INHOMOGENEOUS PHASES IN
THE 1+1 DIMENSIONAL GROSS–NEVEU MODEL
AT FINITE NUMBER OF FERMION FLAVORS∗

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We study the phase diagram of the 1+1 dimensional Gross–Neveu
model at finite number of fermion flavors using the lattice field theory. Nu-
merical results are presented, which indicate the existence of an inhomoge-
neous phase, where the chiral condensate is a spatially oscillating function.

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1. The Gross–Neveu model and its phase diagram
in the limit of infinitely many fermion flavors

Exploring the phase diagram of QCD using lattice computations is cur-
rently restricted to small chemical potential, because of the QCD sign prob-
lem (see e.g., [1, 2] and references therein). There are, however, several QCD-
inspired models e.g., the Gross–Neveu (GN) model [3], which are technically
simpler to treat, and which share certain symmetries with QCD. Studies of
such models might, thus, provide insights concerning the phase diagram of
strongly interacting matter. A notable feature of the GN model in 1+1 di-
mensions in the limit of infinitely many fermion flavors is the existence of
a so-called inhomogeneous phase, where the chiral order parameter is not
a constant, but spatially oscillating [4, 5] (for a review on inhomogeneous
condensates and phases, see [6]). In this work, we perform a lattice field
theory study of the 1+1 dimensional GN model at finite number of fermion
flavors $N_f$, to explore whether inhomogeneous phases also exist at finite $N_f$.  

The Euclidean action of the GN model is

\[
S = \int d^2 x \left( \sum_{n=1}^{N_f} \bar{\psi}_n (\gamma^0(\partial_0 + \mu) + \gamma^1 \partial_1) \psi_n - \frac{\lambda}{2N_f} \left( \sum_{n=1}^{N_f} \bar{\psi}_n \psi_n \right)^2 \right),
\]

(1.1)

where \( \psi \) denotes a fermionic field with \( N_f \) flavors and \( \mu \) is the chemical potential. One can get rid of the four-fermion interaction by introducing a scalar field \( \sigma \), which leads to the following partition function:

\[
Z = \int \mathcal{D}\sigma \exp \left( -N_f \left( \frac{1}{2\lambda} \int d^2 x \sigma^2 - \ln \left( \det \left( (\partial_0 + \mu)\gamma_0 + \partial_1\gamma_1 + \sigma \right) \right) \right) \right),
\]

(1.2)

The effective action has a discrete chiral symmetry, \( S_{\text{eff}}[+\sigma] = S_{\text{eff}}[-\sigma] \), where \( \langle \sigma \rangle \propto \langle \sum_n \bar{\psi}_n \psi_n \rangle \) represents the chiral condensate and indicates whether the symmetry is spontaneously broken.

The phase diagram of the 1+1 dimensional GN model in the limit of \( N_f \to \infty \) has been calculated in [4, 5]. There are three phases (see Fig. 1):

- a **chirally symmetric phase**, where \( \langle \sigma \rangle = 0 \);
- a **homogeneously broken phase**, where \( \langle \sigma \rangle = \text{const} \neq 0 \);
- an **inhomogeneous phase**, where \( \langle \sigma \rangle \) is a spatially oscillating function.

In the inhomogeneous phase, \( \langle \sigma \rangle \) exhibits a periodic kink–antikink structure close to the phase boundary to the homogeneously broken phase and gradually changes into a sin-like structure for increasing \( \mu \).

![Fig. 1. Phase diagram of the GN model in the large-\( N_f \) limit (see [4, 5]).](image-url)
2. The phase diagram at finite number of fermion flavors

We perform lattice Monte Carlo simulations of the 1+1 dimensional GN model defined in Eq. (1.2) at finite \( N_f \in \{8, 16, 24, 32, 48\} \). We use two different discretizations of the fermionic determinant, naive fermions and SLAC fermions (see e.g. [7]), which we consider to be an important cross check of our numerical results: the results obtained with the two discretizations agree within statistical errors. We set the scale via the absolute value of the chiral condensate at chemical potential \( \mu = 0 \) and temperature \( T = 0 \), i.e. \( \sigma_0 = \langle |\bar{\sigma}| \rangle_{\mu=0,T=0} \), where

\[
\bar{\sigma} = \frac{1}{V} \sum_{x,t} \sigma(x,t),
\]

(2.1)

\( V \) is the number of lattice sites and \( \langle \ldots \rangle_{\mu,T} \) denotes the path integral expectation value at chemical potential \( \mu \) and at temperature \( T \), i.e. the average over the generated set of Monte Carlo field configurations. In other words, we express dimensionful quantities in units of \( \sigma_0 \), e.g. \( \mu/\sigma_0 \), \( T/\sigma_0 \).

\( \langle |\bar{\sigma}| \rangle_{\mu,T} \) is also a suitable approximate order parameter to distinguish between a homogeneously broken phase on the one hand (\( \langle |\bar{\sigma}| \rangle_{\mu,T} \neq 0 \)) and a restored or inhomogeneous phase on the other hand (\( \langle |\bar{\sigma}| \rangle_{\mu,T} \approx 0 \)).

Numerical results for \( N_f = 8 \) are shown in Fig. 2, left plot. A homogeneously broken phase is indicated by the light gray/yellow dots at small \( \mu \) and small \( T \), somewhat smaller, but in a similar region as for infinite \( N_f \). Results from analogous computations for \( N_f \in \{16, 24, 32, 48\} \) restricted to \( \mu = 0 \) are shown in Fig. 2, right plot. When increasing \( N_f \), the results approach the numerical result at infinite \( N_f \) (the latter has been obtained using techniques developed and explained in [8–10]).
To check for the existence of an inhomogeneous phase at finite $N_f$, we compute the spatial correlation function of the chiral condensate $\langle C(x) \rangle_{\mu,T}$ and its Fourier transform $\langle \tilde{C}(k) \rangle_{\mu,T}$, where

$$C(x) = \frac{1}{V} \sum_{y,t} \sum_x \sigma(y,t)\sigma(y+x,t).$$  \hspace{1cm} (2.2)$$

Both $\langle C(x) \rangle_{\mu,T}$ and $\langle \tilde{C}(k) \rangle_{\mu,T}$ are suited to distinguish the three phases we are expecting as illustrated by Fig. 3:

— **Chirally symmetric phase**: $\langle C(x) \rangle_{\mu,T}$ quickly approaches 0.0. The Fourier transform is a smooth function close to 0.0 indicating a vanishing chiral condensate.

— **Homogeneously broken phase**: $\langle C(x) \rangle_{\mu,T}$ quickly approaches $\sigma_0^2$. The Fourier transform exhibits a pronounced peak at $k = 0$ representing the non-vanishing constant chiral condensate.

— **Inhomogeneous phase**: $\langle C(x) \rangle_{\mu,T}$ is an oscillating function. The Fourier transform exhibits a pronounced peak at $k \neq 0$ proportional to the inverse wave length of the chiral condensate.

![Fig. 3. $C(x)$ and $\tilde{C}(k)$ for $N_f = 8$. Top: $\mu/\sigma_0 = 0$ and $T/\sigma_0 = 0.988$ (chirally symmetric phase) as well as $T/\sigma_0 = 0.082$ (homogeneously broken phase). Bottom: $\mu/\sigma_0 \in \{0.5, 0.7, 1.0\}$ and $T/\sigma_0 = 0.082$ (inhomogeneous phase).](image-url)
Of particular interest are the plots at the bottom of Fig. 3, because they provide clear evidence for the existence of an inhomogeneous phase at finite $N_f$.

To identify the boundary between the homogeneously broken phase and the inhomogeneous phase, we plot in Fig. 4

$$k_{\text{max}} = \left| \arg \max \left( \langle \tilde{C}(k) \rangle_{\mu,T} \right) \right|$$

(2.3)

as a function of $\mu$ and $T$. The phase boundary is clearly visible at $\approx \mu/\sigma_0 \approx 0.45$ separating the black/blue points ($k_{\text{max}} \approx 0$, homogeneously broken phase) from the gray/red points ($k_{\text{max}} \neq 0$, inhomogeneous phase).

![Fig. 4. (Color online) $k_{\text{max}}/\sigma_0$ as a function of $\mu/\sigma_0$ and $T/\sigma_0$ for $N_f = 8$.](image)

To exhibit the oscillations of the chiral condensate in the inhomogeneous phase in an even more direct way, we compute $\langle \sigma(x+x_{\text{shift}},t) \rangle_{\mu,T}$. Here, $x_{\text{shift}}$ is the phase shift of the spatially oscillating chiral condensate $\sigma(x,t)$ determined individually for each Monte Carlo field configuration by a standard Fourier transform. In this way, destructive interference is excluded, when averaging over the Monte Carlo field configurations. In Fig. 5, we show $\langle \sigma(x+x_{\text{shift}},t) \rangle_{\mu,T}$ at three different $(\mu, T)$. In the left plot (homogeneously

![Fig. 5. $\langle \sigma(x+x_{\text{shift}},t) \rangle_{\mu,T}$ as a function of $x/\sigma_0$ and $t/\sigma_0$ for $(\mu/\sigma_0,T/\sigma_0) = (0.0, 0.038)$ (homogeneously broken phase, left plot) and $(\mu/\sigma_0,T/\sigma_0) \in \{(0.5, 0.038), (0.7, 0.038)\}$ (inhomogeneous phase, center and right plot).]
broken phase), $\langle \sigma(x + x_{\text{shift}}, t) \rangle_{\mu, T}$ is almost constant, close to $\sigma_0$, while in the center plot and the right plot (inhomogeneous phase), spatial oscillations are clearly visible.

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