nuclei $\Psi(t)$ attributes components $\sim \varphi_b$, whereas the amplitude of the component φ_a exponentially attenuates. We shall see that the decay rate $\lambda \sim e^{-2S}$ only far from the Bohr–Sommerfeld restriction. We deal with the decay into levels of the rotational band of the daughter even–even nucleus, having axially symmetric deformed shape ($\gamma = 0$).

Below, we consider the nuclei surrounded by electrons. The Coulomb potential of the spherical nucleus with the charge number Z, screened by electrons, may be written as [5]

$$\Phi^N_{\rm scr}(r) = \frac{Ze}{r} + \Phi_e(r) = \frac{Ze}{r} e^{-r/r_{\rm s}}\,, \tag{2}$$

where $r_{\rm s}$ is the screening radius, $\Phi_e(\boldsymbol{r})$ the potential due to electrons

$$\Phi_e(r) = -\frac{Ze}{r}\left(1 - e^{-r/r_{\rm s}}\right)\,. \tag{3}$$

The energies of the parent and the daughter nuclei in the electronic environment are then given by

$$\epsilon_a = M_{\rm p}c^2 + \mathcal{E}_{I_{\rm p}} - \frac{Z^2e^2}{r_{\rm s}} \tag{4}$$

and

$$\epsilon_b = (M_{\rm d} + M_\alpha)c^2 + \mathcal{E}_{I_{\rm d}} - \frac{Z(Z-2)e^2}{r_{\rm s}} + E\,, \tag{5}$$

where $M_{\rm p(d)}$ and M_α are the masses of the parent (daughter) nuclei and the α particle, respectively, $\mathcal{E}_{I_{\rm p}(I_{\rm d})}$ are the excitation energies of the parent (daughter) nuclei in the states $|I_{\rm p}M_{\rm p}\rangle(|I_{\rm d}M_{\rm d}\rangle)$ with spin I and its projection M, E is the energy of the relative motion of the α particle and the daughter nucleus, and $\boldsymbol{r}$ is their relative radius-vector. According to the energy conservation law $\epsilon_a = \epsilon_b$, one has

$$E \approx E_{I_{\rm d}} = Q_{I_{\rm d}} - \frac{2Ze^2}{r_{\rm s}}\,, \tag{6}$$

where the nuclear energy released during the decay is

$$Q_{I_{\rm d}} = M_{\rm p}c^2 - (M_{\rm d} + M_\alpha)c^2 + \mathcal{E}_{I_{\rm p}} - \mathcal{E}_{I_{\rm d}}\,. \tag{7}$$

Lowering of E by $2Ze^2/r_{\rm s}$ almost completely compensates narrowing of the Coulomb barrier by the electronic screening [6]. More exact analysis of the role of electronic screening has been given in [7].

2. Main definitions

We treat the nucleus as a uniformly charged ellipsoid, whose surface is drawn up by the radius-vector

$$R(\hat{\boldsymbol{r}}) = R_0 \left[1 + \beta \sum_{\mu=-2}^{2} D^2_{\mu 0}(\theta) Y_{2\mu}(\hat{\boldsymbol{r}}) \right] , \tag{8}$$

where β is the quadrupole deformation parameter, $D^2_{\mu 0}(\theta)$ is the rotation matrix, depending on the Eulerian angles $\theta = \theta_1, \theta_2, \theta_3$. The symbols $\hat{\boldsymbol{r}}$ in (8) and $\hat{\boldsymbol{\kappa}}$ in (23) denote spherical angles for the vectors $\boldsymbol{r}$ and $\boldsymbol{\kappa}$, respectively.

The α particle is affected by both the nuclear $V_{\rm N}(\beta;\theta,\boldsymbol{r})$ and Coulomb $V_{\rm C}(\beta;\theta,\boldsymbol{r})$ interactions, depending on the deformation parameter β. Since $\beta \ll 1$, it is convenient to divide the interactions $V_{\rm N(C)}(\beta;\theta,\boldsymbol{r})$ into the large central potential $V^{(0)}_{\rm N(C)}(r)$, associated with the spherical shape ($\beta = 0$) and a small correction $\delta V_{\rm N(C)}(\boldsymbol{r})$ due to deformation

$$V_{\rm N(C)}(\beta;\theta,\boldsymbol{r}) = V^{(0)}_{\rm N(C)}(r) + \delta V_{\rm N(C)}(\theta,\boldsymbol{r}) \, . \tag{9}$$

We approximate the nuclear interaction acting within the volume of the nucleus, $0 \le r < R(\hat{\boldsymbol{r}})$, by the potential well of the depth U_0

$$V^{(0)}_{\rm N}(r) = \begin{cases} -U_0 \, , & 0 \le r < R_0 \, , \\ 0 \, , & r > R_0 \, , \end{cases} \tag{10}$$

where

$$R_0 = r_0 \left[(A-4)^{1/3} + 4^{1/3} \right] , \tag{11}$$

A is the mass number of the parent nucleus and $r_0 = 1.22$ fm [2].

The nuclear perturbation should be equal to $-U_0$ for $R_0 \le r < R(\hat{\boldsymbol{r}})$ and U_0 for $R(\hat{\boldsymbol{r}}) \le r < R_0$, *i.e.*,

$$\delta V_{\rm N}(\theta,\boldsymbol{r}) = U_0 \left[\Theta(R_0 - r)\Theta(r - R(\hat{\boldsymbol{r}})) - \Theta(r - R_0)\Theta(R(\hat{\boldsymbol{r}}) - r) \right] , \tag{12}$$

where

$$\Theta(x) = \begin{cases} 1 \, , & x > 0 \, , \\ 0 \, , & x < 0 \end{cases} \tag{13}$$

is the Heaviside step function.

As to the spherically symmetric part of the Coulomb interaction, it is given by

$$V^{(0)}_{\rm C}(r) = \frac{(Z-2)e^2}{R_0} \left[3 - \frac{r^2}{R_0^2} \right] \Theta(R_0 - r) + \frac{2(Z-2)e^2}{r} e^{-r/r_{\rm s}} \Theta(r - R_0) \, , \tag{14}$$

whereas the correction in the linear approximation in β is [8]

$$\delta V_{\mathrm{C}}(\boldsymbol{r}) = \frac{6(Z-2)e^2}{5}\left[\frac{r^2}{R_0^3}\Theta(R_0-r)+\frac{R_0^2}{r^3}e^{-r/r_{\mathrm{s}}}\Theta(r-R_0)\right] \times\beta\sum_{\mu=-2}^{2}D^2_{\mu 0}(\theta)Y_{2\mu}(\hat{\boldsymbol{r}})\,. \tag{15}$$

The Hamiltonian of the nucleus in the α channel is written as

$$H = H_0 + V\,, \tag{16}$$

where H_0 is the unperturbed Hamiltonian, while the perturbation is

$$V = V' + \delta V(\theta,\boldsymbol{r})\,, \tag{17}$$

where V' is a residual interaction of the nucleons providing α decay, and

$$\delta V(\theta,\boldsymbol{r}) = \delta V_{\mathrm{C}}(\theta,\boldsymbol{r}) + \delta V_{\mathrm{N}}(\theta,\boldsymbol{r})\,. \tag{18}$$

In the α channel, the operator H_0 contains the following terms:

$$H_0 = K + V^{(0)}(r)\,,\qquad V^{(0)}(r) = V_{\mathrm{N}}^{(0)}(r) + V_{\mathrm{C}}^{(0)}(r)\,. \tag{19}$$

Here, $\hat{K}$ is a sum of the kinetic energy operator of the relative motion of the α particle with respect to the daughter nucleus and the Hamiltonians of their internal motion

$$K = -\frac{\hbar^2}{2\mu}\varDelta_{\boldsymbol{r}} + H_{\mathrm{in}}^{(\alpha)} + H_{\mathrm{in}}^{(\mathrm{d})}\,, \tag{20}$$

the reduced mass $\mu = M_{\mathrm{d}}M_\alpha/(M_{\mathrm{d}}+M_\alpha)$.

3. Basis wave functions

The eigenfunction φ_b^+ of H_0 factorizes as follows:

$$\varphi_b^+ = \psi_{\boldsymbol{\kappa}}^+(\boldsymbol{r})|I_{\mathrm{d}}M_{\mathrm{d}}\rangle\,, \tag{21}$$

where the nuclear wave functions for the rotational levels are

$$|I_{\mathrm{d}}M_{\mathrm{d}}\rangle = \sqrt{\frac{2I_{\mathrm{d}}+1}{8\pi^2}}D^{I_{\mathrm{d}}\,*}_{M_{\mathrm{d}}0}(\theta)\,. \tag{22}$$

The wave function of the relative motion $\psi^{+}_{\boldsymbol{\kappa}}(\boldsymbol{r})$ is represented by the expansion in partial waves [4]

$$\psi^{+}_{\boldsymbol{\kappa}}(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^{l} e^{i\delta_l(\kappa)} \frac{w_l(\kappa;r)}{\kappa r} Y^{*}_{lm}(\hat{\boldsymbol{\kappa}}) Y_{lm}(\hat{\boldsymbol{r}}) \,, \tag{23}$$

depending on the phase shift $\delta_l(\kappa)$. Here, the radial functions $w_l(\kappa;r)$ satisfy the equation

$$w_l''(\kappa;r) - \left[\frac{l(l+1)}{r^2} + v(r) - \kappa^2\right] w_l(\kappa;r) = 0 \,, \tag{24}$$

where the reduced potential

$$v(r) = \frac{2\mu}{\hbar^2} V^{(0)}(r) \,. \tag{25}$$

At large r, where $l(l+1)/r^2 + v(r) \approx 0$, their asymptotic is [4]

$$w_l(\kappa;r) \approx \sqrt{\frac{2}{\pi}} \sin\left(\kappa r - \frac{l\pi}{2} + \delta_l(\kappa)\right) \,. \tag{26}$$

It is useful to introduce the total angular momentum $\boldsymbol{I} = \boldsymbol{I}_{\mathrm{d}} + \boldsymbol{l}$ as well as the eigenfunctions of the squared angular momentum $\boldsymbol{I}^2$ and its projection on the quantization axis I_z

$$\mathcal{Y}^{M}_{IlI_{\mathrm{d}}}(\theta;\hat{\boldsymbol{r}}) = \sum_{mM_{\mathrm{d}}} (lI_{\mathrm{d}}mM_{\mathrm{d}}|IM) Y_{lm}(\hat{\boldsymbol{r}}) |I_{\mathrm{d}}M_{\mathrm{d}}\rangle \,, \tag{27}$$

where $(j_1 j_2 m_1 m_2|jm)$ are the Clebsh–Gordan coefficients. Then the wave function (21), (23) is rewritten as

$$\varphi^{+}_{b} = \sum_{IM}\sum_{l=0}^{\infty} \frac{w_l(\kappa;r)}{\kappa r} \mathfrak{Y}^{M*}_{I}(lI_{\mathrm{d}}M_{\mathrm{d}};\hat{\boldsymbol{\kappa}}) \mathcal{Y}^{M}_{IlI_{\mathrm{d}}}(\theta;\hat{\boldsymbol{r}}) \,, \tag{28}$$

where the following notation is introduced

$$\mathfrak{Y}^{M}_{I}(lI_{\mathrm{d}}M_{\mathrm{d}};\hat{\boldsymbol{\kappa}}) = i^{-l} e^{-i\delta_l} \sum_{m=-l}^{l} (lI_{\mathrm{d}}mM_{\mathrm{d}}|IM) Y_{lm}(\hat{\boldsymbol{\kappa}}) \,. \tag{29}$$

4. Quasi-classical approximation

Let us solve the radial equation (24) in the quasi-classical (WKB) approximation, using Langer's substitution [9]

$$\kappa r = e^{x}\,, \qquad w_l(\kappa;r) = e^{x/2} y_l(x)\,, \tag{30}$$

where x varies on the whole axis from $-\infty$ to ∞. Then Eq. (24) transforms to

$$y_l''(x) + q^2(x) y_l(x) = 0 \tag{31}$$

with

$$q^2(x) = e^{2x}\left(1 - \frac{v(x)}{\kappa^2}\right) - \left(l + \frac{1}{2}\right)^2 . \tag{32}$$

The classical turning points x_1, x_2 and x_3 are now the roots of the equation $q(x) = 0$. They are related, respectively, to the points a, R_0 and b on the axis r, for which the wave vector

$$k(r) = \sqrt{\kappa^2 - (l+1/2)^2/r^2 - v(r)} \tag{33}$$

vanishes.

Under the centrifugal barrier on the left-hand side of the turning point x_1, the regular WKB wave function is represented by the attenuating exponent

$$y_l(x) = \frac{C_l}{\sqrt{|q(x)|}} \exp\left(-\int_x^{x_1} \left|q\left(x'\right)\right| \mathrm{d}x'\right), \qquad -\infty < x < x_1\,. \tag{34}$$

Using standard matching rules, one finds the function in the nuclear potential well, where $x_1 < x < x_2$

$$y_l(x) = \frac{2C_l}{\sqrt{q(x)}} \cos\left(\int_{x_1}^{x} q\left(x'\right) \mathrm{d}x' - \frac{\pi}{4}\right). \tag{35}$$

Making simple manipulations, one gets the function under the Coulomb barrier as $x_2 < x < x_3$

$$\begin{aligned} y_l(x) \; = \; \frac{C_l}{\sqrt{|q(x)|}} \Bigg\{ & \cos\alpha e^{-S_l(Q)} \exp\left(\int_x^{x_3} \left|q\left(x'\right)\right| \mathrm{d}x'\right) \\ & - 2\sin\alpha e^{S_l(Q)} \exp\left(\int_x^{x_3} \left|q\left(x'\right)\right| \mathrm{d}x'\right)\Bigg\}\,. \end{aligned} \tag{36}$$

Here, we introduced the action

$$S_l(Q) = \int\limits_{x_2}^{x_3} \left|q\left(x'\right)\right| \mathrm{d}x' \tag{37}$$

and the angle

$$\alpha = \int\limits_{x_1}^{x_2} q(x)\mathrm{d}x - \frac{\pi}{2}\,. \tag{38}$$

Behind the Coulomb barrier, when $x > x_3$,

$$\begin{aligned} y_l(x) \;&=\; -\frac{C_l}{\sqrt{q(x)}}\left\{\cos\alpha e^{-S_l}\sin\left(\int\limits_{x_3}^{x} q\left(x'\right)\mathrm{d}x' - \frac{\pi}{4}\right)\right. \\ &\quad\left. + 4\sin\alpha e^{S_l}\cos\left(\int\limits_{x_3}^{x} q\left(x'\right)\mathrm{d}x' - \frac{\pi}{4}\right)\right\}\,. \end{aligned} \tag{39}$$

By making use of Eq. (30), one can return to the coordinate r. The angle α now reads

$$\alpha = \int\limits_{a}^{R_0} k(r)\mathrm{d}r - \frac{\pi}{2}\,. \tag{40}$$

Note that the equality $\sin\alpha = 0$ is fulfilled if

$$\int\limits_{a}^{R_0} k(r)\mathrm{d}r = \left(n+\tfrac{1}{2}\right)\pi\,, \qquad n = 0,\ \pm 1, \ldots\,, \tag{41}$$

which is nothing but the Bohr–Sommerfeld quantization rule for a quasi-stationary level inside the potential well.

The wave function $w_l(\kappa;r)$ at $r > b$ becomes

$$\begin{aligned} w_l(\kappa;r) \;&=\; C_l\sqrt{\frac{\kappa}{k(r)}}\left[\cos\alpha e^{-S_l}\cos\left(\int\limits_{b}^{r} k\left(r'\right)\mathrm{d}r' + \frac{\pi}{4}\right)\right. \\ &\quad\left. -4\sin\alpha e^{S_l}\sin\left(\int\limits_{b}^{r} k\left(r'\right)\mathrm{d}r' + \frac{\pi}{4}\right)\right]\,. \end{aligned} \tag{42}$$

Comparing it with the asymptotic expression (26), one finds the amplitude of the l-th partial wave

$$C_l = \sqrt{\frac{2}{\pi}} \left[\cos^2 \alpha e^{-2S_l} + 16 \sin^2 \alpha e^{2S_l} \right]^{-1/2}. \tag{43}$$

Usually, the tunneling probability $e^{-2S_l(Q)} \ll 1$. In this case far from the Bohr–Sommerfeld condition (41), as $|\sin \alpha| \gg e^{-2S_l}$, the amplitude is of the order of e^{-S_l}. Otherwise, it attributes large value, $C_l \sim e^{S_l}$. It is a very rare event that a parent compound nucleus has such an energy that allows condition $|\sin \alpha| \sim e^{-2S_l}$. Its probability is also of the order of e^{-2S_l}.

In the approximation $e^{-2S_l} \ll 1$ and $|\sin \alpha| \gg e^{-2S_l}$, one gets the WKB wave function inside the nucleus at $a < r < R_0$

$$w_l(\kappa; r) = \frac{1}{\sqrt{2\pi}} \frac{e^{-S_l}}{\sin \alpha} \sqrt{\frac{\kappa}{k(r)}} \cos \left(\int\limits_a^r k\left(r'\right) \mathrm{d}r' - \frac{\pi}{4} \right) \tag{44}$$

and under the centrifugal barrier at $0 < r < a$,

$$w_l(\kappa; r) = \frac{1}{2\sqrt{2\pi}} \frac{e^{-S_l}}{\sin \alpha} \sqrt{\frac{\kappa}{|k(r)|}} \exp \left(- \int\limits_r^a \left| k\left(r'\right) \right| \mathrm{d}r' \right). \tag{45}$$

In the same approximation $e^{-S_l} \ll 1$, the radial function under the Coulomb barrier, $R_0 < r < b$, is represented by a single exponent

$$w_l(\kappa; r) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\kappa}{|k(r)|}} \exp \left(- \int\limits_r^b \left| k\left(r'\right) \right| \mathrm{d}r' \right). \tag{46}$$

Notice once more that if condition (41) is fulfilled, then the wave function inside the nucleus becomes very large, *i.e.*, $w_l \sim e^{S_l}$.

In full analogy, there can be calculated the irregular WKB solution $z_l(\kappa; r)$ of Eq. (24), having the asymptotic

$$z_l(\kappa; r) \approx -\sqrt{\frac{2}{\pi}} \cos \left(\kappa r - \frac{l\pi}{2} + \delta_l(\kappa) \right) \tag{47}$$

at $r \to \infty$.

5. Decay rate

Let the initial state $|a\rangle$ of the parent nucleus be formed at $t = 0$. Time-evolution of the wave function at $t \geq 0$ is governed by the equation [4]

$$\Psi_a(t) = -\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \mathrm{d}\epsilon e^{-i\epsilon t/\hbar}\mathcal{G}^+(\epsilon)\Psi_a(0)\,, \tag{48}$$

where the retarded Green's operator

$$\mathcal{G}^+(\epsilon) = (\epsilon + i\varepsilon - H)^{-1}\,, \qquad \varepsilon \to +0\,. \tag{49}$$

The partial α-decay rate into the state with spin I_d is yielded by

$$\lambda_{I_\mathrm{d}} = \frac{2\pi}{\hbar}\sum_{M_\mathrm{d}}\int \mathrm{d}\Omega_{\hat{\boldsymbol{\kappa}}}\left|\mathcal{R}^+_{ba}\right|^2 \varrho(\epsilon_b)\,, \tag{50}$$

where the density of final states is given by

$$\varrho(\epsilon_b) = \mu\kappa_{I_\mathrm{d}}/\hbar^2 \tag{51}$$

with the wave number

$$\kappa_{I_\mathrm{d}} = \sqrt{2\mu E_{I_\mathrm{d}}}/\hbar \tag{52}$$

and energy E_{I_d} defined in Eqs. (6), (7). The $\mathcal{R}^+_{ba}(\epsilon_a) = \mathcal{R}_{ba}(\epsilon_a + i\varepsilon)$ represents the matrix for the level shift operator, determined by the expansion [4]

$$\mathcal{R}(\epsilon) = V + V\frac{1-\Lambda_a}{\epsilon - H_0}V + \dots\,, \tag{53}$$

containing the projection operator

$$\Lambda_a = |a\rangle\langle a| \tag{54}$$

on the initial state $|a\rangle$.

First, the transition occurs from the initial state $|a\rangle$ to any state of the continuous spectrum caused by the interaction V' and only then the mixing transitions between the states φ^+_b due to perturbation δV. Hence,

$$\mathcal{R}^+_{ba} = V'_{ba} + \sum_{b'}\frac{\langle b|\delta V|b'\rangle V'_{b'a}}{\epsilon_a + i\varepsilon - \epsilon_{b'}} + \dots \tag{55}$$

Here, the brackets $\langle\dots\rangle$ mean the integration over the Eulerian angles θ' and vector $\boldsymbol{r}'$, while the sum over b' stands for summation over the intermediate nuclear states $|I'_{\rm d}M'_{\rm d}\rangle$ and integration over $\boldsymbol{\kappa}'$. Changing the integration order over $\theta', \boldsymbol{r}'$ and $\boldsymbol{\kappa}'$, we rewrite expression (55) as

$$\begin{aligned}\mathcal{R}^{+}_{ba} &= V'_{ba} + \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\theta \int \mathrm{d}\boldsymbol{r}' \int \mathrm{d}\theta' \varphi_b^{+*}(\theta, \boldsymbol{r}) \delta V(\theta, \boldsymbol{r}) \\ &\quad \times G^{+}\left(\epsilon_a; \theta\boldsymbol{r}, \theta'\boldsymbol{r}'\right) V'\left(\boldsymbol{r}'\right) \varphi_a \, . \end{aligned} \tag{56}$$

Here, the retarded Green's function is given by

$$G^{+}\left(\epsilon_a; \theta\boldsymbol{r}, \theta'\boldsymbol{r}'\right) = -\frac{2\mu}{\hbar^2} \sum_{I'_{\rm d}M'_{\rm d}} \int \mathrm{d}\boldsymbol{\kappa}' \frac{\varphi^{+}_{b'}(\theta, \boldsymbol{r}) \varphi^{+*}_{b'}\left(\theta', \boldsymbol{r}'\right)}{\kappa'^2 - \kappa^2_{I'_{\rm d}} - i\varepsilon} \, . \tag{57}$$

Expression (57) transforms to the form

$$G^{+}\left(\epsilon_a; \theta\boldsymbol{r}, \theta'\boldsymbol{r}'\right) = \frac{2\mu}{\hbar^2} \sum_{I_{\rm d}, l} \frac{G^{+}_{l}\left(\kappa_{I_{\rm d}}; r, r'\right)}{rr'} \sum_{IM} \mathcal{Y}^{(M)}_{IlI_{\rm d}}(\theta, \hat{\boldsymbol{r}}) \mathcal{Y}^{(M)*}_{IlI_{\rm d}}\left(\theta', \hat{\boldsymbol{r}}'\right) \, , \tag{58}$$

containing the partial Green's functions

$$G^{+}_{l}\left(\kappa_{I_{\rm d}}; r, r'\right) = -\int\limits_{0}^{\infty} \frac{w_l(\kappa; r) w_l\left(\kappa; r'\right)}{\kappa^2 - \kappa^2_{I_{\rm d}} - i\varepsilon} \mathrm{d}\kappa \, . \tag{59}$$

In order to calculate this integral, we introduce also the function

$$d^{(\pm)}_{l}(\kappa; r) = w_l(\kappa; r) \pm i z_l(\kappa; r) \, , \tag{60}$$

which is similar to spherical Hankel's function and has the asymptotics

$$d^{(\pm)}_{l}(\kappa; r) \approx \mp i \sqrt{\frac{2}{\pi}} e^{\pm i(\kappa r - l\pi/2 + \delta_l(\kappa))} \, , \qquad r \to \infty \, . \tag{61}$$

It can be shown that

$$d^{(+)}_{l}(-\kappa; r) = (-1)^l d^{(-)}_{l}(\kappa; r) \, . \tag{62}$$

Using this symmetry property, one can transform (59) to

$$G^{+}_{l}(\kappa_{I_{\rm d}}; r, r') = -\frac{1}{2} \int\limits_{-\infty}^{\infty} \frac{w_l(\kappa; r_<) d^{(+)}_{l}(\kappa; r_>)}{\kappa^2 - \kappa^2_{I_{\rm d}} - i\eta} \mathrm{d}\kappa \, , \tag{63}$$

where $r_<$ and $r_>$ denote the least and largest values of r and r'. Enclosing the integration contour at infinity in the upper complex half-plane κ, one finally finds

$$G_l^+\left(\kappa_{I_\mathrm{d}}; r, r'\right) = -\frac{i\pi}{2\kappa_{I_\mathrm{d}}} w_l(\kappa_{I_\mathrm{d}}; r_<) d_l^{(+)}(\kappa_{I_\mathrm{d}}; r_>) \,. \tag{64}$$

Now, we restrict our consideration to decay of the ground state of an even–even nuclei with $I_\mathrm{p} = 0$. In this case, $I_\mathrm{d} = l$ and it is convenient to introduce more short designation

$$F_{I_\mathrm{d}}(\theta; \hat{\boldsymbol{r}}) \equiv \mathcal{Y}^{(0)}_{0 I_\mathrm{d} I_\mathrm{d}}(\theta; \hat{\boldsymbol{r}}) \,. \tag{65}$$

This angular factor is determined by the simple formula

$$F_l(\theta; \hat{\boldsymbol{r}}) = \frac{(-1)^l}{\sqrt{8\pi^2}} \sum_{m=-l}^{l} D^{l*}_{m0}(\theta) Y^*_{lm}(\hat{\boldsymbol{r}}) \tag{66}$$

to be applied in calculations of the $\mathcal{R}$ matrix. Calculating the $\mathcal{R}_{ba}$ by means of Eqs. (58), (64), (56) and (66), one finds the decay rate

$$\lambda_l = \lambda_l^{(0)} + \delta\lambda_l \,. \tag{67}$$

We neglect the contribution of δV_N into $\delta\lambda_l$. For the decay constant of a spherical nucleus, we get

$$\lambda_l^{(0)} = \frac{2\pi}{\hbar} |\mathfrak{I}_l|^2 \varrho(\epsilon_b) \,, \tag{68}$$

where the integral over the volume of the nucleus is

$$\mathfrak{I}_l = \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\theta \frac{w_l(\kappa_l; r)}{\kappa_l r} F_l(\theta; \hat{\boldsymbol{r}}) V' |a\rangle \tag{69}$$

with $w_l(\kappa_l; r)$ given in (45), (46). Strictly speaking, here the integration over intrinsic coordinates of the nuclei should be also added.

For the correction $\sim \beta$, one gets the expression

$$\begin{aligned}\delta\lambda_l \;&=\; 3\eta_l \beta \sqrt{\frac{\pi}{5}} \sum_{l'} \left(\frac{2l+1}{2l'+1}\right)^{\frac{1}{2}} (l200|l'0)^2 \\ &\times \left(\lambda_l^{(0)} \lambda_{l'}^{(0)}\right)^{1/2} \int_{R_0}^{\infty} \frac{R_0^2}{r^3} w_l(\kappa_l; r) z_{l'}(\kappa_{l'}; r) \mathrm{d}r \,,\end{aligned} \tag{70}$$

including the dimensionless Sommerfeld parameter

$$\eta_l = \frac{2(Z-2)e^2\mu}{\hbar^2\kappa_l}\,. \tag{71}$$

The constant $\lambda_l^{(0)}$ is proportional to the probability $\mathcal{P}$ of finding the α particle inside the nucleus. By making use of the wave function (44), we get the decay constant of the spherical nucleus

$$\lambda_l^{(0)} = 4\mathcal{M}^2\frac{\mu}{\hbar^3}\frac{e^{-2S_l}}{\sin^2\alpha_l}\int\limits_a^{R_0}\frac{\mathrm{d}r}{k_l(r)}\cos^2\left(\int\limits_a^r k_l\left(r'\right)\mathrm{d}r' - \frac{\pi}{4}\right), \tag{72}$$

where the fitting factor $\mathcal{M}$ has the dimensionality of energy. In derivation of (72), we neglected small contribution from the region $0 < r < a$ under the centrifugal barrier.

Only in the case of decay into the ground state, as $I_\mathrm{d} = l = 0$, the wave number $k_0(r)$ takes the constant value $K = \sqrt{2\mu E(K)}/\hbar$, where the kinetic energy in the potential well is $E(K) = U_0 + E$. Then in the approximation $KR_0 \gg 1$, we arrive at result (1), where the formation probability

$$p = \left(\frac{\mathcal{M}}{E(K)}\right)^2\frac{(KR_0)^2}{\sin^2\alpha} \tag{73}$$

depends on the parameters of the potential well, and the knocking frequency

$$\nu = \frac{\hbar K}{2R_0\mu}\,. \tag{74}$$

Note also that Eq. (72) is derived far from the Sommerfeld requirement (41), *i.e.* when $|\sin\alpha| \gg e^{-2S_l}$. In the opposite case with $|\sin\alpha| \sim e^{-2S_l}$, the α decay is characterized by unrealistically large decay rate $\lambda_l \sim e^{2S_l}$.

REFERENCES

[1] V.Yu. Denisov, A.A. Khudenko, *Phys. Rev. C* **80**, 034603 (2009).
[2] A. Zdeb, M. Warda, K. Pomorski, *Phys. Scr.* **T154**, 014029 (2013).
[3] G. Gamow, *Z. Phys.* **51**, 204 (1928).
[4] M.L. Goldberger, K.M. Watson, *Collision Theory*, J. Wiley, New York 1964.
[5] A.Ya. Dzyublik, G. Gosselin, V. Mèot, P. Morel, *Europhys. Lett.* **102**, 62001 (2013).
[6] N.T. Zinner, *Nucl. Phys. A* **781**, 81 (2007).
[7] A.Ya. Dzyublik, *Phys. Rev. C* **90**, 054619 (2014).
[8] T. Tamura, *Rev. Mod. Phys.* **37**, 679 (1965).
[9] R.E. Langer, *Phys. Rev.* **51**, 669 (1937).