KINETIC GAS DISKS SURROUNDING SCHWARZSCHILD BLACK HOLES

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We describe stationary and axisymmetric gas configurations surrounding black holes. They consist of a collisionless relativistic kinetic gas of identical massive particles following bound orbits in a Schwarzschild exterior spacetime and are modeled by a one-particle distribution function which is the product of a function of the energy and a function of the orbital inclination associated with the particle’s trajectory. The morphology of the resulting configuration is analyzed.

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1. Introduction

In recent years, there has been interest in analyzing the properties of solutions to the Vlasov equation on a fixed, curved background spacetime. In particular, such an analysis has been performed for a Schwarzschild background with the aim of understanding the Bondi–Michel and Bondi–Hoyle–Littleton accretion models for a collisionless kinetic gas [1–3]. Also, kinetic analogues of the perfect fluid “Polish doughnuts” configurations are discussed in [4]. Similarly to their fluid counterparts, they describe stationary and axisymmetric disks around black holes, where the individual gas particles follow bound timelike geodesics in a Schwarzschild spacetime. In [4], these configurations are modeled by a one-particle distribution function (DF) depending only on the energy $E$, azimuthal $L_z$, and total angular momentum $L$ of the particles. Examples are given in which the DF is described by a generalized polytropic ansatz [5, 6] depending only on $E$ and $L_z$. In this article, we provide additional examples where the DF is a function of $E$ and...
the inclination angle $i$ defined by $\cos i = L_z/L$. We analyze the behavior of the resulting particle density and compute the total number of particles of the gas cloud as a function of the free parameters in our ansatz.

2. The model

We work in the Schwarzschild exterior spacetime, written in the usual coordinates $(t, r, \vartheta, \varphi)$, with metric

\[ g := -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) , \quad N(r) := 1 - \frac{2M}{r} > 0 , \]

where $M > 0$ is the mass of the black hole. Since this spacetime is static and spherically symmetric, the particle’s rest mass $m$ is conserved along with $E$, $L$, and $L_z$. In terms of the orthonormal tetrad $e^\mu_0 = N(r)^{-1/2} \partial_t$, $e^\mu_1 = N(r)^{1/2} \partial_r$, $e^\mu_2 = r^{-1} \partial_\vartheta$, $e^\mu_3 = (r \sin \vartheta)^{-1} \partial_\varphi$, the four-momentum of the particles can be parametrized as $p = p^\mu e^\mu_\mu$ with (see [2, Eq. (58)])

\[ \left( p_\mu^\mu \right) = \left( \frac{E}{\sqrt{N(r)}}, \epsilon_r \sqrt{\frac{E^2 - V_L(r)}{N(r)}}, \epsilon_\vartheta \frac{r}{r} \sqrt{\frac{L^2 - L_z^2}{\sin^2 \vartheta}}, \frac{L_z}{r \sin \vartheta} \right) , \]

where the signs $\epsilon_r = \pm 1$ and $\epsilon_\vartheta = \pm 1$ determine the direction of motion in the radial and polar directions, respectively, and $V_L(r) = N(r)(m^2 + L^2/r^2)$ is the effective potential for the radial motion.

A collisionless relativistic gas consisting of identical massive particles of mass $m$ trapped in $V_L$ is described by a DF which relaxes in time to a DF depending only on integrals of motion. This is due to phase mixing, see e.g. [7, 8] and references therein. Here, we assume, in addition, that the final configuration is axisymmetric, which implies that the DF has the form of

\[ f(x, p) = F(E, L, L_z) \]

for some function $F$ which we shall specify shortly. The relevant spacetime observables are the particle current density vector field $J$ and the energy-momentum-stress tensor $T$ defined by

\[ J_\mu(x) := \int_{P_x^+(m)} f(x, p)p_\mu \text{dvol}_x(p) , \quad T^\mu_\nu(x) := \int_{P^+_x(m)} f(x, p)p^\mu p^\nu \text{dvol}_x(p) , \]

where $\text{dvol}_x(p) = dp^1 \wedge dp^2 \wedge dp^3/p^0$ is the Lorentz-invariant volume form on the future mass hyperboloid $P^+_x(m)$ of mass $m$ at $x$, see [9] for details.

\[ ^1 \text{We use units in which the speed of light and the gravitational constant are one.} \]
For the following, we focus on the particular ansatz

\[ F(E, L, L_z) := F_0(E) \cos^{2s}(i), \quad F_0(E) = \alpha \left( 1 - \frac{E}{m} \right)^{k-\frac{3}{2}}, \tag{5} \]

where \( \alpha > 0, k > 1/2 \) are constants, \( i \) is the inclination angle, and \( s \geq 0 \) is a parameter. The notation \( f_+ \) refers to the positive part of the quantity \( f \), that is \( f_+ = f \) if \( f > 0 \) and \( f_+ = 0 \) otherwise. Here, the function \( F_0 \) is the general relativistic generalization of the polytropic ansatz [10], while the parameter \( s \) controls the concentration of the orbits near the equatorial plane \( \vartheta = \pi/2 \) (see Fig. 1).

For the following, we introduce the dimensionless quantities \( \xi := r/M, \lambda := L/(Mm), \varepsilon := E/m, \) and \( U_\lambda(\xi) := V_L(r)/m^2 \), and parametrize the future mass hyperboloid \( P^+_x(m) \) in terms of the quantities \( (\varepsilon, \lambda, \chi) \), where the angle \( \chi \) is defined by \( (p^2, p^3) = \frac{m \Lambda}{\xi} (\cos \chi, \sin \chi) \) which implies \( \cos i = \sin \vartheta \sin \chi \). For bound orbits, these quantities are restricted to the following domain (see [11, Appendix A] and [4, Appendix A]):

\[ \varepsilon_c(\xi) < \varepsilon < 1, \quad \lambda_c(\varepsilon) \leq \lambda \leq \lambda_{\text{max}}(\varepsilon, \xi), \quad \text{and} \quad 0 \leq \chi \leq 2\pi, \tag{6} \]

where \( \varepsilon_c(\xi) \) is the minimum energy at radius \( \xi \), \( \lambda_c(\varepsilon) \) is the critical value for the total angular momentum for which the maximum of the potential barrier in \( U_\lambda(\xi) \) is exactly equal to \( \varepsilon^2 \), and \( \lambda_{\text{max}}(\varepsilon, \xi) \) is the maximum angular momentum permitted at the energy \( \varepsilon \) and radius \( \xi \). Note that the domain (6) is empty if \( \xi < 4 \), since for a Schwarzschild black hole, the minimum radius for bound orbits is \( r = 4M \).
For the ansatz (5), the fibre integrals in Eq. (4) yield

\[ J^\hat{\mu}(x) = \frac{m^2 \sin^{2s} \vartheta}{\xi^2} \sum_{\epsilon_r = \pm 1} \int_{\epsilon_c(\xi)}^{1} \int_{\lambda_c(\epsilon)}^{\lambda_{\max}(\epsilon, \xi)} 2\pi p_{\hat{\mu}} F_0(E) \sin^{2s} \chi \frac{d\epsilon \lambda d\lambda d\chi}{\sqrt{\epsilon^2 - U_\lambda(\xi)}}, \]  

(7)

and similarly for \( T_{\hat{\mu}\hat{\nu}}(x) \). Using expressions (2) for the four-momentum, the non-vanishing orthonormal components of \( J^\hat{\mu} \) and \( T_{\hat{\mu}\hat{\nu}} \) are

\[ J^\hat{0} = 4\sqrt{\pi} \frac{\sin^{2s} \vartheta}{N^{3/2}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} m^3 \int_{\epsilon_c(\xi)}^{1} d\epsilon \epsilon Y(\epsilon, \xi)^{1/2} F_0(m\epsilon), \]  

(8)

\[ T^{\hat{0}\hat{0}} = -4\sqrt{\pi} \frac{\sin^{2s} \vartheta}{N^2} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} m^4 \int_{\epsilon_c(\xi)}^{1} d\epsilon \epsilon^2 Y(\epsilon, \xi)^{1/2} F_0(m\epsilon), \]  

(9)

\[ T^{\hat{1}\hat{1}} = 4\sqrt{\pi} \frac{\sin^{2s} \vartheta}{3 N^{2}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} m^4 \int_{\epsilon_c(\xi)}^{1} d\epsilon Y(\epsilon, \xi)^{3/2} F_0(m\epsilon), \]  

(10)

\[ T^{\hat{2}\hat{2}} = 4\sqrt{\pi} \frac{\sin^{2s} \vartheta}{3 N^{2}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 2)} m^4 \int_{\epsilon_c(\xi)}^{1} d\epsilon Y(\epsilon, \xi)^{1/2} Z(\epsilon, \xi) F_0(m\epsilon), \]  

(11)

\[ T^{\hat{3}\hat{3}} = (2s + 1)T^{\hat{2}\hat{2}}, \]  

(12)

where we have introduced the shorthand notation

\[ Y(\epsilon, \xi) := \epsilon^2 - N(r) \left[ 1 + \frac{\lambda_c(\epsilon)^2}{\xi^2} \right], \quad Z(\epsilon, \xi) := \epsilon^2 - N(r) \left[ 1 - \frac{\lambda_c(\epsilon)^2}{2\xi^2} \right]. \]  

(13)

The quantities (8)–(12) determine the relevant macroscopic observables, namely the particle density \( n = J^\hat{0} \), energy density \( E = -T^{\hat{0}\hat{0}} \), and the principal pressures \( P_1 = T^{\hat{1}\hat{1}} \), \( P_2 = T^{\hat{2}\hat{2}} \), and \( P_3 = (2s + 1)P_2 \). Note that all of these quantities have the dependency of \( \sin^{2s} \vartheta \) with respect to the polar angle \( \vartheta \). In the limit \( s = 0 \), the configurations describe a spherical shell of gas trapped in the region of \( \xi > 4 \), while for \( s = 1/2, 1, 3/2, ... \), they are axisymmetric, the macroscopic variables being zero for \( \xi \leq 4 \) and along the axis \( \vartheta = 0, \pi \). In the next section, we analyze the morphology of these configurations as a function of the parameters \( k \) and \( s \) for a fixed total particle number.
3. Total particle number and behavior of the particle density

The (conserved) total particle number $\mathcal{N}$ is defined as minus the flux integral of the current density vector field with respect to a Cauchy surface. This in turn can be rewritten as an integral over the six-dimensional phase space parametrized by $(x^i, p_i)$. To compute this integral, it is convenient to transform $(x^i, p_i)$ to action-angle variables $(Q^i, J_i)$. The integral over the angle variables $Q^i$ yields a factor $(2\pi)^3$, while the integral over the action variables can be rewritten in terms of the conserved quantities $(E, L, L_z)$, taking into account that $\int d^3J = T(E,L) dE dL dL_z/2\pi$, where $T(E,L)$ is the period function for the radial motion. For the Schwarzschild spacetime, this function can be expressed in terms of elliptic integrals and has the form $T(E,L) = 2M\varepsilon (H_2 - H_0)$ (see [7, Appendix A] and [4] for the explicit form of $H_2$ and $H_0$ in the Schwarzschild case). For ansatz (5), this yields the following expression for the total particle number:

$$\mathcal{N} = \frac{16\pi^2}{2s+1} (Mm)^3 \alpha \int_{\varepsilon_{\text{min}}}^{1} d\varepsilon \varepsilon (1-\varepsilon)^{k-\frac{3}{2}} \int_{\lambda_c(\varepsilon)}^{\lambda_{\text{ub}}(\varepsilon)} d\lambda \lambda (H_2 - H_0), \quad (14)$$

where $\varepsilon_{\text{min}} = \sqrt{8/9}$ and $\lambda_{\text{ub}}(\varepsilon)$ is given in [11, Appendix A]. To compute this integral, it is convenient to re-parametrize the orbits in terms of their eccentricity $e$ and “semi-latus rectum” $P$, related to the turning points $(\xi_1, \xi_2)$ by $\xi_1 = P/(1+e)$ and $\xi_2 = P/(1-e)$, and to the conserved quantities $(\varepsilon, \lambda)$ according to [4, 7, 12, 13] $$(\varepsilon^2, \lambda^2) = (P^{-1}[(P-2)^2 - 4e^2], P^2)/(P-e^2-3).$$

Here, $(P, e)$ are restricted to the domain $0 < e < 1$ and $P > 6 + 2e$. The resulting integral is then calculated numerically using Mathematica. The total mass is simply $m\mathcal{N}$ and the total energy is given by the same expression as in Eq. (14) with an extra factor $m\varepsilon$ inside the integral.

![Fig. 2. Left panel: Dimensionless profile of the particle density in the equatorial plane for $k = 3, 4, 5$ and $s = 1$ in a logarithmic scale. Right panel: The same quantity multiplied with $\xi^2$ which shows that even though configurations with higher values of $k$ have a larger maximum, they have a faster decay at infinity.](image-url)
In Fig. 2, we show the dimensionless quantity $M^3n/N$ in the equatorial plane for several values of $k$ and $s = 1$. In Fig. 3, we show contour plots of the same quantity in the $xz$-plane for $k = 3$ and two different values of $s$.

![Contour plots for the particle density in the $xz$-plane for the configurations with $k = 3$ and $s = 1$ (left panel) and $k = 3$ and $s = 3$ (right panel). Here, $(x, z) = r(\sin \vartheta, \cos \vartheta)$, the black region represents the black hole interior and the dashed black circles the interior boundary of the disk. As it is visible from these plots, the configuration with higher $s$ yields a thinner disk.]

Fig. 3. Contour plots for the particle density in the $xz$-plane for the configurations with $k = 3$ and $s = 1$ (left panel) and $k = 3$ and $s = 3$ (right panel). Here, $(x, z) = r(\sin \vartheta, \cos \vartheta)$, the black region represents the black hole interior and the dashed black circles the interior boundary of the disk. As it is visible from these plots, the configuration with higher $s$ yields a thinner disk.

4. Conclusions

We described a family of stationary and axisymmetric collisionless gas configurations which are trapped in the gravitational potential of a Schwarzschild black hole. This family depends on two parameters $s$ and $k$ which control the thickness of the disk and its radial density distribution. An alternative model is discussed in detail in [4]. We expect these configurations to serve as a first approximation for the description of low-luminosity disks surrounding black holes.

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REFERENCES


