

# COMPUTING RADIATIVE CORRECTIONS IN FOUR DIMENSIONS\*

ROBERTO PITTAU

Departamento de Física Teórica y del Cosmos and CAFPE  
Universidad de Granada, Campus Fuentenueva s.n.  
18071 Granada, Spain

*(Received January 10, 2018)*

I comment and summarize the principles underlying the Four Dimensional Regularization/Renormalization (FDR) approach to the UV and IR infinities. A few recent results are also reviewed.

DOI:10.5506/APhysPolBSupp.11.387

## 1. Introduction

The overwhelming complexity of the perturbative calculations performed nowadays to cope with the precision required by the present and future experimental measurements in High Energy Particle Physics makes it advisable to try alternative approaches to this problem. The source of many complications is the presence of divergent integrals in the intermediate steps of the calculations, that need to be regulated and removed from the physical predictions.

In this contribution, I review the present status of the FDR approach [1], with special emphasis on the mechanisms which should be used as guidelines when defining divergent integrals in unitary gauge theories.

## 2. FDR

The main aim of FDR is embedding the UV subtraction directly in the definition of the loop integration. In that way, the renormalized Green's functions are directly computed in four dimensions, without adding counterterms in the Lagrangian  $\mathcal{L}$  [2]. FDR can also be used to regulate IR divergences [3]. In the following two subsections, I review the main features of FDR.

---

\* Presented at the Final HiggsTools Meeting, Durham, UK, September 11–15, 2017.

### 2.1. UV infinities

The FDR UV subtraction works at the integrand level. Consider, for example, a UV divergent one-loop integrand

$$J(q) = \frac{1}{q^2 D_p}, \quad D_p = (q+p)^2. \quad (1)$$

The FDR loop integration over  $J(q)$  is defined as follows:

$$\int [d^4 q] J(q) \equiv \lim_{\mu \rightarrow 0} \int_R \left( J(q) - \frac{1}{\bar{q}^4} \right) \equiv \int [d^4 q] \frac{1}{\bar{q}^2 \bar{D}_p}, \quad (2)$$

where

$$\bar{q}^2 \equiv q^2 - \mu^2, \quad \bar{D}_p \equiv D_p - \mu^2. \quad (3)$$

In Eq. (2)  $R$  is an arbitrary UV regulator and  $\mu^2$  regulates the IR behavior induced by the subtraction term  $1/\bar{q}^4$ . Tensors are defined likewise. Given, for instance,

$$J^{\alpha\beta}(q) = \frac{q^\alpha q^\beta}{q^2 D_{p_1} D_{p_2}}, \quad (4)$$

one has

$$\int [d^4 q] J^{\alpha\beta}(q) = \lim_{\mu \rightarrow 0} \int_R \left( J^{\alpha\beta}(q) - \frac{q^\alpha q^\beta}{\bar{q}^6} \right). \quad (5)$$

This definition can be extended to more loops [4]. The subtracted integrands are dubbed FDR vacua, or simply vacua, and do not depend on physical scales.

### 2.2. Virtual IR divergences

For IR convergent loop integrals,  $q^2$  in the original integrands — such as  $J(q)$  in Eq. (1) — can be left unbarred. Barring it regulates virtual IR divergences, giving rise to IR logarithms of  $\mu$ . As an example, the fully massless scalar one-loop triangle is defined in FDR as [3]

$$\int [d^4 q] \frac{1}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}} \equiv \lim_{\mu \rightarrow 0} \int d^4 q \frac{1}{q^2 \bar{D}_{p_1} \bar{D}_{p_2}} = \frac{i\pi^2}{2s} \ln^2 \left( \frac{\mu^2}{-s - i0} \right), \quad (6)$$

with  $s = (p_1 - p_2)^2 = -2(p_1 \cdot p_2)$ .

2.3. Real IR divergences

Virtual and real IR divergences are matched by a consistent treatment of the real radiation. The cutting rule

$$\frac{i}{\bar{q}^2 + i0^+} \rightarrow (2\pi) \delta_+ (\bar{q}^2) \tag{7}$$

establishes the needed connection between barred loop propagators and massive external particles. As a consequence, the logarithms of  $\mu$  in Eq. (6) can be rewritten as counterterms integrated over a  $\mu$ -massive phase-space  $\bar{\Phi}_3$

$$\int_{\bar{\Phi}_2} \Re \left( \int [d^4 q] \frac{1}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}} \right) = \int_{\bar{\Phi}_3} \frac{1}{\bar{s}_{13} \bar{s}_{23}} \begin{cases} \bar{s}_{ij} = (\bar{p}_i + \bar{p}_j)^2 \\ \bar{p}_{i,j}^2 = \mu^2 \end{cases} . \tag{8}$$

Thus,  $m$ -body virtual and  $(m + 1)$ -body real IR divergences compensate each other, as depicted in Fig. 1. In both cases, the divergent splitting is regulated by  $\mu$ -massive unobserved particles, denoted by thick lines. This treatment has been shown to work at NLO [3]. The corresponding NNLO Ansatz is illustrated in Fig. 2.

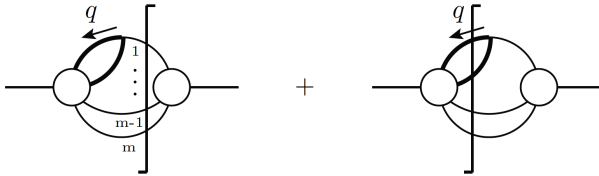


Fig. 1. Cancellation of NLO final-state IR singularities in FDR.

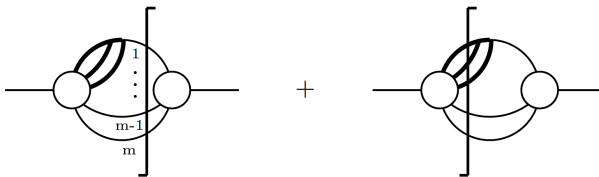


Fig. 2. Cancellation of doubly unresolved final-state IR singularities in FDR (Ansatz).

3. Fundamental properties of the loop integration

In this section, I enumerate the three key properties that must be maintained by any consistent definition of loop integration and show how they are obeyed in Dimensional Regularization (DReg) and FDR. The properties are:

1. Shift invariance;
2. The possibility of cancelling numerators and denominators;
3. The possibility of inserting sub-loop expressions in higher loop calculations (sub-integration consistency).

When the above requirements hold, r.h.s. and l.h.s. coincide in Eq. (9), Eq. (10) and Fig. 3, respectively,

$$\int_R d^4q_1 \cdots d^4q_\ell J(q_1, \dots, q_\ell) \stackrel{?}{=} \int_R d^4q_1 \cdots d^4q_\ell J(q_1 + p_1, \dots, q_\ell + p_\ell), \tag{9}$$

$$\int_R d^4q_1 \cdots d^4q_\ell \frac{\not{D}_i}{D_0 \cdots \not{D}_i \cdots D_k} \stackrel{?}{=} \int_R d^4q_1 \cdots d^4q_\ell \frac{1}{D_0 \cdots D_k}, \tag{10}$$

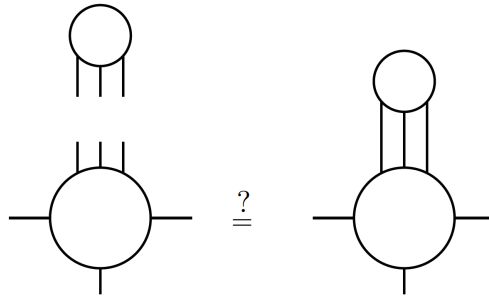


Fig. 3. Schematic representation of the sub-integration consistency requirement.

The first condition guarantees routing invariance, the second one maintains the needed gauge cancellations, while the third requirement is essential to ensure unitarity. In fact, the unitarity equation

$$T - T^\dagger = iT^\dagger T$$

mixes different loop orders, so that it is essential that the result of a sub-loop integration stays the same also when embedded in higher loop computations.

### 3.1. DReg

In DReg, the first two conditions are fulfilled by construction. On the other hand, preserving the sub-integration consistency requires introducing order-by-order counterterms (CTs) in  $\mathcal{L}$ . For example, without CTs, one has

$$\int d^n q_1 d^n q_2 \frac{1}{(q_1^2 - M^2)^2} \frac{1}{(q_2^2 - M^2)^2} \Big|_{\frac{1}{\epsilon}=0} \neq \left( \int d^n q \frac{1}{(q^2 - M^2)^2} \Big|_{\frac{1}{\epsilon}=0} \right)^2, \tag{11}$$

which prevents one from defining loop integrals as DReg integrals devoid of  $1/\epsilon$  poles. The role of the CTs is precisely subtracting UV poles in such a way to restore the equality in Eq. (11).

### 3.2. FDR

FDR integrals are shift invariant, *e.g.*

$$\int [d^4 q] \frac{1}{\bar{q}^2 \bar{D}_p} = \int [d^4 q] \frac{1}{\bar{q}^2 \bar{D}_{-p}}, \tag{12}$$

because both sides share the same subtraction term. As for the numerator/denominator cancellation, one has to distinguish self-contractions of loop momenta generated by tensor decomposition from the case when they originate from Feynman rules. In the former case, no cancellation must occur<sup>1</sup>. On the other hand, gauge invariance prescribes cancellation in the latter situation. FDR deals with both circumstances thanks to the introduction of the so-called Extra Integrals (EI). Consider, for instance,

$$\int [d^4 q] \frac{q^2}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}} \neq \int [d^4 q] \frac{\bar{q}^2}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}}. \tag{13}$$

The inequality holds because the l.h.s. subtracts  $q^2/\bar{q}^6$ , whilst  $1/\bar{q}^4$  is subtracted in the r.h.s. The difference can be computationally encoded in an EI, defined as the difference between the two subtraction terms surviving the  $\mu \rightarrow 0$  limit

$$\int [d^4 q] \frac{\mu^2}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}} \equiv \int_R d^4 q \frac{\bar{q}^2 - q^2}{\bar{q}^6} = -\mu^2 \int d^4 q \frac{1}{\bar{q}^6} = \frac{i\pi^2}{2}. \tag{14}$$

Thus, it is possible to write the following algebraic equation:

$$\int [d^4 q] \frac{q^2}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}} = \int [d^4 q] \frac{\bar{q}^2}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}} + \int [d^4 q] \frac{\mu^2}{\bar{q}^2 \bar{D}_{p_1} \bar{D}_{p_2}}. \tag{15}$$

---

<sup>1</sup> This is a consequence of requiring the result of the decomposition to coincide with the original tensor.

From all of this, it is clear that preserving gauge cancellations prescribes the replacement  $q^2 \rightarrow \bar{q}^2$  both in denominators and numerators whenever  $q^2$  does not originate from tensor reduction [5]. This operation is called Global Prescription (GP).

As for the unitarity condition, the FDR counterpart of Eq. (11)

$$\int [d^4q_1] [d^4q_2] \frac{1}{(\bar{q}_1^2 - M^2)^2} \frac{1}{(\bar{q}_2^2 - M^2)^2} = \left( \int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} \right)^2 \quad (16)$$

holds without the addition of CTs. However, the equality in Fig. 3 is fulfilled only if the GP at the level of the sub-amplitude on the left does not clash with the GP at the level of the full amplitude on the right. This is not always the case, but it is possible to correct for the mismatch and ensure sub-integration consistency by adding “Extra” Extra Integrals (EEI) derived by solely analyzing the loop diagrams on the right [6].

### 4. Results

In the following, I review a few recent results obtained in the framework of FDR.

#### 4.1. DReg versus FDR @NLO

A one-to-one correspondence exists between DReg and FDR for both UV and IR divergent loop integrals [7]

$$\Gamma(1 - \epsilon) \pi^\epsilon \int \frac{d^n q}{\mu_R^{-2\epsilon}} (\dots) \Big|_{\mu_R = \mu \text{ and } \frac{1}{\epsilon^i} = 0} = \int [d^4q] (\dots) . \quad (17)$$

Analogously, for the real contribution

$$\begin{aligned} & \left( \frac{\mu_R^2}{s} \right)^\epsilon \int_{\phi_3} dx dy dz (\dots) \delta(1 - x - y - z) (xyz)^{-\epsilon} \Big|_{\mu_R = \mu \text{ and } \frac{1}{\epsilon^i} = 0} \\ &= \int_{\bar{\phi}_3} dx dy dz (\dots) \delta(1 - x - y - z + 3\mu^2/s) , \end{aligned} \quad (18)$$

where  $\phi_3$  and  $\bar{\phi}_3$  are massless and  $\mu$ -massive three-body phase spaces, respectively.

4.2. DReg versus FDR @NNLO

FDR has been proven to renormalize consistently off-shell QCD up to two loops [6]. The  $\alpha_s$  and  $m_q$  shifts necessary to translate FDR to  $\overline{\text{MS}}$  in DReg have been determined by analyzing the FDR vacua of the two-loop 2- and 3-point QCD correlators  $G^{(2\text{-loop})}$  given in Fig. 4.

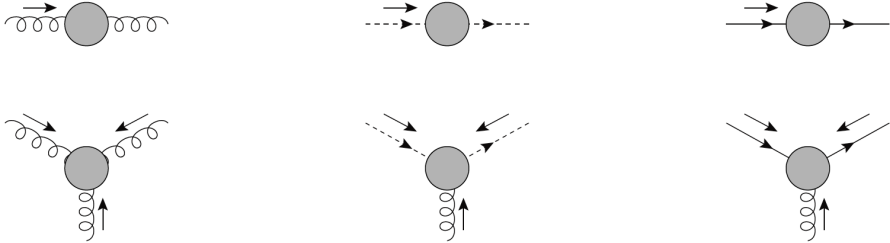


Fig. 4. Irreducible 2- and 3-point QCD Green’s functions.

4.3. EEIs

Analyzing the FDR EEIs led to a fix of two-loop “naive” FDH in DReg [6]

$$G_{\text{bare, DReg}}^{(2\text{-loop})}|_{n_s=4} \rightarrow G_{\text{bare, DReg}}^{(2\text{-loop})}|_{n_s=4} + \sum_{\text{Diag}} \text{EEI}_b|_{n_s=4}, \tag{19}$$

where  $n_s = \gamma_\mu \gamma^\mu = g_{\mu\nu} g^{\mu\nu}$ . In the above equation, EEI<sub>b</sub>s are DReg integrals obtained from FDR EEIs by dropping the subtraction term, *e.g.*

$$\int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_p} \rightarrow \int d^n q \frac{1}{q^2 D_p}. \tag{20}$$

The EEI<sub>b</sub>s reproduce the effect of the evanescent operators needed in FDH and dimensional reduction to restore renormalizability, at least off shell. A preliminary study of the two-loop QCD vertices in Fig. 5 indicates that the same phenomenon is likely to be observed on-shell as well [8].

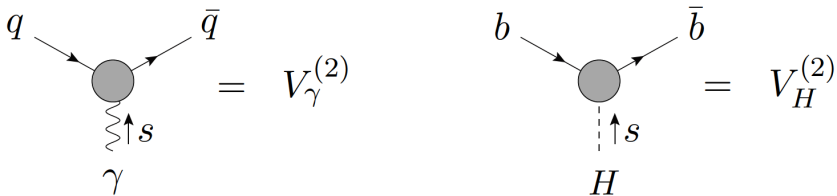


Fig. 5. On-shell two-loop  $\gamma^* \rightarrow q\bar{q}$  and  $H \rightarrow b\bar{b}$  QCD vertices.

#### 4.4. Local subtraction of IR divergences @NLO

It is possible to set up a local FDR subtraction of the final-state IR infinities by rewriting the virtual logarithms as counterterms to be added to the real radiation [7], in the same spirit of Eq. (8). Schematically

$$\begin{aligned} \sigma_{\text{NLO}} = & \int_{\Phi_2} \left( |M|_{\text{Born}}^2 + \underbrace{|M|_{\text{Virt}}^2}_{\text{devoid of logs of } \mu} \right) F_J^{(2)}(p_1, p_2) \\ & + \int_{\substack{\Phi_3 \\ \mu \rightarrow 0 \text{ here}}} \left( |M|_{\text{Real}}^2 F_J^{(3)}(p_1, p_2, p_3) - |M|_{\text{CT}}^2 F_J^{(2)}(\underbrace{\hat{p}_1, \hat{p}_2}_{\text{mapped kinematics}}) \right), \end{aligned} \quad (21)$$

where  $F_J$  are jet functions. For instance, in the case of  $e^+e^- \rightarrow \gamma^* \rightarrow q\bar{q}(g)$ , the explicit form of the local counterterm is

$$|M|_{\text{CT}}^2 = \frac{16\pi\alpha_S}{s} C_F |M|_{\text{Born}}^2(\hat{p}_1, \hat{p}_2) \left( \frac{s^2}{s_{13}s_{23}} - \frac{s}{s_{13}} - \frac{s}{s_{23}} + \frac{s_{13}}{2s_{23}} + \frac{s_{23}}{2s_{13}} - \frac{17}{2} \right), \quad (22)$$

and the mapping reads

$$\hat{p}_1^\alpha = \kappa \Lambda_\beta^\alpha p_1^\beta \left( 1 + \frac{s_{23}}{s_{12}} \right), \quad \hat{p}_2^\alpha = \kappa \Lambda_\beta^\alpha p_2^\beta \left( 1 + \frac{s_{13}}{s_{12}} \right), \quad (23)$$

where  $\kappa = \sqrt{\frac{ss_{12}}{(s_{12}+s_{13})(s_{12}+s_{23})}}$  and  $\Lambda_\beta^\alpha$  is the boost that brings the sum of  $\hat{p}_1$  and  $\hat{p}_2$  back to the center-of-mass frame:  $\hat{p}_1 + \hat{p}_2 = (\sqrt{s}, 0, 0, 0)$ .

The inclusive  $\sigma_{\text{NLO}} = \sigma_0 \left( 1 + C_F \frac{3}{4} \frac{\alpha_S}{\pi} \right)$  cross section is reproduced by a numerical implementation of Eq. (21). In addition, successful comparisons [9] with `MadGraph5_aMC@NLO` [10] interfaced with `FastJet` [11] have been attained for realistic jet observables.

## 5. Outlook

FDR is turning to a competitive tool to compute radiative corrections. The UV subtraction is incorporated, at the integrand level, in the definition of the loop integration. As a consequence, one directly deals with four-dimensional integrals, without introducing UV counterterms in  $\mathcal{L}$ . This has been shown to be a workable alternative to DReg up to two loops for off-shell quantities.

The FDR regularization of the IR divergences is well-understood at NLO, and a completely local subtraction of final-state IR infinities has been worked out for two-jet cross sections.



Going on-shell at NNLO seems feasible. In fact, as a by-product of the FDR UV treatment, a fix to two-loop “naïve” FDH avoiding evanescent couplings is available for on-shell observables.

Future investigations include an extension of FDR to initial-state IR singularities and a complete two-loop calculation [12] of the QCD form factors in Fig. 5. Finally, it would be interesting to investigate FDR integration as a new mathematical tool to be used also in other branches of physics where divergent integrals occur.

## REFERENCES

- [1] R. Pittau, *J. High Energy Phys.* **1211**, 151 (2012).
- [2] R. Pittau, *Fortsch. Phys.* **63**, 132 (2015).
- [3] R. Pittau, *Eur. Phys. J. C* **74**, 2686 (2014).
- [4] A.M. Donati, R. Pittau, *Eur. Phys. J. C* **74**, 2864 (2014).
- [5] A.M. Donati, R. Pittau, *J. High Energy Phys.* **1304**, 167 (2013).
- [6] B. Page, R. Pittau, *J. High Energy Phys.* **1511**, 183 (2015).
- [7] C. Gnendiger *et al.*, *Eur. Phys. J. C* **77**, 471 (2017).
- [8] C. Gnendiger, A. Signer, private communication.
- [9] M. Moretti, R. Pittau, in preparation.
- [10] J. Alwall *et al.*, *J. High Energy Phys.* **1407**, 079 (2014).
- [11] M. Cacciari, G.P. Salam, G. Soyez, *Eur. Phys. J. C* **72**, 1896 (2012).
- [12] B. Page, R. Pittau, work in progress.