

FLUCTUATIONS AND CONSERVATION LAWS*

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We report on recent progress concerning effects of global conservation laws on cumulants of conserved quantities. Specifically, we will relate — for an arbitrary equation of state — cumulants of a conserved charge measured in a subvolume of a thermal system with the corresponding grand-canonical susceptibilities, taking into account exact global conservation of that charge. Applications to actual measurement at the RHIC and LHC as well as extensions to multiple conserved charges will be discussed.

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1. Introduction

The studies of fluctuations of conserved charges have been central to the goal of exploring the QCD phase diagram. They represent a crucial observable in the RHIC beam energy scan, where one is searching for a possible critical point in the high baryon density region of the QCD phase diagram [1–3]. Fluctuations of conserved charges are also studied in heavy-ion experiments at the highest RHIC and LHC energies with the goal to experimentally find remnants of the chiral criticality at vanishing chemical potential [4, 5].

Theoretically fluctuations, or rather the various susceptibilities characterizing them, are calculated either using first-principle lattice QCD simulations [6, 7], or in various effective QCD approaches [8, 9]. These calculations are typically carried out in the grand canonical ensemble (GCE), where the

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charges are only conserved on the average. Since in experiment the charges of the system, baryon number, strangeness and electric charge are conserved globally, it is essential to establish how these susceptibilities are related to experimental measurements [10–14]. The measurements typically have limited acceptance and only cover a fraction of the total momentum space, which, for simplicity, we assume to be characterized by a finite acceptance window in rapidity, ΔY_{acc} . As discussed *e.g.* in [15], for a sufficiently small acceptance window, $\Delta Y_{\text{acc}} \ll \Delta Y_{4\pi}$, conditions corresponding to the GCE may be imitated, *i.e.* effects of global charge conservation become negligible. However, to capture all the physics governing the behavior of thermal fluctuations, the acceptance window ΔY_{acc} must be much larger than the correlation length ΔY_{cor} . It turns out, however, that the deviations of various cumulants obtained in lattice QCD from that of the hadron resonance gas model are rather small. Indeed, they are of the same order of magnitude as charge conservation effects already for acceptances as small as $\Delta Y_{\text{acc}}/\Delta Y_{4\pi} \sim 0.1$ [12]. Therefore, in order to capture the physics of *e.g.* chiral criticality, the effect of charge conservation needs to be understood very well, since simply reducing the acceptance window even further risks eliminating all the non-trivial effects associated with the relevant QCD dynamics [16].

The subensemble acceptance method (SAM) [17, 18] that we present in this contribution is able to relate cumulants of a subsystem subject to global charge conservation to the susceptibilities obtained in the GCE for any equation of state, in particular that of QCD.

2. The subensemble acceptance method (SAM)

Let us briefly sketch the essential steps in deriving the SAM. For more details, we refer to Refs. [17, 18]. For simplicity, we discuss here the case of a single conserved charge. The generalization to multiple charges can be found in [18].

Let us consider a spatially uniform thermal system at a fixed temperature T , volume V , and total net charge, say net-baryon number, B . It is characterized by its canonical partition function, $Z(T, V, B)$. We pick a subsystem of a fixed volume $V_1 = \alpha V$ within the whole system, which can freely exchange the conserved charge B with the rest of the system. Our goal is to evaluate the cumulants $\kappa_n[B_1]$ of the distribution of charge B_1 within that subsystem.

Assuming the subvolume V_1 as well as the remaining volume $V_2 = (1-\alpha)V \equiv \beta V$ to be large compared to correlation length ξ , $V_1 \gg \xi^3$ and $V_2 \gg \xi^3$, the canonical ensemble partition function of the *total* system with total baryon number B is well-approximated by

$$Z(T, V, B) = \sum_{B_1} Z(T, \alpha V, B_1) Z(T, \beta V, B - B_1). \tag{1}$$

In the thermodynamic limit, $Z(T, V, B) = \exp[-\frac{V}{T} f(T, \rho_B)]$ with f and ρ_B the free energy and baryon density, respectively. Therefore, the probability $P(B_1)$ to find B_1 baryons in the subsystem is

$$P(B_1) \propto Z(T, \alpha V, B_1) Z(T, \beta V, B - B_1). \tag{2}$$

Given the cumulant generation function

$$G_{B_1}(t) \equiv \ln \sum_{B_1} \exp(tB_1) P(B_1), \tag{3}$$

we see that all cumulants $\kappa_n[B_1]$ of the order of $n \geq 2$ may be expressed as derivatives of the generalized “ t -dependent” first-order cumulant $\tilde{\kappa}_1[B_1(t)] = \partial G(t)/\partial t$

$$\kappa_n[B_1] = \left. \frac{\partial^n G_{B_1}(t)}{\partial t^n} \right|_{t=0} = \left. \frac{\partial^{n-1} \tilde{\kappa}_1[B_1(t)]}{\partial t^{n-1}} \right|_{t=0}. \tag{4}$$

Obviously,

$$\tilde{\kappa}_1[B_1(t)] = \frac{\partial G_{B_1}(t)}{\partial t} = \frac{\sum_{B_1} B_1 \tilde{P}(B_1; t)}{\sum_{B_1} \tilde{P}(B_1; t)} \equiv \langle B_1(t) \rangle \tag{5}$$

with the (un-normalized) t -dependent probability

$$\tilde{P}(B_1; t) = \exp \left\{ tB_1 - V \frac{\alpha f(T, \rho_{B_1}) + \beta f(T, \rho_{B_2})}{T} \right\}. \tag{6}$$

In the thermodynamic limit, $V \rightarrow \infty$, \tilde{P} has a sharp maximum at the mean value of B_1 , $\langle B_1(t) \rangle$ [19]. The condition $\partial \tilde{P}(B_1; t)/\partial B_1 = 0$ determines the location of this maximum resulting in an implicit relation that determines $\langle B_1(t) \rangle$

$$t = \hat{\mu}_B[T, \rho_{B_1}(t)] - \hat{\mu}_B[T, \rho_{B_2}(t)]. \tag{7}$$

Here, $\hat{\mu}_B = \mu_B/T$, and $\rho_{B_1}(t) = \langle B_1(t) \rangle / (\alpha V)$, $\rho_{B_2}(t) = [B - \langle B_1(t) \rangle] / [(1 - \alpha)V]$. We also used the thermodynamic relation $[\partial f(T, \rho_B) / \partial \rho_B]_T = \mu_B(T, \rho_B)$. It follows from Eq. (7) that $\rho_{B_1} = \rho_{B_2} = B/V$ for $t = 0$, *i.e.* the net-baryon number is uniformly distributed between the two subsystems, as it should be by construction. Therefore,

$$\kappa_1[B_1] = \alpha \kappa_1[B] = \alpha VT^3 \chi_1^B. \tag{8}$$

The second order cumulant is given by $\tilde{\kappa}_2[B_1(t)] = \partial \tilde{\kappa}_1[B_1(t)]/\partial t = \langle B'_1(t) \rangle$. $\langle B'_1(t) \rangle$ can be obtained implicitly by differentiating Eq. (7) with respect to t . Applying the chain rule and using the identity $[\partial \hat{\mu}_B(T, \rho_{B_{1,2}})/\partial \rho_{B_{1,2}}]_T = [T^3 \chi_2^B(T, \rho_{B_{1,2}})]^{-1}$, we find

$$\tilde{\kappa}_2[B_1(t)] = \langle B'_1(t) \rangle = \frac{V T^3}{[\alpha \chi_2^B(T, \rho_{B_1})]^{-1} + [\beta \chi_2^B(T, \rho_{B_2})]^{-1}} \quad (9)$$

which at $t = 0$ gives the 2nd order cumulant

$$\kappa_2[B_1] = \alpha(1 - \alpha) V T^3 \chi_2^B. \quad (10)$$

The higher-order cumulants $\kappa_n[B_1]$ for $n \geq 3$ then result from iteratively differentiating the t -dependent cumulants $\tilde{\kappa}_n[B_1(t)]$ with respect to t , starting from $\tilde{\kappa}_2[B_1(t)]$, and making use of expression (9) for $\langle B'_1(t) \rangle$. The result for the cumulants up to the 4th order is the following:

$$\begin{aligned} \frac{\kappa_3[B_1]}{\alpha V T^3} &= \beta(1 - 2\alpha) \chi_3^B, \\ \frac{\kappa_4[B_1]}{\alpha V T^3} &= \beta \left[(1 - 3\alpha\beta) \chi_4^B - 3\alpha\beta \frac{(\chi_3^B)^2}{\chi_2^B} \right]. \end{aligned} \quad (11)$$

The commonly studied cumulant ratios are

$$\frac{\kappa_2[B_1]}{\kappa_1[B_1]} = (1 - \alpha) \frac{\chi_2^B}{\chi_1^B}, \quad (12)$$

$$\frac{\kappa_3[B_1]}{\kappa_2[B_1]} = (1 - 2\alpha) \frac{\chi_3^B}{\chi_2^B}, \quad (13)$$

$$\frac{\kappa_4[B_1]}{\kappa_2[B_1]} = (1 - 3\alpha\beta) \frac{\chi_4^B}{\chi_2^B} - 3\alpha\beta \left(\frac{\chi_3^B}{\chi_2^B} \right)^2. \quad (14)$$

In the $\alpha \rightarrow 0$ limit, the cumulant ratios reduce to those in the grand canonical limit, as expected, since in this limit effects of global conservation become negligible. Note, however, that the $\alpha \rightarrow 0$ limit discussed here assumes that the condition $V_1 \gg \xi^3$ still holds no matter how small the value of α is. Such a scenario can be realized by holding the subsystem volume fixed to a sufficiently large value and increasing the total volume, *i.e.* $V_1 = \text{const.} \gg \xi^3$ and $V \rightarrow \infty$.

In heavy-ion collisions, on the other hand, a different scenario is realized. The total volume is fixed, while the volume of the subsystem is regulated by the measurement acceptance, for example, in longitudinal rapidity. This implies that the $\alpha \rightarrow 0$ limit corresponds to $V = \text{const.}$ and $V_1 \rightarrow 0$, meaning

that our assumption of the subsystems being close to the thermodynamic limit breaks down, as the subsystem becomes much smaller than the correlation length, $\alpha V \ll \xi^3$. The cumulants then approach the Poisson limit [20] rather than the GCE limit.

So far, we have only considered a single conserved charge. Extension to multiple charges is possible and has been carried out in Ref. [18]. One finds that for cumulants of the order of $n \geq 4$, the grand-canonical susceptibility of charges other than that under consideration enters. For example, when considering electric charge conservation in addition to baryon number conservation, the fourth-order cumulant of the baryon number is given by

$$\frac{\kappa_4 [B^1]}{\alpha V T^3} = \beta \left[(1 - 3\alpha\beta) \chi_4^B - 3\alpha\beta \frac{(\chi_3^B)^2}{\chi_2^B} \left(\frac{1 - 2\frac{\chi_{21}^{BQ}\chi_{11}^{BQ}}{\chi_2^Q\chi_3^B} + \frac{(\chi_{21}^{BQ})^2\chi_2^B}{\chi_2^Q(\chi_3^B)^2}}{1 - \frac{(\chi_{11}^{BQ})^2}{\chi_2^B\chi_2^Q}} \right) \right] \tag{15}$$

which, absent of BQ -correlations, reduces to the above result with only baryon number conservation, Eq. (11). The corrections due to additional conserved charges, while clearly present, are rather small in practice. This is shown in the right panel of Fig. 1, where we show the κ_4/κ_2 ratio for baryon number, strangeness and electric charge for both the case of single conserved charge and for the case where all three charges of QCD are conserved. Here, the GCE susceptibilities of a HRG model at finite baryon density, namely at $T = 160$ MeV and $\mu_B = 100$ MeV, were used in order to verify the SAM results via Monte Carlo samplings, which are shown as squares in the figure.

3. Results

Let us next discuss some of the consequences of these new developments. In the left panel of Fig. 1, we show the dependence of the κ_6/κ_2 ratio for net-baryon fluctuations in the case where only baryon number conservation is considered. We show the SAM results based on lattice QCD results from Ref. [7] for temperatures $T = 155$ and 160 MeV. We also show, as dashed lines, the results obtained in a hadron resonance gas model with global charge conservation. As already pointed out, in the $\alpha \rightarrow 0$ limit, the ratio κ_6/κ_2 obtained with SAM approaches its GCE value. We note that the SAM results for κ_6/κ_2 lie below the binomial (HRG) acceptance baseline for all values of α , which reflects the suppression of the lattice values for χ_6^B/χ_2^B relative to the HRG baseline. Interestingly, the difference between the HRG and QCD is the smallest at $\alpha = 0.5$, where the effects of baryon conservation are the strongest. Actually, in the entire region of $0.2 < \alpha < 0.8$, the difference is so small that it may be difficult to distinguish the true dynamics

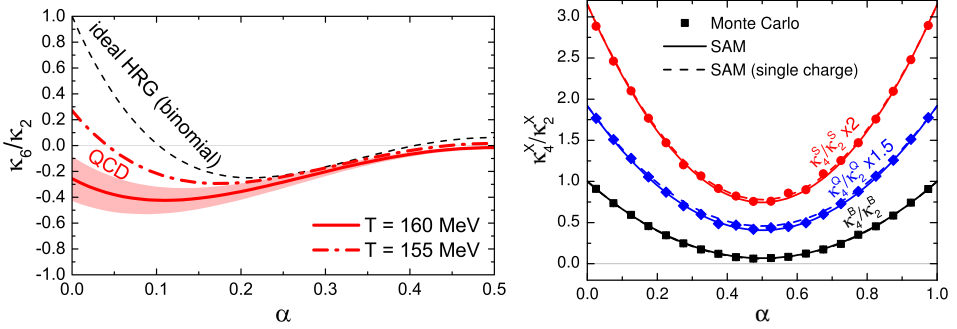


Fig. 1. (Colour on-line) Left panel: Dependence of the κ_6/κ_2 ratio calculated for net-baryon fluctuations inside a subvolume on the fraction α of the total system volume covered for QCD matter at the LHC conditions at the chemical freeze-out at $T = 160$ MeV (solid lines) and $T = 155$ MeV (dash-dotted lines). The band corresponds to the error propagation of the lattice data. The dashed line depicts the ideal hadron resonance gas results. The results are symmetric with respect to a change $\alpha \rightarrow 1 - \alpha$ and thus shown up to $\alpha = 0.5$. Right panel: Dependence of the cumulant ratios κ_4/κ_2 of net-baryon (black), net-charge (blue), and net-strangeness (red) numbers on the acceptance fraction α , as calculated in the hadron resonance gas model using a canonical ensemble Monte Carlo sampler of the FIST package [21] with 10^8 events (symbols) and analytically in the framework of the subensemble acceptance method (solid lines). The dashed lines depict SAM calculations for a single conserved charge [Eq. (11)].

of QCD from that of an ideal HRG¹. Measurements in this region of α , on the other hand, may serve as a model-independent test of baryon number conservation effects. For $\alpha < 0.2$, however, the measurable ratio κ_6/κ_2 becomes sensitive to the equation of state, *i.e.* to the actual value for χ_6^B/χ_2^B . We find that a negative κ_6/κ_2 for $\alpha \lesssim 0.1$ is consistent with χ_6^B/χ_2^B which is either negative or close to zero. Such a measurement would constitute a potentially unambiguous experimental signature of the QCD chiral crossover transition.

If we apply these conditions to actual experiments such as ALICE and STAR, it translates into the following: At the LHC (ALICE) with $\sqrt{s_{NN}} = 5.02$ TeV, the beam rapidity is $y_{\text{beam}} \simeq \ln(\sqrt{s_{NN}}/m_N) \simeq 8.5$, while for the top RHIC energy ($\sqrt{s_{NN}} = 200$ GeV), one has $y_{\text{beam}} \simeq 5.4$. Thus, $\alpha \lesssim 0.1$ would correspond to measurements within approximately two (one) units of rapidity for LHC (RHIC).

¹ We note that the cumulant ratios are symmetric with respect to $\alpha \leftrightarrow (1 - \alpha)$ [20].

As discussed above, for α below a certain value, $\alpha < \alpha_{\text{lim}}$, our formalism breaks down and the cumulants approach the Poisson limit. Comparing the volumes used in the lattice QCD calculations with those at chemical freeze-out at the LHC energies, one finds that $\alpha_{\text{lim}} \sim 10^{-3}$ [17]. Even an order of magnitude error in this estimate implies that $\alpha_{\text{lim}} \lesssim 10^{-2}$ so that our method is applicable for virtually the entire linear scale shown in Fig. 1. Of course, since the SAM works in configuration space and experimental cuts are in momentum space, these results are only valid if one has perfect space-momentum correlations, such as in a Bjorken fireball without any thermal smearing. Additional thermal smearing will likely modify the results for small α to be somewhere between the SAM and the HRG result in Fig. 1.

Turning next to the case of multiple conserved charges, one finds that the effects are rather modest as already discussed in the context of the fourth-order cumulants. Additionally, one finds that any ratio of second-order cumulants, such as $\kappa_{11}^{BS}/\kappa_2^S$, and also for third-order cumulants, is independent of charge conservation effects. This is demonstrated in the left panel of Fig. 2. In addition, one may extend the SAM to study cumulants of non-conserved quantities such as the proton number. Again, in this case, the ratios of second cumulants are not affected by charge corrections as long they involve one conserved charge and one non-conserved quantity, such as $\kappa_{11}^{pQ}/\kappa_2^Q$. This is shown in the right panel of Fig. 2.

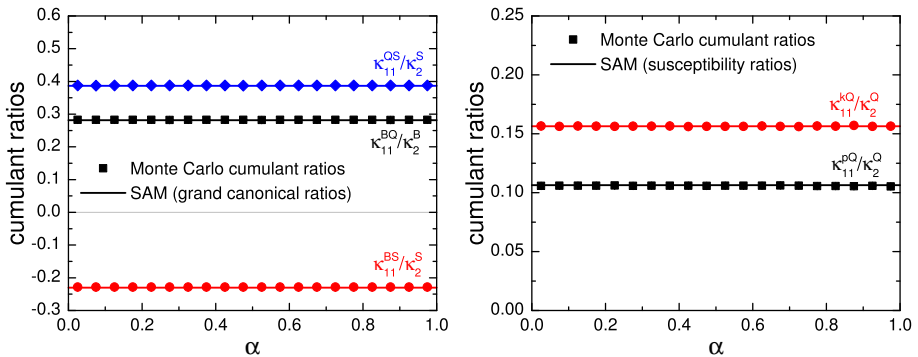


Fig. 2. (Colour on-line) Dependence of combinations of second-order cumulants involving conserved (left panel) and partially non-conserved charges (right panel) on the acceptance fraction α , as calculated in the hadron resonance gas model using canonical ensemble Monte Carlo sampler (symbols), and analytically in the framework of the subensemble acceptance method (lines). Left panel: Off-diagonal to diagonal conserved charge cumulant ratios $\kappa_{11}^{BQ}/\kappa_2^B$ (black), $\kappa_{11}^{QS}/\kappa_2^S$ (blue), and $\kappa_{11}^{BS}/\kappa_2^S$ (red). Right panel: Mixed conserved non-conserved off-diagonal cumulant ratios $\kappa_{11}^{kQ}/\kappa_2^Q$ (red) and $\kappa_{11}^{pQ}/\kappa_2^Q$ (black).

4. Summary

In summary, we have presented the subensemble acceptance method (SAM) which allows to correct susceptibilities obtained from *any* equation of state for global charge conservation. This method can be applied for any number of conserved charges. Since the SAM is formulated in the configuration space, its application to measurements in heavy-ion collisions requires strong space — momentum space correlation, as is the case at mid-rapidity for the highest collisions energies. Using the lattice QCD result for the baryon number hyperkurtosis, κ_6/κ_2 , we find an acceptance fraction of roughly $\alpha \sim 0.1$ corresponding to a rapidity window of $\Delta Y_{\text{acc}} \simeq 2$ at the LHC energies to be a sweet spot for observing the negative value predicted by chiral criticality. We further found that any ratio of second- and third-order cumulants is *not* affected by charge conservation. This is also the case if one considers correlations of non-conserved quantities such as the proton number with any conserved charge.

The next step is to explore corrections from thermal smearing and to extend the method to non-homogeneous system in order to provide guidance for measurements at lower energies.

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