TWO-SIDED WALKER AND PARA-KÄHLER SPACES AS REAL SLICES OF HYPERHEAVENLY SPACES*

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Complex, 4-dimensional spaces which are equipped with congruences of self-dual and anti-self-dual null strings are considered. Criteria of a classification of such spaces are given. Some interesting classes of two-sided Walker and para-Kähler spaces are presented.

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1. Introduction

The paper is devoted to complex and real neutral 4-dimensional geometries. By *complex geometry*, we understand the geometry of a 4-dimensional complex manifold $\mathcal{M}$ equipped with a holomorphic metric. Hence, local coordinates are complex and functions are holomorphic. A great advantage of the complex 4-dimensional geometry is the fact that any real 4-dimensional space can be obtained from a generic complex space as a *real slice* of such a complex space [1]. In what follows, we focus only on *real neutral slices*, *i.e.*, slices of complex spaces which lead to real spaces equipped with a metric of the neutral signature $(++--)$.

Especially we are interested in complex spaces equipped with additional structures, so-called *congruences of null strings* [2] (see Section 2.1). Congruences of null strings are families of totally null and totally geodesic 2-dimensional surfaces such that for every point $p \in \mathcal{M}$, there exists only one surface of this family such that p belongs to this surface. If such a congruence is parallelly propagated then a space is called *weak nonexpanding hyperheavenly space ($\mathcal{HH}$-space)* [3, 4]. Weak $\mathcal{HH}$-space with an orientation

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chosen in such a manner that the congruence of null strings is self-dual (SD) is a starting point for our considerations (see Section 3.1). Note, that a real neutral counterpart of weak $\mathcal{HH}$-space is called *Walker space* [5].

In the second step, we equip a weak $\mathcal{HH}$-space with an additional congruence of anti-self-dual (ASD) null strings. Thus, we arrive at the *(complex) sesqui-Walker spaces* [6]. In Section 3.2, some generalizations of the results presented in [6] are given. Note, that congruences of SD and ASD null strings necessarily intersect and this intersection constitutes the congruence of complex null geodesics. The properties of such intersections are investigated in Section 2.2.

Then we equip a weak $\mathcal{HH}$-space with two different and parallely propagated congruences of ASD null strings (see Section 3.3). For such spaces, the ASD part of the Weyl tensor is of the Petrov–Penrose type [D]. Hence, such spaces are complex counterparts of a special family of *para-Kähler spaces*. Finally, we consider also para-Kähler Einstein spaces (Section 3.4). Our interest in para-Kähler Einstein spaces is motivated by the recent paper [7]. The authors of [7] found a very interesting relation between para-Kähler Einstein spaces and (2,3,5)-distributions. Thus, we believe that a more detailed investigation of para-Kähler Einstein spaces is justified.

All considerations are purely local. All metrics presented in our paper are complex. They have neutral slices which can be obtained from complex metrics by replacing all complex coordinates by real ones and all holomorphic functions by real smooth ones. In Section 2.1, the spinorial formalism in the Infeld–Van der Waerden–Plebański notation is used (see [8, 9] for details).

2. Congruences of null strings and their intersection

2.1. Congruences of null strings

Consider a nowhere vanishing undotted 1-index spinor field m_A such that

$$m^A m^B \nabla_{A\dot{M}} m_B = 0\,. \tag{1}$$

If Eqs. (1) are satisfied, then a 2-dimensional totally null distribution $\mathcal{D}_{m^A} = \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{A}} b^{\dot{B}} \neq 0$ is completely integrable. Its integral manifolds are 2-dimensional, totally null and totally geodesic surfaces called *null strings*. A family of such surfaces constitutes *a congruence (foliation) of SD null strings*. Equations (1) are called *the SD null string equations* and we say that the spinor m^A *generates the congruence of SD null strings*.

Equations (1) can be rewritten in the equivalent form of

$$m^B \nabla_{A\dot{M}} m_B = m_A M_{\dot{M}}\,, \tag{2}$$

where 1-index dotted spinor field $M_{\dot{M}}$ is an *expansion* of the congruence of SD null strings [2]. The expansion $M^{\dot{M}}$ describes the most important property of the congruence of the null strings. If $M_{\dot{M}} = 0$, then the 2-dimensional distribution $\mathcal{D}_{m^A}$ is parallely propagated. It means that $\nabla_X V \in \mathcal{D}_{m^A}$ for every vector field $V \in \mathcal{D}_{m^A}$ and for arbitrary vector field X. Such a congruence is called *nonexpanding*. If $M_{\dot{M}} \neq 0$, then a congruence is *expanding*. Note, that real spaces equipped with nonexpanding congruences of null strings are called *Walker spaces* [5].

Congruences of ASD null strings are generated by 1-index dotted spinors. If a spinor $m^{\dot{A}}$ generates a congruence of ASD null strings, then it satisfies *the ASD null string equations*

$$m^{\dot{B}}\,\nabla_{A\dot{M}} m_{\dot{B}} = m_{\dot{M}} M_A\,, \tag{3}$$

where M_A is *an expansion of congruence of ASD null strings.*

2.2. Intersection of congruences of null strings

If a space admits congruences of both SD and ASD null strings, then these congruences intersect. The intersection constitutes a congruence of complex null geodesic lines. The idea how to describe properties of congruences of null geodesics is well known in the general theory of relativity. Let K_a be a null vector field along the intersection of congruences of null strings generated by spinors m_A and $m_{\dot{A}}$. Then $K_a \sim m_A m_{\dot{A}}$. Define scalars θ and ϱ as follows:

$$\theta \;:=\; \tfrac{1}{2}\nabla^a K_a\,, \tag{4a}$$

$$\varrho^2 \;:=\; \tfrac{1}{2}\nabla_{[a}K_{b]}\,\nabla^a K^b\,. \tag{4b}$$

In the Lorentzian geometry, θ is called *an expansion* and ϱ is called *a twist* of a congruence of null geodesics. Following this terminology, we also call θ — *an expansion* and ϱ — *a twist* of a congruence of complex null geodesics. However, it must be pointed out that in complex geometry and in real neutral geometry, the scalars θ and ϱ do not have such a transparent geometrical interpretation like in the Lorentzian case. Note also that an expansion of a congruence of SD (ASD) null strings $M^{\dot{A}}$ (M^A) and an expansion of a congruence of null geodesics θ are different concepts. We call both these quantities "expansion" but we believe it will not be misleading.

It has been proven in [10] that if a congruence of null geodesics is in an affine parametrization, then θ and ϱ are proportional to the following scalars:

$$\begin{aligned}\theta &\sim m_A M^A + m_{\dot{A}} M^{\dot{A}}\,,\\ \varrho &\sim m_A M^A - m_{\dot{A}} M^{\dot{A}}\,.\end{aligned} \tag{5}$$

There are four possibilities for which we propose the following symbols:

$$\begin{aligned} [++] &: \theta \neq 0\,,\ \varrho \neq 0\,, \\ [+-] &: \theta \neq 0\,,\ \varrho = 0\,, \\ [-+] &: \theta = 0\,,\ \varrho \neq 0\,, \\ [--] &: \theta = 0\,,\ \varrho = 0\,. \end{aligned} \tag{6}$$

Consequently, one arrives at Table I.

TABLE I

Possible types of congruences of null geodesics via properties of congruences of null strings.

Expansions	$M^A = 0$	$M^A \neq 0$
$M^{\dot{A}} = 0$	$[--]$	$[--]$, $[++]$
$M^{\dot{A}} \neq 0$	$[--]$, $[++]$	$[--]$, $[-+]$, $[+-]$, $[++]$

2.3. Nomenclature

In what follows, we classify spaces according to three criteria. Namely: (*i*) the Petrov–Penrose classification of the SD and ASD parts of the Weyl tensor; (*ii*) properties of congruences of null strings; (*iii*) properties of congruences of null geodesics. Thus, we introduce the following symbol:

$$\left\{[\mathrm{SD_{type}}]^{i_1 i_2 \cdots} \otimes [\mathrm{ASD_{type}}]^{j_1 j_2 \cdots}, [k_{11}, k_{12}, \ldots, k_{21}, k_{22}, \ldots]\right\} . \tag{7}$$

$\mathrm{SD_{type}}, \mathrm{ASD_{type}} = \{\mathrm{I}, \mathrm{II}, \mathrm{D}, \mathrm{III}, \mathrm{N}, \mathrm{O}\}$ are the Petrov–Penrose types of the SD and ASD parts of the Weyl tensor. Also, the abbreviation *any* means that the Petrov–Penrose type is arbitrary, while *deg* means that it is algebraically degenerated. Note, that in complex and in neutral spaces, the SD and ASD parts of the Weyl tensor are independent. Hence, spaces with "mixed" conformal curvature (for example $[\mathrm{II}] \otimes [\mathrm{D}]$) appear in complex and neutral geometries and thus it is necessary to use separate symbols for the SD and ASD parts of the Weyl tensor.

The number of superscripts $i(j)$ carries information about the number of SD (ASD) congruences of null strings. $i_1, i_2, \ldots, j_1, j_2, \ldots = \{n, e\}$, where n stands for nonexpanding congruences while e stands for expanding congruences. Finally, $k_{11}, k_{12}, \ldots, k_{21}, k_{22}, \ldots = \{--, -+, +-, ++\}$ tell us about the properties of intersections of congruences of null strings. The properties of the intersection of i_m-congruence of SD null strings with j_n-congruence of ASD null strings are gathered in the symbol k_{mn}. Thus, an order of the symbols k_{mn} in the square bracket in the symbol (7) is important.

To understand better the meaning of the symbol (7), we give an example. Consider the type $\{[\mathrm{N}]^n \otimes [\mathrm{II}]^{ne}, [--,++]\}$. It means that the SD part of the Weyl tensor is of the type [N], while ASD part is of the type [II]. There is a single congruence of SD null strings which is nonexpanding and two congruences of ASD null strings. The first one is nonexpanding, while the second one is expanding. The intersection of the congruence of SD null strings with the first congruence of ASD null strings is nonexpanding and nontwisting, while the second intersection is expanding and twisting.

Detailed considerations prove that if a space is equipped with a single congruence of SD null strings and a single congruence of ASD null strings, there are 7 subtypes [10]. If there is a single SD congruence and two ASD congruences, then the number of subtypes is 24. Spaces with two SD and two ASD congruences of null strings can be divided into 89 different subtypes. The details of these classifications will be presented elsewhere (see [11]).

3. The metrics

3.1. Weak nonexpanding $\mathcal{HH}$-spaces

A starting point for our considerations are spaces which are called *weak nonexpanding hyperheavenly spaces* (*weak nonexpanding $\mathcal{HH}$-spaces*). Such spaces are equipped with a single nonexpanding congruence of SD null strings. There is no additional ASD structure, so weak nonexpanding $\mathcal{HH}$-spaces are spaces of the types $[\mathrm{deg}]^n \otimes [\mathrm{any}]$. It is well known [2, 3] that the metric of such spaces can be brought to the form of

$$\tfrac{1}{2}\mathrm{d}s^2 = \mathrm{d}q\,\mathrm{d}y - \mathrm{d}p\,\mathrm{d}x + \mathcal{A}\,\mathrm{d}p^2 - 2\mathcal{Q}\,\mathrm{d}q\,\mathrm{d}p + \mathcal{B}\,\mathrm{d}q^2\,, \tag{8}$$

where (q,p,x,y) are local coordinates; $\mathcal{A} = \mathcal{A}(q,p,x,y)$, $\mathcal{Q} = \mathcal{Q}(q,p,x,y)$, and $\mathcal{B} = \mathcal{B}(q,p,x,y)$ are arbitrary holomorphic functions of their variables.

3.2. Two-sided Walker and sesqui-Walker spaces

If a weak $\mathcal{HH}$-space is equipped with a congruence of ASD null strings, we arrive at the three different families of spaces. They are[1]

$$\begin{aligned}
\text{sesqui-Walker spaces}: &\quad \{[\mathrm{deg}]^n \otimes [\mathrm{any}]^e, [++]\}\,,\\
\text{integrable sesqui-Walker spaces}: &\quad \{[\mathrm{deg}]^n \otimes [\mathrm{any}]^e, [--]\}\,,\\
\text{two-sided Walker spaces}: &\quad \{[\mathrm{deg}]^n \otimes [\mathrm{deg}]^n, [--]\}\,.
\end{aligned} \tag{9}$$

It is quite easy to specialize the metric (8) for the integrable sesqui-Walker spaces [6] and for the two-sided Walker spaces [3]. Indeed, one finds

[1] Names *sesqui-Walker* spaces and *integrable sesqui-Walker* spaces have been introduced in [6].

that the metric of such spaces has the form of (8) with

$$\text{for the integrable sesqui-Walker spaces}: \quad \mathcal{B} = \mathcal{B}(q,p,y)\,, \quad \mathcal{Q}_x \neq 0\,, \tag{10a}$$
$$\text{for the two-sided Walker spaces}: \quad \mathcal{B} = \mathcal{B}(q,p,y)\,, \quad \mathcal{Q} = \mathcal{Q}(q,p,y)\,. \tag{10b}$$

A little more complicated task is to find the metric for the sesqui-Walker spaces. Skipping all the details, one finds that the metric of such spaces can be always brought to the form of

$$\begin{aligned} \tfrac{1}{2}\mathrm{d}s^2 \;=\; & -\mathrm{d}p\,\mathrm{d}x - z\,\mathrm{d}q\,\mathrm{d}x - (x-\Sigma_z)\,\mathrm{d}q\,\mathrm{d}z + \mathcal{A}\,\mathrm{d}p^2 \\ & +(\Sigma_p - 2\mathcal{Q})\,\mathrm{d}p\,\mathrm{d}q + \big((x-\Sigma_z)\,\Omega + z\Sigma_p - 2z\mathcal{Q} - z^2\mathcal{A}\big)\,\mathrm{d}q^2\,, \end{aligned} \tag{11}$$

where (q,p,x,z) are local coordinates; $\mathcal{A} = \mathcal{A}(q,p,x,z)$, $\mathcal{Q} = \mathcal{Q}(q,p,x,z)$, $\Sigma = \Sigma(q,p,z)$ and $\Omega = \Omega(q,p,z)$ are arbitrary holomorphic functions of their variables. Note, that the metric (11) has not been found in [6]. It is a new result a little stronger (more general) then the metric (8) constraint by (10a).

3.3. Para-Kähler spaces

If a two-sided Walker space is equipped with one more congruence of ASD null strings which is expanding, one arrives at the space of the types $\{[\mathrm{deg}]^n \otimes [\mathrm{any}]^{ne}, [--,++]\}$ or $\{[\mathrm{deg}]^n \otimes [\mathrm{any}]^{ne}, [--,--]\}$. We do not consider such spaces here, for details, see [11]. However, if the second ASD congruence is nonexpanding, then the space is of the types $\{[\mathrm{deg}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$ and it belongs to the para-Kähler class.

Remark. Para-Kähler spaces are characterized by the existence of two nonexpanding congruences of null strings of the same duality. In our paper, the orientation is chosen in such a manner that these congruences are ASD. Hence, para-Kähler spaces are of the types $[\mathrm{any}] \otimes [\mathrm{D}]^{nn}$. If an algebraic degeneration of the SD part of the Weyl tensor and the existence of a congruence of SD null strings is also assumed, then there are three families of para-Kähler spaces. These are the types $\{[\mathrm{deg}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$, $\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [--,++]\}$ and $\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [++,++]\}$. In this paper we consider the first family.

The metric of a space of the types $\{[\mathrm{deg}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$ can be brought to the form of

$$\tfrac{1}{2}\mathrm{d}s^2 = \mathrm{d}q\,\mathrm{d}y - \mathrm{d}p\,\mathrm{d}x + \mathcal{A}(q,p,x)\,\mathrm{d}p^2 + \mathcal{B}(q,p,y)\,\mathrm{d}q^2\,, \tag{12}$$

where (q,p,x,y) are local coordinates; $\mathcal{A} = \mathcal{A}(q,p,x)$ and $\mathcal{B} = \mathcal{B}(q,p,y)$ are arbitrary holomorphic functions of their variables.

As long as $\mathcal{A}_{xx} + \mathcal{B}_{yy} \neq 0$, the metric (12) is of the type $\{[\mathrm{II}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$ or $\{[\mathrm{D}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$. Type $\{[\mathrm{D}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$ is characterized by surprisingly complicated equation

$$(\mathcal{A}_{qx} + \mathcal{B}_{py})^2 - (\mathcal{A}_{xx} + \mathcal{B}_{yy})(\mathcal{B}_{pp} + \mathcal{A}_{qq} - \mathcal{B}_p \mathcal{A}_x + \mathcal{B}_y \mathcal{A}_q) = 0\,. \tag{13}$$

A special solution of Eq. (13) is $\mathcal{A} = 0$, $\mathcal{B}_p = f(\mathcal{B}_y, q)$, where f is an arbitrary holomorphic function.

If $\mathcal{A}_{xx} + \mathcal{B}_{yy} = 0$, then the structural functions $\mathcal{A}$ and $\mathcal{B}$ can be brought to the form of

$$\mathcal{A} = Mx^2 + Px + \Omega\,, \qquad \mathcal{B} = -My^2 + Ny\,, \tag{14}$$

where M, P, Ω, and N are arbitrary functions of variables (q, p). In this case, the ASD part of the Weyl tensor vanishes and the space is SD. As long as $|M_p| + |M_q| + |N_p| + |P_q| \neq 0$, the space is of the type $[\mathrm{III}]^n \otimes [\mathrm{O}]^n$, while for the type $[\mathrm{N}]^n \otimes [\mathrm{O}]^n$ one finds $M = M_0 = \text{const.}$, $P = \Sigma_p$, $N = -\Sigma_q$ with $\Sigma = \Sigma(q, p)$.

Note, that SD spaces of the type $[\mathrm{N}]^n \otimes [\mathrm{O}]^n$ are two-sided conformally recurrent [12]. However, in [12], three classes of such spaces have been found and all of them depend on five arbitrary functions of two variables. The question which of these arbitrary functions are gauge-dependent has not been answered in [12]. Our results prove that all of these classes can be, in fact, reduced to a single class with one constant and two arbitrary functions of two variables.

3.4. Para-Kähler Einstein spaces

The metric of para-Kähler Einstein spaces of the types $\{[\mathrm{II}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}$ or $\{[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn}, [--,--,--,--]\}$ can be brought to the form of

$$\frac{1}{2}\mathrm{d}s^2 = \mathrm{d}q\,\mathrm{d}y - \mathrm{d}p\,\mathrm{d}x + \left(\frac{\Lambda}{2}x^2 + \Omega\right)\mathrm{d}p^2 + \left(\frac{\Lambda}{2}y^2 + \Sigma\right)\mathrm{d}q^2\,, \tag{15}$$

where (q, p, x, y) are local coordinates; the cosmological constant $\Lambda \neq 0$; $\Omega = \Omega(q, p)$ and $\Sigma = \Sigma(q, p)$ are functions such that

$$\begin{aligned} \text{for the type } \{[\mathrm{II}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}: &\quad |\Sigma_p| + |\Omega_q| \neq 0\,, \\ \text{for the type } \{[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn}, [--,--,--,--]\}: &\quad \Sigma = \Omega = 0\,. \end{aligned} \tag{16}$$

If $\Lambda = 0$, one arrives at the SD Einstein spaces of the types $[\mathrm{III}, \mathrm{N}]^n \otimes [\mathrm{O}]^n$. The metric of such spaces has the form of

$$\frac{1}{2}\mathrm{d}s^2 = \mathrm{d}q\,\mathrm{d}y - \mathrm{d}p\,\mathrm{d}x + (\Phi_p\, x + \Omega)\,\mathrm{d}p^2 + \Phi_q\, y\,\mathrm{d}q^2\,, \tag{17}$$

where (q,p,x,y) are local coordinates; $\mathit{\Phi} = \mathit{\Phi}(q,p)$ and $\mathit{\Omega} = \mathit{\Omega}(q,p)$ are functions such that

$$\begin{aligned} &\text{for the type } \{[\mathrm{III}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}: \quad \mathit{\Phi}_{pq} \neq 0 \text{ and } \mathit{\Omega} \text{ is arbitrary}\,, \\ &\text{for the type } \{[\mathrm{N}]^n \otimes [\mathrm{D}]^{nn}, [--,--]\}: \quad \mathit{\Phi} = 0 \text{ and } \Omega_{qq} \neq 0\,. \end{aligned} \tag{18}$$

All metrics of the algebraically degenerated SD Einstein spaces have been found for the first time in [13, 14]. Thus, (17) is not a new result but we present it here for completeness.

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