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ON SOME SOLUTIONS OF THE TYPE [D] SELF-DUAL SPACES*

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Complex, 4-dimensional, self-dual spaces which are two-sided conformally recurrent are considered. The explicit metrics of the spaces of the type $[\mathrm{D}] \otimes [-]$ are presented.

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1. Introduction

This paper is devoted to the manifolds which are 4-dimensional in a complex sense and which are equipped with a holomorphic metric. Such spaces are generalizations of the 4-dimensional real manifolds, equipped with a real smooth metric. There are many advantages of the analysis of 4-dimensional complex spaces. The most important advantage is that a real smooth metric can be obtained from a complex metric by the procedure of a *real slice* of the complex space [8]. Moreover, the technique of a real slice leads to the real smooth metrics of different signatures. Hence, from complex spaces real Lorentzian, neutral or Riemannian spaces can be obtained[1].

The spaces which we analyze in this paper are *two-sided conformally recurrent*. Two-sided conformally recurrent spaces were defined in [5].

Definition 1.1 *Let $\mathcal{M}$ be a four-dimensional real smooth or complex analytic differential manifold equipped with a real or holomorphic metric ds^2.*

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[1] By *Lorentzian space*, we understand a real space equipped with a metric of the signature $(+++-)$; by *neutral space* — a real space equipped with a metric of the signature $(++--)$; by *Riemannian space* — a real space equipped with a metric of the signature $(++++)$.

Then the pair $(\mathcal{M}, ds^2)$ *is two-sided conformally recurrent Riemannian manifold if there exist vectors* r_m *and* $\dot{r}_m$ *such that*

$$\nabla_m C_{ABCD} = r_m C_{ABCD} \,, \tag{1.1a}$$

$$\nabla_m C_{\dot{A}\dot{B}\dot{C}\dot{D}} = \dot{r}_m C_{\dot{A}\dot{B}\dot{C}\dot{D}} \,, \tag{1.1b}$$

and C_{ABCD} *and* $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ *do not vanish simultaneously.*

C_{ABCD} and $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ are spinorial images of the self-dual (SD) and anti-self-dual (ASD) parts of the Weyl tensor, respectively.

Two-sided conformally recurrent 4-dimensional manifolds which are equipped with a real metric of the Lorentzian signature have been analyzed in many papers. McLenaghan and Leroy devoted to this problem paper [3]. They found all Lorentzian metrics which are two-sided conformally recurrent. It appeared that if a space is not conformally flat, then these metrics are of the Petrov–Penrose-types [D] or [N].

The generalization of this problem to the complex case has been presented in the distinguished paper by Plebański and Przanowski [5]. They proved that if a space is two-sided conformally recurrent, then both SD and ASD Weyl spinors must be of the Petrov–Penrose-types [D], [N] or [−]. Hence, the only complex spaces which can be two-sided conformally recurrent are spaces of the types $[\mathrm{D}]\otimes[\mathrm{D}]$, $[\mathrm{N}]\otimes[\mathrm{N}]$, $[\mathrm{D}]\otimes[-]$ and $[\mathrm{N}]\otimes[-]$. The generalization of the results of [3] leads to the complex spaces of the types $[\mathrm{D}]\otimes[\mathrm{D}]$ and $[\mathrm{N}]\otimes[\mathrm{N}]$. Real neutral and Riemannian slices of these types of spaces have been found in [5]. In addition, two-sided conformally recurrent self-dual metrics of the type $[\mathrm{N}]\otimes[-]$ have been explicitly presented, but Plebański and Przanowski wrote: "Up to now we have not succeeded in integrating the type $[\mathrm{D}]\otimes[-]$. It seems to be a rather hard problem". The main aim of our paper is to fill this gap.

The paper is organized as follows. In Section 2, the definition of the *congruence of the null strings* is given and the relation between such congruences and two-sided conformally recurrent spaces is found. In Section 3, the general structure of the *weak hyperheavenly* ($\mathcal{HH}$) *spaces* is presented. Finally, Section 4 contains the main results — explicit self-dual metrics which are two-sided conformally recurrent. All considerations are purely local. We use the spinorial formalism [4].

2. Congruences of the null strings

From Eqs. (1.1a)–(1.1b) it follows that the spaces which are two-sided conformally recurrent are equipped with a 2-dimensional completely integrable distributions. The integral manifolds of such distributions are 2-dimensional, totally null and totally geodesic surfaces, called *the null strings*. The family of such surfaces constitute *the congruence of the null strings*.

Definition 2.1 *Congruence (foliation) of SD null strings in a complex manifold $\mathcal{M}$ is a family of totally null and totally geodesics 2-dimensional holomorphic surfaces, such that for every point $p \in \mathcal{M}$, there exists only one surface of this family such that p belongs to this surface.*

It has been proved in [6] that a manifold $\mathcal{M}$ admits a congruence of the SD null strings if and only if there exists a nowhere vanishing undotted 1-index spinor field m_A such that

$$m^A m^B \nabla_{A\dot{M}} m_B = 0\,. \tag{2.1}$$

Equations (2.1) are called *the SD null string equations.* From (2.1), we find

$$\nabla_{A\dot{M}} m_B = Z_{A\dot{M}} m_B + \in_{AB} M_{\dot{M}}\,, \tag{2.2}$$

where $Z_{A\dot{M}}$ is the *Sommers vector* and the spinor field $M_{\dot{M}}$ is the *expansion* of the congruence of SD null strings [6]. The expansion describes the most important property of the congruence of the null strings. If $M_{\dot{M}} = 0$, then the 2-dimensional distribution $\mathcal{D}_{m^A} := \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{B}} b^{\dot{B}} \neq 0$ is parallely propagated. It means that $\nabla_X V \in \mathcal{D}_{m^A}$ for every vector field $V \in \mathcal{D}_{m^A}$ and for arbitrary vector field X. Such congruences are called *nonexpanding.* If $M_{\dot{M}} \neq 0$, then we deal with *expanding* congruences.

It can be easily proved that if a spinor m_A generates a congruence of SD null strings, then it is a Penrose spinor, *i.e.* $C_{ABCD} = m_{(A} a_B b_C c_{D)}$, see, *e.g.* [1]. Moreover, if a spinor m_A generates a nonexpanding congruence of SD null strings, then it is a multiple Penrose spinor, *i.e.* $C_{ABCD} = m_{(A} m_B a_C b_{D)}$. Thus, we arrive at the following:

Theorem 2.2 *Let $C_{ABCD} \neq 0$ and $\nabla_m C_{ABCD} = r_m C_{ABCD}$. Then a manifold $\mathcal{M}$ is equipped with one nonexpanding congruence of the SD null strings and C_{ABCD} is of the Petrov–Penrose type $[N]$, or a manifold $\mathcal{M}$ is equipped with two nonexpanding congruences of the SD null strings and C_{ABCD} is of the Petrov–Penrose type $[D]$.*

In self-dual spaces, the ASD part of the Weyl spinor vanishes, $C_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$. It implies that there are infinitely many congruences of the ASD null strings and if the curvature scalar $R \neq 0$, then all of them are expanding.

It follows from Theorem 2.2 that the only self-dual spaces which can be two-sided conformally recurrent are spaces of the types $[\mathrm{N}]^n \otimes [-]^n$ or $[\mathrm{D}]^{nn} \otimes [-]^e$ (where the upper index $n(e)$ means that the corresponding congruence of the null strings is nonexpanding (expanding)).

3. Weak hyperheavenly spaces

It appears that the best formalism to find the general metrics of the type $[\mathrm{D}]^{nn} \otimes [-]^{e}$ which are two-sided conformally recurrent is the formalism of the *weak hyperheavenly ($\mathcal{HH}$) spaces* [2].

Definition 3.1 *Weak hyperheavenly space (weak $\mathcal{HH}$ space) is a 4-dimensional complex analytic differential manifold $\mathcal{M}$ endowed with a holomorphic metric ds^2 satisfying the following conditions:*

- *there exists a congruence of the SD null strings generated by the spinor m_A,*
- *the SD Weyl spinor C_{ABCD} is algebraically degenerate and m_A is a multiple Penrose spinor i.e. $C_{ABCD} = m_{(A} m_B a_C b_{D)}$.*

Definition 3.1 implies that weak hyperheavenly spaces are spaces of the types $[\deg]^n \otimes [\mathrm{any}]$ or $[\deg]^e \otimes [\mathrm{any}]$ (where *deg* means that SD part of the Weyl spinor is algebraically degenerated and *any* means that ASD part of the Weyl spinor is of the arbitrary Petrov–Penrose type). The metric of the weak hyperheavenly spaces can be easily specialized to the case of the spaces of the types $[\deg]^n \otimes [-]$. Indeed, one finds that the metric of the space of the type $[\deg]^n \otimes [-]$ without any loss of generality can be brought to the form of

$$\mathrm{d}s^2 = 2\left(-\mathrm{d}p^{\dot{A}}\mathrm{d}q_{\dot{A}} + Q^{\dot{A}\dot{B}}\,\mathrm{d}q_{\dot{A}}\mathrm{d}q_{\dot{B}}\right), \tag{3.1}$$

where $(q_{\dot{A}}, p_{\dot{B}})$ are local coordinates and

$$Q^{\dot{A}\dot{B}} = A^{\dot{N}} p_{\dot{N}} p^{\dot{A}} p^{\dot{B}} + B^{\dot{N}(\dot{A}} p^{\dot{B})} p_{\dot{N}} + B p^{\dot{A}} p^{\dot{B}} + C^{\dot{A}\dot{B}\dot{N}} p_{\dot{N}} + C^{(\dot{A}} p^{\dot{B})} + E^{\dot{A}\dot{B}}, \tag{3.2}$$

where $A^{\dot{N}}$, $B^{\dot{N}\dot{A}} = B^{(\dot{N}\dot{A})}$, B, $C^{\dot{A}\dot{B}\dot{N}} = C^{(\dot{A}\dot{B}\dot{N})}$, $C^{\dot{A}}$, $E^{\dot{N}\dot{A}} = E^{(\dot{N}\dot{A})}$ are arbitrary functions of the variables $q^{\dot{A}}$. The scalar curvature R, nonzero SD conformal curvature coefficient $C^{(i)}$ and nonzero components of the traceless Ricci tensor $C_{AB\dot{C}\dot{D}}$ read

$$\begin{aligned}
\frac{R}{6} &= C^{(3)} = 4A_{\dot{N}} p^{\dot{N}} - 2B\,, \qquad & C^{(2)} = -\partial^{\dot{A}} Q_{\dot{A}}\,, \\
\frac{1}{2}C^{(1)} &= -\eth^{\dot{A}} Q_{\dot{A}} + Q_{\dot{B}} \partial_{\dot{C}} Q^{\dot{B}\dot{C}}\,, \qquad & C_{12\dot{A}\dot{B}} = 2A_{(\dot{A}} p_{\dot{B})} + B_{\dot{A}\dot{B}}\,, \\
C_{22\dot{A}\dot{B}} &= -\partial_{(\dot{A}} Q_{\dot{B})}\,,
\end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
Q_{\dot{B}} := {} & \frac{\partial}{\partial q_{\dot{A}}}\Big(A^{\dot{N}} p_{\dot{N}} p_{\dot{A}} p_{\dot{B}} + B^{\dot{N}}{}_{(\dot{A}} p_{\dot{B})} p_{\dot{N}} + B p_{\dot{A}} p_{\dot{B}} + C^{\dot{N}}{}_{\dot{A}\dot{B}} p_{\dot{N}} + C_{(\dot{A}} p_{\dot{B})} + E_{\dot{A}\dot{B}}\Big) \\
& - A_{\dot{C}} C^{\dot{C}\dot{A}\dot{X}} p_{\dot{X}} p_{\dot{A}} p_{\dot{B}} + \frac{1}{2} A^{\dot{C}} C^{\dot{A}} p_{\dot{C}} p_{\dot{A}} p_{\dot{B}} - \frac{1}{2} B B^{\dot{A}\dot{C}} p_{\dot{C}} p_{\dot{A}} p_{\dot{B}} \\
& - B_{\dot{B}\dot{C}} C^{\dot{A}\dot{C}\dot{X}} p_{\dot{A}} p_{\dot{X}} - A_{\dot{C}} E^{\dot{A}\dot{C}} p_{\dot{A}} p_{\dot{B}} + A^{\dot{N}} E^{\dot{A}}{}_{\dot{B}} p_{\dot{A}} p_{\dot{N}} - \frac{3}{4} C_{\dot{X}} B^{\dot{X}\dot{Z}} p_{\dot{Z}} p_{\dot{B}} \\
& - \frac{1}{4} C^{\dot{A}} B^{\dot{X}}{}_{\dot{B}} p_{\dot{A}} p_{\dot{X}} - p_{\dot{X}} \left(C^{\dot{A}\dot{C}\dot{X}} C_{\dot{A}\dot{C}\dot{B}} + \frac{1}{4} C_{\dot{B}} C^{\dot{X}} - B E^{\dot{X}}{}_{\dot{B}} \right) \\
& - E^{\dot{A}\dot{C}} B_{\dot{A}\dot{C}} p_{\dot{B}} - \frac{1}{2} C_{\dot{B}}{}^{\dot{A}\dot{C}} C_{\dot{A}} p_{\dot{C}} - E^{\dot{A}\dot{C}} C_{\dot{A}\dot{C}\dot{B}} - \frac{1}{2} C^{\dot{A}} E_{\dot{A}\dot{B}}
\end{aligned} \tag{3.4}
$$

and

$$\partial_{\dot{A}} := \frac{\partial}{\partial p^{\dot{A}}}\,, \qquad \eth^{\dot{A}} := \frac{\partial}{\partial q_{\dot{A}}} + Q^{\dot{A}\dot{B}} \partial_{\dot{B}}\,. \tag{3.5}$$

The last step is to specialize metric (3.1) to be of the type $[\mathrm{D}]^{nn} \otimes [-]^{e}$. We proved that without any loss of generality it can be done by putting $Q_{\dot{B}} = 0$. Then (3.4) becomes a third-order polynomial in $p^{\dot{A}}$ and it splits to the system of 15 equations for 15 functions. This system can be completely solved.

4. The metrics

Finally, we arrive at the following results. There are three classes of the spaces of the type $[\mathrm{D}]^{nn} \otimes [-]^{e}$ which are two-sided conformally recurrent. The metric of such spaces has *double Kerr–Schild* form [7]

$$\frac{1}{2}\mathrm{d}s^2 = \mathrm{d}y\mathrm{d}q - \mathrm{d}x\mathrm{d}p + \mathcal{A}\,\mathcal{B}^2 + \mathcal{P}\,\mathrm{d}p^2\,, \tag{4.1}$$

where (x, y, p, q) are local coordinates. Metric (4.1) becomes the *single Kerr–Schild metric* if and only if $\mathcal{P} = 0$. For the first class of solutions, one finds

$$
\begin{aligned}
& \mathcal{A} := -xy^2 - M_0 x - 3N_0 y\,, \qquad \mathcal{B} := \mathrm{d}q + \frac{-x^2 y + \frac{1}{2}y^2 + 3P_0 y - \frac{1}{2}M_0}{xy^2 + M_0 x + 3N_0 y}\mathrm{d}p\,, \\
& \mathcal{P} := -x^3 + xy + 3P_0 x + \frac{3}{2}N_0 + \frac{\left(-x^2 y + \frac{1}{2}y^2 + 3P_0 y - \frac{1}{2}M_0\right)^2}{xy^2 + M_0 x + 3N_0 y}\,,
\end{aligned} \tag{4.2}
$$

where M_0, N_0 and P_0 are arbitrary constants. The traceless Ricci tensor C_{ab} of (4.2) and the function $\mathcal{P}$ are necessarily nonzero.

The second class has the form of (4.1) with

$$\begin{aligned}\mathcal{A} &:= -x - B_0\,, \qquad \mathcal{B} := y\,\mathrm{d}q + \frac{3P_0 - x^2}{x + B_0}\,\mathrm{d}p\,,\\ \mathcal{P} &:= \frac{\left(3P_0 - x^2\right)\left(3P_0 - B_0^2\right)}{x + B_0}\,,\end{aligned} \tag{4.3}$$

where P_0 and B_0 are arbitrary constants. This solution is characterized by $C_{ab} \neq 0$. For $3P_0 = B_0^2$, the metric becomes single Kerr–Schild metric.

The third class corresponds to the Einstein case. It can be brought to the form of (4.1) with

$$\begin{aligned}&\mathcal{A} := B_0\,, \qquad \mathcal{B} := y\,\mathrm{d}q - \frac{1}{B_0 y}\left(B_0 x y + f\left(qy - \frac{1}{B_0}\right)\right)\mathrm{d}p\,,\\ &\mathcal{P} := -fqx - \left(qy - \frac{1}{B_0}\right)\left(f^2 q^2 + \frac{\mathrm{d}f}{\mathrm{d}p}q + \frac{2xf}{y} + \frac{f^2}{B_0 y^2}\left(qy - \frac{1}{B_0}\right)\right)\,,\end{aligned} \tag{4.4}$$

where $f = f(p)$ is an arbitrary function of one variable and $3B_0 = \Lambda \neq 0$ where Λ is cosmological constant. If $f = 0$, then the metric becomes single Kerr–Schild metric.

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