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ON RELATIONS BETWEEN ELLIPTIC AND ELEMENTARY FUNCTIONS*

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V.I. Arnold suggested that an elliptic Weierstrass function cannot be reduced to an elementary one. We prove this conjecture by demonstration that the Weierstrass $\wp$-function cannot be homeomorphically transformed to any elementary function. This implies the general observation that the physical world cannot be described only by elementary functions up to appropriate coordinate transformations.

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1. Introduction

One of the Arnold's problem [7, Problem 2003-8, pp. 168–170] concerns relations between elliptic and elementary functions. Before its precise statement and solution, we want to explain its role in modern science following Arnold's talks and discussions. We address to a function arisen as a solution of the physical problem, for instance, to a function satisfying an ordinary differential equation. It is known that not all of these solutions are considered as elementary functions. The question consists in the principal possibility of the scale transformation of its argument $\boldsymbol{x} \in \mathbb{R}^n$ and values $f(\boldsymbol{x})$ to a new function which becomes elementary in the transformed coordinates. Hence, the question may be stated as follows. Whether the world is described in terms of elementary functions up to coordinate transformations? In the present paper, we give the negative answer by demonstration that an elliptic Weierstrass function cannot be reduced to an elementary one.

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One can hear first from the secondary school that it is impossible to express roots of the algebraic equations of degree 5 or higher in terms of the coefficients using only arithmetic operations and radicals. This assertion were proved by Ruffini in 1799 with minor gaps (see historical notes [3]) and it is known in our days as the Abel–Ruffini theorem.

In 1963, Vladimir Igorevich Arnold gave special course *Abel's theorem* for pupils of the College of the Moscow State University. Later, Alekseev prepared the book [2] according to this course. In 1963–1964, Arnold proved that equation $x^5 + ax + 1 = 0$ cannot be solved in a wider sense, namely the roots of this equation cannot be presented as a topologically elementary function $x(a)$. In 1963, Arnold stated the following question whether the elliptic integral and the Weierstrass $\wp$-function are topologically elementary or not? (see the definition of the topologically elementary function below in Introduction, *cf.* Definition 2.1 in Sec. 2). Arnold conjectured that the answer to this question is negative and proposed a plan of the long proof of this conjecture which has not been realized yet. One can find a discussion devoted to this question in the book [7, Problem 2003-8, pp. 168–170] and other interesting facts addressed to this question in [2–6].

For brevity, we denote the set of rational functions by $Q[x]$. In real analysis, the basic elementary functions are the following functions $Q \in Q[x]$, exp, log, sin, cos, tan, cot, arcsin, arccos, arctan, arccot. Functions which can be built from basic elementary functions by using a finite number of composition and four arithmetic operations are called *real elementary compositions*. This collection is surely excessive. Thus *e.g.* usually the radical x^α is referred to the basic elementary functions. However, it can be expressed through the above functions: $x^\alpha = \exp(\alpha \log x)$.

Conception of complex elementary functions is due to Liouville (see *e.g.* [8]). All complex elementary functions can be built by the basic functions Q, exp, log (below, this class is denoted by $\boldsymbol{E}[Q, \exp, \log]$), since all other basic in the real case functions are expressed through these functions by the formulas

$$\begin{aligned} \sin\, z &= \frac{1}{2i}\left(\mathrm{e}^{iz} - \mathrm{e}^{-iz}\right), \qquad \cos\, z = \frac{1}{2}\left(\mathrm{e}^{iz} + \mathrm{e}^{-iz}\right), \\ \arcsin\, z &= -i \log\left[iz + \sqrt{1-z^2}\right], \\ \arctan\, z &= \frac{i}{2}\left[\log(1-iz) - \log(1+iz)\right]. \end{aligned} \tag{1}$$

Similar to (1), one can also easily express tan, cot, arccos, arccot through Q, exp, log. In what follows, elementary functions of complex variable are those which can be obtained from the basic elementary functions Q, exp, log by using a finite number of arithmetic operations and compositions ($\boldsymbol{E}[Q, \exp, \log]$).

Let us introduce now a subclass of *elementary complex functions continuous in $\mathbb{C}$ except a discreet set of isolated in $\mathbb{C}$ points*, where a function can be equal to infinity. For shortness, we call this class *elementary continuous complex functions.* It is worth noting that the function log is discontinuous, where log is defined as a fixed branch of the complex logarithm [11]. The discontinuity set of log is the negative half-axis, *i.e.*, it is not a set of isolated points.

In paper [10], we have proved that the Weierstrass $\wp$-function cannot be homeomorphically transformed to any elementary complex function from $\boldsymbol{E}[Q, \exp]$. We took in [10] this class of elementary functions, since the Weierstrass function is continuous except a discreet set of points. Arnold noted that the logarithm has to be added to our class of elementary functions and made other remarks concerning [10]. Our proof from [10] for the class $\boldsymbol{E}[Q, \exp]$ can be extended to the continuous functions of $\boldsymbol{E}[Q, \exp, \log]$. This enables us to give the complete modified proof of Arnold's conjecture presented in the next section.

Before the proof, we recall that the elliptic integral is defined as follows [1]:

$$u(w) = \int_w^\infty \frac{\mathrm{d}t}{\sqrt{4t^3 - g_2 t - g_3}}\,, \tag{2}$$

where g_2 and g_3 are given constants. The Weierstrass $\wp$-function with the periods ω_1, ω_2 ($\operatorname{Im} \omega_2/\omega_1 > 0$) can be defined as the series

$$\wp(z) = \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right). \tag{3}$$

It satisfies the differential equation

$$\mathrm{d}z = -\frac{\mathrm{d}\wp(z)}{\sqrt{4[\wp(z)]^3 - g_2\wp(z) - g_3}}\,, \tag{4}$$

where g_2 and g_3 are related to the periods ω_1 and ω_2. Comparing (2) and (4), one can see that the elliptic integral $u(w)$ is the inverse function to $\wp(z)$.

2. Main result

Let $\mathbb{C}_j$ ($j = 1, 2, 3, 4$) be four copies of the complex plane $\mathbb{C}$.

Definition 2.1. *A function $\widetilde{f}\colon \mathbb{C}_3 \to \mathbb{C}_4$ is called the topologically elementary if there exists an elementary function $f(z)$ and homeomorphisms h, k of the complex plane such that the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathbb{C}_1 & \overset{f}{\longrightarrow} & \mathbb{C}_2 \\
h \downarrow & & \downarrow k \\
\mathbb{C}_3 & \overset{\widetilde{f}}{\longrightarrow} & \mathbb{C}_4
\end{array}
\tag{5}
$$

We say that diagram (5) is commutative and the functions f and $\widetilde{f}$ are topologically conjugated, if $k \circ f = \widetilde{f} \circ h$.

Using the results and designation presented in Introduction, we state Arnold's problem about the Weierstrass function in the form of the following theorem.

Theorem 2.1. *The Weierstrass $\wp$-function is not topologically elementary, i.e., it is not topologically conjugated to any continuous function from $\boldsymbol{E}[Q, \exp, \log]$.*

Before the proof, we introduce the following definition.

Definition 2.2. *Let a continuous curve γ be defined by the parametrization $x \mapsto g(x)$, where $0 \le x < \infty$, $g(x) \in \mathbb{C}$. One says that γ goes by infinity if for any $R > 0$ there exist points of γ which do not lie in the disk $|z| < R$.*

One says that a continuous curve γ *goes to* infinity if the following relation holds $\lim_{x\to+\infty} g(x) = \infty$. Here, the second infinity belongs to the extended complex plane.

The first part of definition is equivalent to the standard definition of unbounded sets. This definition is introduced for shortness to describe a curve γ near infinity.

Proof. The proof is based on the assertion that for any function $f \in \boldsymbol{E}[Q, \exp, \log]$ and any homeomorphisms h, k of the complex plane, there exists such a curve $\gamma_1 \subset \mathbb{C}_1$ that the restriction of diagram (5) to γ_1 cannot be commutative.

Let us consider functions from $\boldsymbol{E}[Q, \exp, \log]$ which are continuous in $\mathbb{C}$ except a discreet set of isolated points in $\mathbb{C}$. We denote this set by S. One can see that these functions are single-valued analytic in $\mathbb{C}\backslash S$. It is worth noting that despite it is not forbidden for log to participate in a combination, the resulting function has only isolated singularities. According to the general classification of the isolated singularities (see *e.g.* [11, pp. 210–211]), an isolated singularity cannot be logarithmic. Hence, actually, we arrive at the class of elementary functions discussed in [10], *i.e.* the class $\boldsymbol{E}[Q, \exp]$ does not contain any function having at least one logarithmic singularity.

There are unbounded domains which contain curves going by infinity along which the functions exp and polynomials tends to ∞. It is true also for compositions, sums, differences and products of these functions. Thus, we will consider all those functions $f \in \boldsymbol{E}[Q, \exp]$ mapping every neighbourhood of infinity $|z| > r$ to a neighbourhood of infinity. Analogous consideration is valid also for (transcendental) meromorphic functions and for rational functions $\frac{P}{Q}$, deg $P >$ deg Q. In the case of rational functions $\frac{P}{Q}$, deg $P \leq$ deg Q, we have uniformly $\lim_{|z|\to\infty} \frac{P(z)}{Q(z)} = a \in \mathbb{C}$. The discussion of this case is presented at the end of the paper.

We assume that diagram (5) is commutative. It is possible to find the curves $\gamma_j \in \mathbb{C}_j$ $(j = 1, 2, 3, 4)$ having the following properties. The curve γ_4 goes to infinity, its end-points z_0 and ∞ lay on the extended complex plane $\widehat{\mathbb{C}}_4$. This behaviour of γ_4 will lead to a contradiction with the following properties of the corresponding curves γ_j on other complex planes. The curve $\gamma_2 = k^{-1}(\gamma_4)$ goes by infinity on $\mathbb{C}_2$ since k is a homeomorphism. The curve $\gamma_1 = f^{-1}(\gamma_2)$ goes by infinity on $\mathbb{C}_1$ again due to the Fragmen–Lindelöf theorem. Here, $f^{-1}(\gamma_2)$ means the full pre-image of γ_2. The curve $\gamma_3 = h(\gamma_1)$ goes by infinity on $\mathbb{C}_3$ because k is a homeomorphism. The mapping $k \circ f = \wp \circ h\colon \gamma_1 \to \gamma_4$ homeomorphically transforms the curve γ_1 onto the curve $\gamma_4 := k \circ f(\gamma_1) \subset \mathbb{C}_4$.

Consider a doubly periodic family, $\Pi_{(l_1,l_2)}$, $(l_1, l_2) \in \mathbb{Z}^2$, of parallelograms on $\mathbb{C}_3$ such that the Weierstrass $\wp$-function is univalent in the interior of each parallelogram. It follows from the properties of $\wp$-function that such a family exists, cover the whole complex plane and on one of four sides of any parallelogram, there is a pole of $\wp$-function (see [1]). One of such a parallelogram with sides Γ_1, Γ_2, Γ_3 and Γ_4 is displayed in Fig. 1. Let for definiteness the vertices of this parallelogram be the points $-\frac{1}{2}(\omega_1 + \omega_2)$, $-\frac{1}{2}\omega_2$, $\frac{1}{2}\omega_2$ and $\frac{1}{2}(-\omega_1 + \omega_2)$. Let us consider the second parallelogram adjusted to the first one along Γ_2 (the red parallelogram in Fig. 1 in on-line version of the paper). These two parallelograms form the periodicity cell of $\wp(\zeta)$ denoted by $\mathcal{P}$. Due to periodicity of the Weierstrass function, one can note that its values on the boundaries $\partial\mathcal{P} + l_1\omega_1 + l_2\omega_2$, are connected to each other in such a way that the sets

$$L := \wp(\partial\mathcal{P} + l_1\omega_1 + l_2\omega_2) \tag{6}$$

do not depend on l_1, l_2. Moreover, L is a bounded set on the complex plane $\mathbb{C}_4$.

Let us examine the behaviour of the curve γ_3 and its image $\wp(\gamma_3)$. Let us take an infinite sequence $\{\mathcal{P}_m\}_{m=1}^{\infty}$ of the family of parallelograms $\mathcal{P} + l_1\omega_1 + l_2\omega_2$, for which γ_3 intersects their sides. Choosing at least one crossing point ζ_m on $\gamma_3 \cap \partial\mathcal{P}_m$, we order them in accordance with the orientation on γ_3. The obtained sequence $(\zeta_{m_p})_{p=1}^{\infty}$ tends to infinity.

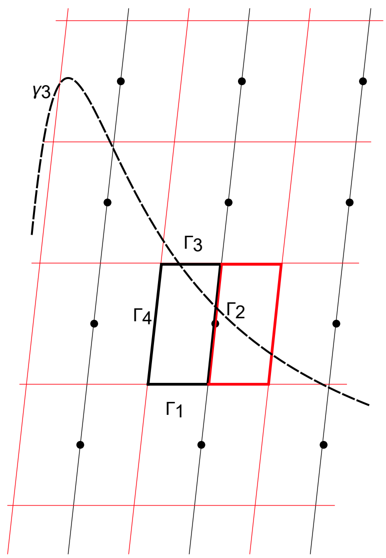

Fig. 1. (Colour on-line) Complex plane $\mathbb{C}_3$. Univalent parallelogram of the $\wp$-function with sides Γ_1, Γ_2, Γ_3, Γ_4. The side Γ_2 contains the pole of the $\wp$-function. Two univalent parallelograms generate a periodicity parallelogram of the $\wp$-function.

The curve $\gamma_4 \equiv \wp(\gamma_3)$ goes to infinity. At the same time, the points $\wp(\zeta_{m_p})$ belong to the bounded set L. This contradiction completes the proof when γ_2 goes by infinity.

In the remaining case of rational functions $f = \frac{P}{Q}$, deg $P \leq$ deg Q, we have

$$\lim_{t\to\infty, t\in\gamma_1} f(t) = a \in \mathbb{C}$$

for any unbounded curve γ_1, hence, $\lim_{t\to\infty, t\in\gamma_1} (k\circ f)(t) = b \in \mathbb{C}$. The same argument as before shows that the function $\wp \circ h$ cannot satisfy the relation

$$\lim_{t\to\infty, t\in\gamma_1} (\wp \circ h)(t) = b\,.$$

Therefore, diagram (5) cannot be commutative in this case too. □

The proof of Theorem is completed.

Before Professor Vladimir Igorevich Arnold passed away, we had opportunity to discuss with him the content of the paper. He made some remarks concerning the class $\boldsymbol{E}[Q, \exp, \log]$. We highly appreciate his advices and took them into account in the present version.

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