FERMION CONDENSATION UNDER ROTATION ON ANTI-DE SITTER SPACE*

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Due to the local curvature, the fermion condensate (FC) for a free Dirac field on anti-de Sitter (adS) space becomes finite, even in the massless limit. Employing the point splitting method using an exact expression for the Feynman two-point function, an expression for the local FC is derived. Integrating this expression, we report the total FC in the adS volume and on its boundary.

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1. Introduction

Over the past couple of decades, the analysis of quantum field theory (QFT) on the anti-de Sitter (adS) background space-time has received much attention due to the conjectured adS/CFT correspondence [1]. Through this conjecture, important insight into the properties of the quark–gluon plasma formed in relativistic heavy-ion collisions was drawn [2].

Recent experiments performed by the STAR Collaboration revealed the polarisation of the QGP in non-central collisions [3]. One mechanism that could lead to this polarisation is the chiral vortical effect, due to the spin–orbit coupling predicted through the Dirac equation [4].

In this contribution, we present a study of thermal states of fermions undergoing rigid rotation on the anti-de Sitter space. The focus of this study is the fermion condensate (FC) induced by the coupling to curvature. The discussion is restricted to massless particles in the absence of interaction.

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2. Finite temperature expectation values

The line element of adS can be written as

\[ ds^2 = \frac{1}{\cos^2 \omega r} \left[ -dt^2 + dr^2 + \sin^2 \omega r \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right], \tag{1} \]

where \( t \in (-\infty, \infty) \), \( 0 \leq \omega r < \frac{\pi}{2} \) and the inverse radius of curvature \( \omega \) is related to the Ricci scalar through \( R = -12\omega^2 \). We further employ the following Cartesian gauge tetrad \([5]\):

\[
e_i = \cos \omega r \partial_t, \quad e_i = \cos \omega r \left[ \frac{\omega r}{\sin \omega r} \left( \delta_{ij} - \frac{x^i x^j}{r^2} \right) + \frac{x^i x^j}{r^2} \right] \partial_j, \tag{2}\]

by which the local gamma matrices \( \gamma^\mu = e^\mu_{\hat{\alpha}} \gamma^{\hat{\alpha}} \) are written in terms of the Minkowski ones, which satisfy \( \{ \gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}} \} = -2\eta^{\hat{\alpha}\hat{\beta}} \). At finite temperature \( \beta_0^{-1} \) and in rigid rotation with angular velocity \( \Omega = \Omega k \), we have \([6]\)

\[
\langle \hat{\Psi} \hat{\Psi} \rangle_{\beta_0,\Omega} = Z^{-1} \text{tr} \left( \hat{\rho} \hat{\Psi} \hat{\Psi} \right), \quad \hat{\rho} = e^{-\beta_0 (\hat{H} - \Omega \hat{M}^z)} , \tag{3}\]

where \( \hat{H} = i\partial_t, \hat{M}^z = -i\partial_\varphi + S^z, S^z = \frac{i}{2} \gamma_1 \gamma^2 \) and \( Z = \text{tr}(\hat{\rho}) \).

To evaluate Eq. (3), we take the point-splitting approach, by which \([7]\)

\[
\langle \hat{\Psi} \hat{\Psi} \rangle_{\beta_0,\Omega} = - \lim_{x' \to x} [i S^F_{\beta_0,\Omega} (x, x') \Lambda (x', x)] , \tag{4}\]

where \( S^F_{\beta_0,\Omega}(x, x') \) is the thermal two-point function and \( \Lambda(x, x') \) is the bispinor of parallel transport, given by \([11]\)

\[
\Lambda(x, x') = \frac{\sec(\omega s/2)}{\sqrt{\cos \omega r \cos \omega r'}} \times \left[ \frac{\cos \omega \Delta t}{2} \left( \cos \frac{\omega r}{2} \cos \frac{\omega r'}{2} + \sin \frac{\omega r}{2} \sin \frac{\omega r'}{2} \frac{x \cdot \gamma x' \cdot \gamma}{r'} \right) \right.
\]

\[
+ \sin \frac{\omega \Delta t}{2} \left( \sin \frac{\omega r}{2} \cos \frac{\omega r'}{2} \frac{x \cdot \gamma i}{r'} + \sin \frac{\omega r'}{2} \cos \frac{\omega r}{2} \frac{x' \cdot \gamma i}{r'} \right) \right]. \tag{5}\]

Using the property \( \hat{\rho} \hat{\Psi}(t, \varphi) \hat{\rho}^{-1} = e^{-\beta_0 \Omega S^z} \hat{\Psi}(t + i\beta_0, \varphi + i\beta_0 \Omega) \), together with the imaginary time anti-periodicity of the two-point function \([8]\), it is possible to compute \( S^F_{\beta_0,\Omega}(x, x') \) via \([9]\)

\[
S^F_{\beta_0,\Omega} (x, x') = \sum_{j=-\infty}^{\infty} (-1)^j e^{-i j \beta_0 \Omega S^z} S^F_{\text{vac}} (t + i j \beta_0, \varphi + i j \beta_0 \Omega; t', \varphi') , \tag{6}\]

\(^1\) We consider the covering space of adS.
where $S_{\text{vac}}^F(x, x')$ is the vacuum two-point function. The above expression is valid only when the vacua ($\beta_0 \rightarrow 0$) corresponding to the rotating (finite $\Omega$) and non-rotating ($\Omega = 0$) cases coincide. This is ensured on adS when $|\Omega| \leq \omega$ \cite{10}, which we assume to hold for the remainder of this paper.

Due to the maximal symmetry of adS, $S_{\text{vac}}^F(x, x')$ can be written as \cite{12}

$$iS_{\text{vac}}^F(x, x') = [\mathcal{A}(s) + \mathcal{B}(s) \Phi] \Lambda(x, x'),$$

where $n_\mu = \nabla_\mu s(x, x')$ is the normalised tangent at $x$ to the geodesic connecting $x$ and $x'$, while the geodesic interval $s$ is given through

$$\cos \omega s = \frac{\cos \omega t}{\cos \omega r \cos \omega r'} - \cos \gamma \tan \omega r \tan \omega r',$$

where $\gamma$ is the angle between $x$ and $x'$, such that $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \Delta \varphi$. For massless fermions, the functions $\mathcal{A}$ and $\mathcal{B}$ are \cite{11}

$$\mathcal{A}|_{M=0} = \frac{\omega^3}{16\pi^2} \left(\cos \frac{\omega s}{2}\right)^{-3}, \quad \mathcal{B}|_{M=0} = \frac{i\omega^3}{16\pi^2} \left(\sin \frac{\omega s}{2}\right)^{-3}.$$

### 3. Analysis and conclusions

Without presenting the details of the computation, we find \cite{10}

$$\left\langle \hat{\Psi} \hat{\Psi} : \right\rangle_{\beta_0, \Omega} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \omega^3 (\cos \omega r)^4 \cosh \frac{\omega j \beta_0}{2} \cosh \frac{\Omega j \beta_0}{2}}{2\pi^2 \sinh^2 \left(\frac{\omega j \beta_0}{2}\right) + \cos^2 \omega r - \sin^2 \omega r \sin^2 \theta \sinh^2 \left(\frac{\Omega j \beta_0}{2}\right)}.$$  

The total FC can be obtained by integrating Eq. (10) over the whole space

$$V_{\beta_0, \Omega}^{\text{FC}} = \int d^3x \sqrt{-g} \left\langle \hat{\Psi} \hat{\Psi} : \right\rangle_{\beta_0, \Omega} = -\sum_{j=1}^{\infty} \frac{(-1)^j \cosh \left(\frac{\Omega j \beta_0}{2}\right) / \sinh \left(\frac{\omega j \beta_0}{2}\right)}{\cosh(\omega j \beta_0) - \cosh(\Omega j \beta_0)}$$

$$\approx \frac{3\zeta(3)T_0^3}{\omega (\omega^2 - \Omega^2)} - \frac{3\omega^2 - \Omega^2}{6\omega (\omega^2 - \Omega^2)} T_0 \ln 2 + O(T_0^{-1}).$$  

On the boundary, the following result is obtained:

$$S_{\beta_0, \Omega}^{\text{FC}} = \int d\Omega \sqrt{-g} \left\langle \hat{\Psi} \hat{\Psi} : \right\rangle_{\beta_0, \Omega}$$

$$\approx \frac{7\pi^3 T^4}{45(\omega^2 - \Omega^2)^{3/2}} \left[\frac{\omega}{\Omega} \tan^{-1} \left(\frac{\Omega/\omega}{\sqrt{1 - \frac{\Omega^2}{\omega^2}}}\right) + \sqrt{1 - \frac{\Omega^2}{\omega^2}}\right] + O(T^2).$$  


Fig. 1. Dependence of (a) $V_{\beta_0,\Omega}^{FC}$ and (b) $S_{\beta_0,\Omega}^{FC}/\omega$ with respect to $(1 - \Omega^2/\omega^2)^{-1}$, in logarithmic scale. The dotted lines and symbols are numerical results obtained using Eq. (10), while the analytic curves correspond to Eqs. (11) and (12).

Both $V_{\beta_0,\Omega}^{FC}$ (11) and $S_{\beta_0,\Omega}^{FC}$ are amplified due to the rotation through the prefactors $(1 - \Omega^2/\omega^2)^{-1}$ and $(1 - \Omega^2/\omega^2)^{-3/2}$, respectively. Figures 1 (a) and (b) show the dependence of $V_{\beta_0,\Omega}^{FC}$ and $S_{\beta_0,\Omega}^{FC}$ on $(1 - \Omega^2/\omega^2)^{-1}$, for various values of the temperature $T_0 = \beta_0^{-1}$. It can be seen that the analytic results (11) and (12) (shown with solid black lines) match well the numerical results (dotted lines and symbols) computed using Eq. (10).

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