

# SHEAR VISCOSITY IN QCD AND WHY IT IS HARD TO CALCULATE\*

GUY D. MOORE

Institut für Kernphysik, Technische Universität Darmstadt  
Schlossgartenstraße 2, 64289 Darmstadt, Germany

(Received November 25, 2020)

Shear viscosity is a dynamical property of fluid systems close to equilibrium, describing resistance to shear flow. After reviewing the physics of viscosity and the reason why it is usually difficult to compute, I discuss its importance within the theory of QCD and the obstacles to carrying out such a computation. A diagrammatic analysis requires extensive resummations and even then convergence is poor at physically relevant couplings. Lattice approaches require a poorly controlled analytical continuation of data from the Euclidean to the Minkowski domain. At present, our best results for QCD shear viscosity come from the hydrodynamical interpretations of experiments, with first-principles calculations trailing behind.

DOI:10.5506/APhysPolBSupp.14.261

## 1. Introduction

The topic of this conference is the thermodynamics of the Quark–Gluon Plasma. This paper will not quite be about that — it will be “thermodynamics adjacent,” talking about something closely related. Specifically, viscosity is a property of fluids (anything without crystallization, that is, long-range ordering and spontaneous rotation symmetry breaking, so the medium is free to move around with no long-term “memory” of its previous form), when the fluid is close to equilibrium. Technically, we can relate it, by fluctuation–dissipation, to a fluid *in* equilibrium, but its unequal-time correlation functions which are technically beyond what people usually mean when they say “thermodynamics.” Thus, it is “thermodynamics adjacent” in the sense that you can talk about it for an equilibrium system, but it is not strictly a thermodynamical property.

---

\* Presented at the on-line meeting on *Criticality in QCD and the Hadron Resonance Gas*, Wrocław, Poland, 29–31 July 2020.

Viscosity plays a big role in the development of real systems. In the case of the Quark–Gluon Plasma, we are talking about heavy-ion collisions, where viscosity is important in the hydrodynamic development of the system [1, 2]. It might also be important in the biggest heavy-ion collisions of all, the neutron star mergers [3]. My goal in this paper is to review what shear viscosity is, and to lay out what we do and do not know about it in the case of Quantum Chromodynamics and the Quark–Gluon Plasma. This is only a review of the established literature; nothing in this work will be new, but I hope it will give a nice perspective and introduction to the topic, which is what the conference organizers requested from me.

In the next section, I will remind the reader of the physical meaning, and then of the definition of shear viscosity. Then I will explain why, for many systems, it is really not easy to compute it from first principles. In Section 4, I will show that for QCD at weak coupling, we can compute it analytically; but the computation requires very extensive resummations of the naive perturbative expansion, and the convergence of the series leaves a lot to be desired. Then Section 5 reviews attempts to get at the shear viscosity nonperturbatively from the lattice. Finally, I give a brief summary in the conclusions.

## 2. What is viscosity?

To illustrate what shear viscosity is about, let us think about a gas trapped between two plates. The plate at the top is moving with a constant velocity  $v_0$ ; a distance  $L$  below it, the lower plate is at rest. The fluid then develops a space-dependent velocity  $v(z) = v_0 z/L$ , and the individual atoms develop a space-dependent directional distribution, as illustrated in figure 1.

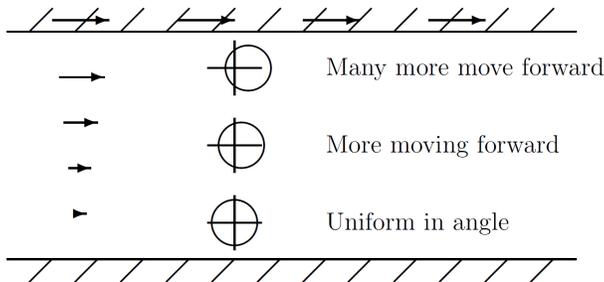


Fig. 1. Gas between two plates, with the top plate moving forward and the bottom plate at rest. The velocity distribution is hinted at the left; the microscopic distribution of particle number with direction is illustrated to the right.

Now remember that this is a gas; the individual molecules are free to fly, and will typically fly a distance  $\lambda$  (the mean free path) between scatterings with other gas molecules. That means that the molecules at any location actually came from somewhere else. At the center of the gas, the downward-moving molecules came from higher up, and the upward-moving molecules came from lower down. Since the molecules above this point tend to move forward, the downward-moving molecules tend to move forward; but the upward-moving molecules come from somewhere with a smaller forward velocity, and they are less likely to move forward. Therefore, there is a correlation between transverse and vertical velocity in the middle, as illustrated in figure 2. The same correlation occurs everywhere in the fluid.

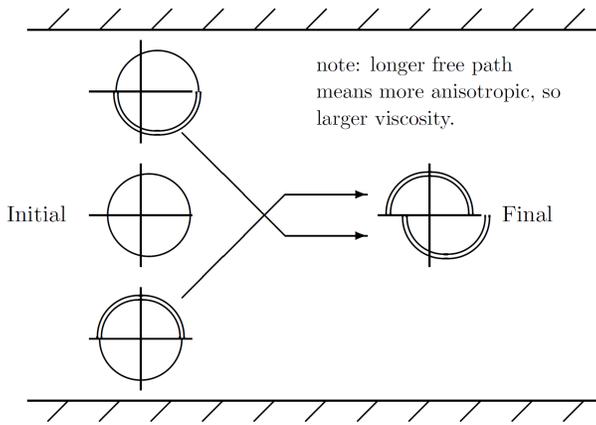


Fig. 2. Velocity distribution in more detail, remembering that molecules have a finite mean free path. Free propagation and space-dependent velocity lead to a locally anisotropic velocity distribution.

The particles which strike the bottom plate tend to move forward and, therefore, impart a forward-pointing force on that plate. At the top, the upward-going particles are moving forward at a smaller velocity than the plate, and impart a net backwards force. This force is proportional to the plate area  $A$  and to  $v_0/L$  the velocity gradient; we can write  $F = \eta \frac{Av_0}{L}$  or  $T_{zx} = -\eta \partial_z v_x$ , where  $\eta$  is some material-dependent coefficient, called the *shear viscosity*. In the last step, we recalled that a force per area is the definition of the stress tensor, and  $z$  and  $x$  are the directions of the surface-normal and the force respectively. More generally, for a fluid undergoing generalized inhomogeneous flow, we can write

$$T_{ij} - T_{ij,eq} = -\eta (\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k) - \zeta \delta_{ij} \partial_k v_k . \quad (1)$$

The  $\delta_{ij} \partial_k v_k$  term separates the pure-divergence part of the flow, which has an independent coefficient  $\zeta$  called the bulk viscosity, which I will not discuss.

$\eta$  is by definition a linearization about equilibrium; we assume  $\partial_i v_j$  is “small” in some sense — generally, it is enough that the resulting  $T_{ij}$  is small compared to the pressure. Moreover, we are assuming that the gradients do not change rapidly, that is, we assume a hierarchy in which  $\lambda \partial_i v_j \ll \lambda^2 \partial_i \partial_j v_k$ . If one or the other condition does not apply, shear viscosity — and hydrodynamics — is generally not a useful concept.

One last point: the fluid flow  $\partial_i v_j$  is itself related to the stress tensor  $T_{ij}$ . For instance, if we had a system in perfect equilibrium, we could generate fluid flow with nonvanishing  $\partial_i v_j$  by *distorting the metric*  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , and in the path integral this would arise from an interaction Hamiltonian of  $H_I = h_{\mu\nu} T^{\mu\nu}$ . Following this logic (see [4]), one can derive a Kubo relation for the shear viscosity in terms of equilibrium, unequal time, retarded correlation functions

$$\eta = i\partial_\omega \int d^3x \int_0^\infty dt e^{i\omega t} \langle [T^{xy}(x, t), T^{xy}(0, 0)] \rangle |_{\omega=0} . \quad (2)$$

We will need this relation repeatedly in what follows.

### 3. Why is it hard to compute?

What is the shear viscosity of air at STP? What about water? These sound like innocent questions, but answering them is hard. Naively, we should be able to: we know the theory describing each system. It is QED with electrons, oxygen nuclei, nitrogen nuclei, and hydrogen nuclei as the particle species. The theory has a weak coupling  $\alpha_{EM} = 1/137$  and it is nonrelativistic, so it *sounds naively* like this is a regime where we can do the calculation. However, that manifestly does not work. The problem is hard because:

- The theory is not actually weakly coupled in a useful way; potential energies are of the order of kinetic energies which are of the order of the temperature, while perturbation theory generally works by perturbing in potential energies being smaller than the other two.
- It is a many-body problem. More is different [5].
- We are after a low-frequency and long-distance property.

To see the challenge, think first of air. This is a system of weakly coupled, well-separated diatomic molecules. If we could compute the differential scattering cross sections between molecules, we could write a Boltzmann equation which could solve for the viscosity. However in practice, molecular scattering is a fairly hard atomic and molecular physics problem; first-principles

calculations are challenging, and they are made more so because we have to account for the rotational states of the molecules. However, if we had scattering data, we could predict the viscosity without too much effort.

The case of water is harder because the molecules are all pressed against each other in a highly correlated system. No amount of molecule-on-molecule scattering data will tell us how to deal with this system, and in the end, the most efficient thing to do is simply to measure the viscosity and give up on a first-principles calculation.

We might worry that the same kind of complexity hits us in QCD. At physically interesting temperatures  $T < 1 \text{ GeV}$ , the coupling is not small,  $\alpha_s \simeq 0.3$ , so there will be strong correlations and we do not have a good perturbative expansion. The theory is fully relativistic, so it is not clear that particles or particle number are useful concepts. Thus, to begin with, let us ask a simpler question: what happens at high temperatures where the coupling is actually weak?

#### 4. Weak coupling: resummations and series convergence

The case of high-temperature QCD, where the coupling is small, should be more tractable. Weak coupling means that there are long-lived quasiparticles, and the behavior is loosely analogous to a gas like the air. However, unlike molecular gases, we have an efficient tool for computing scattering cross sections, perturbation theory. Then this case should be tractable.

We want to evaluate the correlation function shown in Eq. (2). Naively, at leading order, we should only need to calculate the leftmost diagram in figure 3. If we were computing at generic off-shell  $(\omega, p)$  with  $|\omega^2 - p^2| \sim T^2$ , this would be correct. However, we do not want that case; we want  $p = 0$  and the small- $\omega$  limiting behavior, specifically the linear-in- $\omega$  term at  $\omega = 0$ . If the external momentum inserted by the  $T^{xy}$  operator is almost zero, then the upper propagator is on shell exactly where the lower propagator is; pinching poles. Computing this correctly requires including self-energies to push the poles off the real frequency axis. The result is that there is an  $\mathcal{O}(1/\alpha_s)$  enhancement. Inserting a vertical line, as in the right diagram in figure 3, gives another pair of lines which are on shell, and another  $1/\alpha_s$

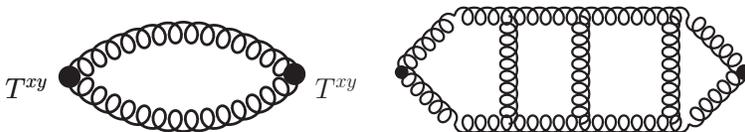


Fig. 3. Left: the naive leading-order diagram. Right: an example of the family of diagrams which actually determines the *leading-log* viscosity.

enhancement. Each vertical line costs  $\alpha_s$  for the vertices, but gives another  $1/\alpha_s$  enhancement, so we need to resum any number of such diagrams; for a detailed discussion, see [6–8]. The short version is that such diagrams, together with a “pinching pole” approximation in which we expand about the two propagators being nearly on-shell, resum into kinetic theory.

Nevertheless, the story is still more complex. Even at leading-logarithmic order [7], it is necessary to include self-energy corrections on low-momentum gluonic lines, the so-called Hard Thermal Loop resummation [9], see figure 4. At leading order [8], one needs to include a *much* more complex set of processes; inelastic splitting, induced by any number of scatterings, correctly accounting for the Landau–Pomeranchuk–Migdal destructive interference effect [10, 11], see figure 5. For a (much) more detailed exposition on this issue, see the original literature [8, 12, 13].

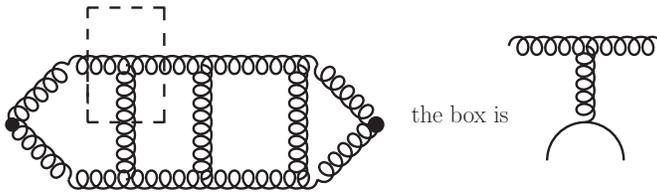


Fig. 4. Illustration of what kinetic theory has to do with ladder graphs. The indicated part of the ladder graph corresponds to the scattering matrix element shown; the lower half of the rung is the starred matrix element. The propagator must always be “cut” through a self-energy insertion, corresponding to a second particle against which the particle in consideration scatters.

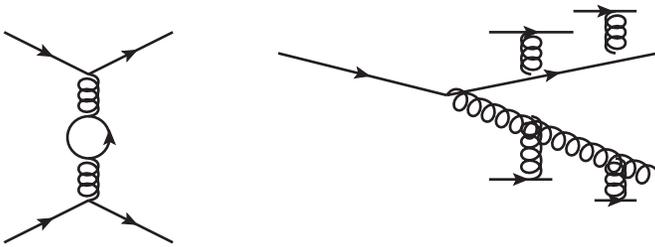


Fig. 5. Other relevant scattering processes. Left: normal scattering requires “HTL” resummations on propagators. Right: inelastic splitting with multiple scatterings must also be accounted for.

What about next-to-leading order? At this order, successive scatterings temporally overlap and interfere with each other, creating complex effects, see figure 6. Since the involved particles move at close to the speed of light, there are resummation methods which can be applied, rendering these NLO effects calculable [14–16]. The first interesting result is that, contrary to vacuum perturbation theory, the first corrections do not appear at  $\mathcal{O}(\alpha_s)$ ; they occur at  $\mathcal{O}(\sqrt{\alpha_s})$ . Second, the effects are surprisingly large. A comparison of the leading-order and next-to-leading order (LO and NLO) results for the shear viscosity are shown in figure 7.

The result is that perturbation theory for the shear viscosity only converges at temperatures well above 100 GeV.

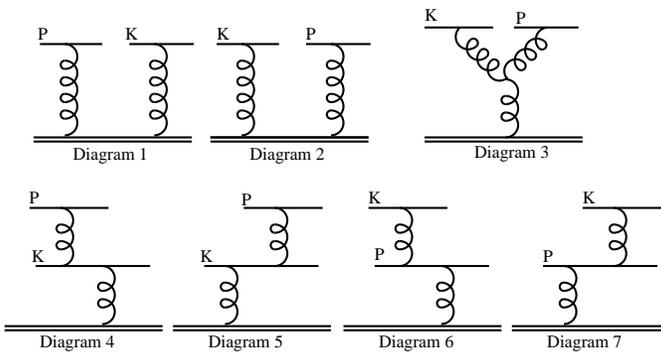


Fig. 6. The types of effect which arise at next-to-leading order. Two successive scatterings overlap in time, thus the interference between various orderings, connections, and non-Abelian effects must be included.

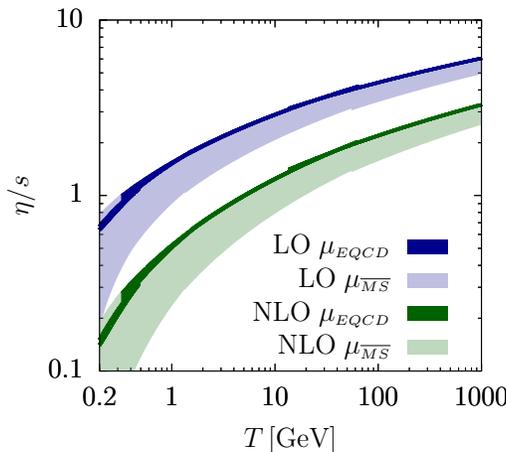


Fig. 7. Shear viscosity to entropy density ratio at LO and NLO in QCD as a function of temperature. The difference between orders is already a factor of 2 at  $T = 100$  GeV.

### 5. What about determining it from the lattice?

So perturbation theory does not work well. What about a first-principles, nonperturbative approach, Lattice QCD? The challenge is that the lattice deals with correlation functions in Euclidean time, and Eq. (2) is written in terms of Minkowski time. What is more, Eq. (2) refers to a commutator of operators. The lattice treats the path integral

$$Z = \int \mathcal{D} (A^\mu, \bar{\psi}, \psi) e^{-S_E[A, \bar{\psi}, \psi]} \tag{3}$$

which converges absolutely and has a positive integrand. That is what makes numerical integration possible. However, the Minkowski correlation functions are determined by the path integral

$$Z = \int \mathcal{D} (A^\mu, \bar{\psi}, \psi) e^{iS_M[A, \bar{\psi}, \psi]} \tag{4}$$

which does not converge absolutely, and has extremely important phases and phase cancellations. This (so far) makes Minkowski lattice techniques unavailable. However, computing a correlation function from the first path integral gives us  $G(\tau) \equiv \langle T^{xy}(x, i\tau) T^{xy}(0, 0) \rangle$ , that is, the correlation function at imaginary time. This is not the correlator we want, but at least, it is analytically related to our correlator. Generalizing Eq. (2) slightly, we can define

$$\sigma(\omega) = i \int d^3x \int_{-\infty}^{\infty} dt e^{i\omega t} \langle [T^{xy}(x, t), T^{xy}(0, 0)] \rangle, \quad \eta = \lim_{\omega \rightarrow 0} \frac{\sigma(\omega)}{\omega}. \tag{5}$$

This is related to the correlator we can measure via

$$G(\tau) = \int \frac{d\omega}{2\pi} \frac{\sigma(\omega)}{\omega} K(\omega, \tau), \quad K(\omega, \tau) = \frac{\omega \cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)} \tag{6}$$

and represents an “inverse problem.” We measure  $G(\tau)$  at a series of  $\tau$  values, and it tells us a series of integrals of  $\sigma(\omega)$ , each with different integral kernels, and with error bars. From this limited and incomplete information, we need to reconstruct the original  $\sigma(\omega)$ .

The reconstruction is theoretically ill posed. Worse, the kernel functions, shown in figure 8, have something in common; they all have the same value, and zero slope, at the point  $\omega = 0$ , exactly where we want to know our function. Therefore, if there is nontrivial structure in  $\sigma(\omega)$  close to  $\omega = 0$ , we will miss it. Perturbatively, there is such a structure. Nonperturbatively, we just do not know. In an analog theory,  $\mathcal{N} = 4$  SYM theory, we know that there is *no* such a structure [17]; yet the reconstructed  $\sigma(\omega)$  is almost the same as in weak-coupling QCD, as illustrated in figure 9.

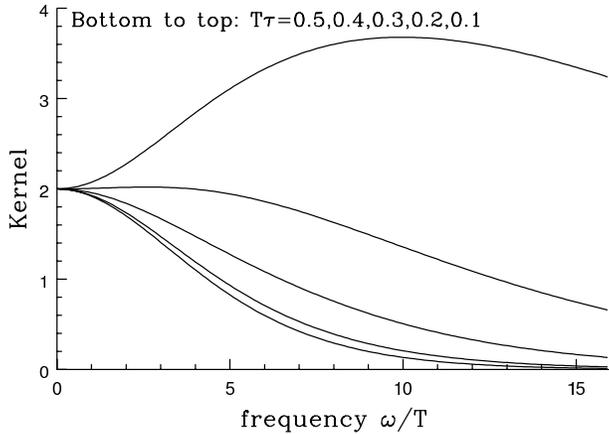


Fig. 8. The integral kernel  $\omega \cosh(\omega(\tau - 1/2T)) / \sinh(\omega/2T)$  for several  $\tau$  values.

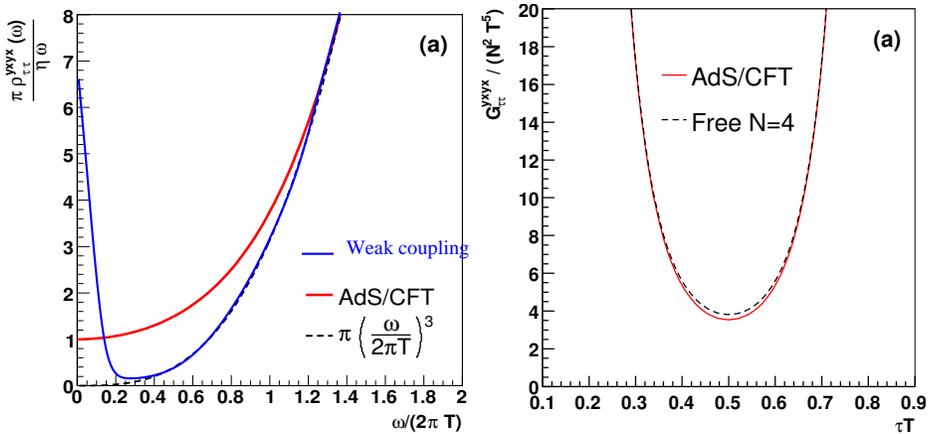


Fig. 9. Left: the (very different) spectral functions  $\sigma(\omega)/\omega$  for two theories, weakly-coupled QCD and strongly-coupled  $\mathcal{N} = 4$  SUSY QCD. Right: their almost identical Euclidean correlation functions  $G(\tau)$ . Figures from Ref. [17].

### 6. Conclusions

Shear viscosity is an important property of QCD, which plays a central role in its dynamics. Computing shear viscosity from first principles is generally hard, and it seems to be especially hard in the theory of QCD. We showed how the perturbative approach encounters challenges associated with diagrammatic resummation, and results in a poorly convergent perturbation theory. The prospects of determining QCD on the lattice are also challenged by the need to reconstruct a Minkowski function from its Euclidean continuation with error bars. In known examples, very distinct Minkowski functions

arise from nearly identical Euclidean continuations. Nevertheless, we see this as the second-best approach to determine shear viscosity for QCD at this time. The best approach is to try to extract it from data. This is also challenging but a different speaker should address this issue.

## REFERENCES

- [1] T. Schäfer, D. Teaney, *Rep. Prog. Phys.* **72**, 126001 (2009).
- [2] D.A. Teaney, «Viscous Hydrodynamics and the Quark Gluon Plasma», in: R.C. Hwa, X.-N. Wang (Eds.) «Quark–Gluon Plasma 4», *World Scientific*, 2010, pp. 207–266.
- [3] B. Abbott *et al.*, *Phys. Rev. Lett.* **119**, 161101 (2017).
- [4] G.D. Moore, K.A. Sohrabi, *Phys. Rev. Lett.* **106**, 122302 (2011).
- [5] P. Anderson, *Science* **177**, 393 (1972).
- [6] S. Jeon, *Phys. Rev. D* **52**, 3591 (1995).
- [7] P.B. Arnold, G.D. Moore, L.G. Yaffe, *J. High Energy Phys.* **0011**, 001 (2000).
- [8] P.B. Arnold, G.D. Moore, L.G. Yaffe, *J. High Energy Phys.* **0305**, 051 (2003).
- [9] E. Braaten, R.D. Pisarski, *Nucl. Phys. B* **337**, 569 (1990).
- [10] L. Landau, I. Pomeranchuk, *Dokl. Akad. Nauk Ser. Fiz.* **92**, 535 (1953).
- [11] A. Migdal, *Phys. Rev.* **103**, 1811 (1956).
- [12] P.B. Arnold, G.D. Moore, L.G. Yaffe, *J. High Energy Phys.* **0111**, 057 (2001).
- [13] P.B. Arnold, G.D. Moore, L.G. Yaffe, *J. High Energy Phys.* **0301**, 030 (2003).
- [14] S. Caron-Huot, *Phys. Rev. D* **79**, 065039 (2009).
- [15] J. Ghiglieri, G.D. Moore, D. Teaney, *J. High Energy Phys.* **03**, 095 (2016).
- [16] J. Ghiglieri, G.D. Moore, D. Teaney, *J. High Energy Phys.* **1803**, 179 (2018).
- [17] D. Teaney, *Phys. Rev. D* **74**, 045025 (2006).