ROBINSON–TRAUTMAN SPACETIMES COUPLED TO CONFORMALLY INVARIANT ELECTRODYNAMICS IN HIGHER DIMENSIONS*

David Kokoška[a,b,†], Marcello Ortaggio[b,‡]

[a]Institute of Theoretical Physics, Faculty of Mathematics and Physics
Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
[b]Institute of Mathematics of the Czech Academy of Sciences
Žitná 25, 115 67 Prague 1, Czech Republic

Received 29 November 2021, accepted 29 November 2021, published online 31 January 2022

We summarize recent results on D-dimensional Robinson–Trautman solutions of Einstein's gravity in the presence of a conformally invariant non-linear electromagnetic field and a cosmological constant. These spacetimes contain static dyonic black holes with various horizon geometries and their time-dependent radiating generalizations, as well as a class of stealth solutions. Extensions to $f(R)$ and the Gauss–Bonnet gravity are mentioned.

DOI:10.5506/APhysPolBSupp.15.1-A8

1. Introduction

The Robinson–Trautman (RT) spacetimes [1, 2] are characterized by the existence of an expanding, shearfree, and twistfree congruence of null geodesics, and provide a natural arena for the study of static black holes and their time-dependent generalizations, as well as other radiative spacetimes (see reviews [3, 4] and references therein). This class of metrics can be defined in an arbitrary dimension D [5, 6], and $D > 4$ Einstein–Maxwell solutions have been studied in [6–8].

* Presented at the 7th Conference of the Polish Society on Relativity, Łódź, Poland, 20–23 September 2021.

† `david.kokoska@matfyz.cz`

‡ `ortaggio@math.cas.cz`

While the linear Maxwell theory is not conformally invariant when $D \neq 4$, a conformally invariant non-linear electrodynamics in D dimensions has been proposed in [9]. Considering a minimal coupling to the Einstein gravity, the action is given by

$$S = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda) - 2\beta F^{D/4} \right] , \qquad F \equiv F_{\mu\nu} F^{\mu\nu} , \tag{1.1}$$

where κ and β are coupling constants, and $\boldsymbol{F} = \mathrm{d}\boldsymbol{A}$.

The corresponding equations of motion read

$$\frac{1}{\kappa} (G_{\mu\nu} + \Lambda g_{\mu\nu}) = \beta F^{D/4-1} \left(D F_{\mu\rho} F_\nu{}^\rho - g_{\mu\nu} F \right) , \tag{1.2}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\frac{D}{4}-1} F^{\mu\nu} \right) = 0 . \tag{1.3}$$

The RHS of (1.2) is traceless, so that the Ricci scalar is a constant proportional to Λ.

Let us observe that for $D \neq 4$, solutions of the theory (1.1) are *stealth* precisely when $F = 0$, which also ensures that Eq. (1.3) is identically satisfied. In other words, any closed 2-form $\boldsymbol{F}$ provides a solution to the theory (1.1) in any Einstein spacetime. For $D = 4$, Eq. (1.1) reduces to the Einstein–Maxwell action.

Since $F^{\frac{D}{4}}$ must be a real quantity, D must be a multiple of 4 when $F < 0$. Requiring the energy density to be non-negative (weak energy condition WEC) means $\beta F^{D/4-1} \geq 0$, so that in the following we will assume: *(i)* $\beta > 0$ if $F > 0$, or if $F < 0$ with $D/4$ being odd; *(ii)* $\beta < 0$ if $F < 0$ with $D/4$ being even; *(iii)* β can have any sign in the stealth case $F = 0$. Since $T_{\mu\nu}$ is traceless, the strong energy condition becomes equivalent to the WEC and is thus also satisfied.

In our recent work [10], we studied the class of RT solutions to the theory (1.1) under the assumption that the RT null vector field $\boldsymbol{k}$ is an eigenvector of the electromagnetic field $\boldsymbol{F}$ (*i.e.*, it is aligned). In the following, we summarize the main results obtained for non-stealth fields (some comments on the stealth case can be found in [10]).

2. Static black holes

The RT solutions of [10] contain, in particular, a family of dyonic black holes. The metric reads

$$\mathrm{d}s^2 = r^2 h_{ij} \mathrm{d}x^i \, \mathrm{d}x^j - 2 \, \mathrm{d}u \, \mathrm{d}r - 2H \, \mathrm{d}u^2 , \tag{2.1}$$

$$2H = K - \lambda r^2 - \frac{\mu}{r^{D-3}} + \frac{Q^2}{r^{D-2}} , \tag{2.2}$$

and the electromagnetic field is

$$\boldsymbol{F} = \frac{e}{r^2}\mathrm{d}r \wedge \mathrm{d}u + \frac{1}{2}F_{ij}(x)\mathrm{d}x^i \wedge \mathrm{d}x^j \,, \tag{2.3}$$

with

$$Q^2 \equiv 2\kappa\beta F_0^{\frac{D}{4}-1}\left(\frac{b^2}{D-2}+e^2\right) , \qquad F_0 \equiv b^2 - 2e^2 \,, \qquad b^2 \equiv F_{ik}F_{jl}h^{ij}h^{kl} \,. \tag{2.4}$$

In the above expressions, $K = 0, \pm 1$, $\lambda = \frac{2\Lambda}{(D-2)(D-1)}$, μ, e, and b are constants, Latin indices $i, j, \ldots = 1, \ldots, D-2$ label the spatial coordinates x^i (also denoted collectively simply as x), and the base-space metric $h_{ij}(x)$ represents a Riemannian Einstein space of dimension $D-2$ and scalar curvature $\mathcal{R} = K(D-2)(D-3)$, with $h \equiv \det h_{ij}$.

The spatial part of $\boldsymbol{F}$ and the base-space metric must obey the following conditions

$$\left(\sqrt{h}h^{ik}h^{jl}F_{kl}\right)_{,j} = 0 \,, \qquad F_{[ij,k]} = 0 \,, \qquad b^2 h_{ij} = (D-2)F_{ik}F_{jl}h^{lk} \,. \tag{2.5}$$

This implies that, when $F_{ij} \neq 0$, the base space must be *almost-Kähler* [11] (in addition to being Einstein) and, in particular, it cannot be a round sphere (but it can be, *e.g.*, flat, *cf.* [7, 10]). This means that dyonic (or purely magnetic) solutions cannot be asymptotically flat. However, in the purely electric case ($F_{ij} = 0$), the base manifold can be any Einstein space, and asymptotically flat solutions (with $\Lambda = 0$) have been known for some time [9].

The above spacetimes generically represent black holes, which are static in regions where $H > 0$. There is a timelike curvature singularity at $r = 0$ and, in the region of $r > 0$, positive values of r for which $H = 0$ represent the Killing horizons. Similarly as for the four-dimensional (A)dS–Reissner–Nordström metrics, the structure and number of horizons depend on the signs of the parameters Λ, K, and μ, and are essentially D-independent — *cf.* [10] for details and plots of $H(r)$ for various values of the parameters.

Interestingly, thanks to the fact that the Ricci scalar of (2.2) is constant, the above black hole solutions can be easily extended to theories for which the Einstein term in (1.1) is replaced by a generic $f(R)$ scalar — examples have been given in [12, 13] (see also [14])[1]. Extensions to the Gauss–Bonnet gravity are also known, for which the metric function H is instead modified in a non-trivial way [13, 18].

[1] For the same reason, metrics (2.1), (2.2) can also be interpreted as vacuum solutions of certain $f(R)$ gravities (in which case Q^2 is an integration constant), *cf.* [15–17].

3. General (non-stealth) solution

The family of black holes described in Section 2 is a subset of more general RT solutions of (1.1). The complete RT class is still described by metric (2.1), but now the electromagnetic field is

$$\boldsymbol{F} = \frac{e}{r^2}\mathrm{d}r \wedge \mathrm{d}u + \left(\frac{e_{,i}}{r} - \xi_i\right)\mathrm{d}u \wedge \mathrm{d}x^i + \frac{1}{2}F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j\,, \tag{3.1}$$

where $\xi_i = \xi_i(u,x)$, and H is defined by

$$2H = K + \frac{2}{D-2}\left(\ln\sqrt{h}\right)_{,u} r - \lambda r^2 - \frac{\mu}{r^{D-3}} + \frac{Q^2}{r^{D-2}}\,, \tag{3.2}$$

where (2.4) still applies. K and λ are as in Section 2, whereas here the Einstein metric $h_{ij} = h^{1/(D-2)}(u,x)\gamma_{ij}(x)$ and the quantities e, F_{ij}, μ, and b in general depend on (u,x).

In addition to the second and third equations of (2.5), one now needs to solve the more complicated set (where (3.4) replaces the first of (2.5))

$$F_{ij,u} = \xi_{i,j} - \xi_{j,i}\,, \qquad \left(F_0^{\frac{D}{4}-1}\sqrt{h}h^{ij}\xi_j\right)_{,i} = \left(F_0^{\frac{D}{4}-1}\sqrt{h}e\right)_{,u}\,, \tag{3.3}$$

$$F_0^{\frac{D}{4}-1}\sqrt{h}h^{ij}e_{,j} = \left(F_0^{\frac{D}{4}-1}\sqrt{h}h^{ik}h^{jl}F_{kl}\right)_{,j}\,, \tag{3.4}$$

$$\mu_{,i} = 2\kappa\beta D F_0^{\frac{D}{4}-1}\left(e\xi_i - F_{ik}\xi_j h^{kj}\right)\,, \tag{3.5}$$

$$(D-2)\mu_{,u} = -(D-1)\mu\left(\ln\sqrt{h}\right)_{,u} - 2\kappa D\beta F_0^{\frac{D}{4}-1}h^{ij}\xi_i\xi_j\,. \tag{3.6}$$

Here, the base space is *almost-Hermitian*. Note that Eq. (3.6) must be modified in the special case of $D=4$ [2–4, 7, 10].

The line element of the complete RT class is thus in general time-dependent. Furthermore, apart from the electric and magnetic components e and F_{ij}, the electromagnetic field (3.1) may contain also a radiative null term F_{ui}, which is related to a possible mass loss (or gain) as the retarded time u evolves, encapsulated in Eq. (3.6). The energy flux along the RT null vector field $\boldsymbol{k} = \partial_r$ is given by $\beta D F_0^{D/4-1}h^{ij}\xi_i\xi_j r^{2-D}$ [10].

For the sake of definiteness, a simple explicit time-dependent solution (not presented in [10]) with a flat base space $h_{ij} = \delta_{ij}$ is given for $D=8$ by

$$F_{ij} = 0\,, \qquad e = e_0(c_1u+1)^{1/5}\,, \qquad \xi_i\,\mathrm{d}x^i = \frac{3c_1e_0x_1 + 5c_0}{5\,(c_1u+1)^{4/5}}\,\mathrm{d}x^1\,,$$

$$\mu = \frac{16e_0^2\beta\kappa}{3c_1}\left[-\frac{(3c_1e_0x_1+5c_0)^2}{5(c_1u+1)^{1/5}} + 5{c_0}^2\right] + \mu_0\,, \qquad (D=8)\,, \tag{3.7}$$

where e_0, c_0, c_1, and μ_0 are constants. Note that the limit $c_1 \to 0$ gives rise to a function μ linear in u (and to a static solution if one sets, further, $c_0 = 0$).

Equations defining marginally trapped surfaces [19] and dynamical horizons [20] in the above spacetimes have been also obtained in [10]. Those are to be understood as preliminary results needed in order to define an analog of the past horizon in RT spacetimes, in the spirit of [21]. See also [22–27] for related results.

This work has been supported by research plan RVO: 67985840 and research grant GAČR 19-09659S.

REFERENCES

[1] I. Robinson, A. Trautman, «Spherical Gravitational Waves», *Phys. Rev. Lett.* **4**, 431 (1960).

[2] I. Robinson, A. Trautman, «Some spherical gravitational waves in general relativity», *Proc. R. Soc. A* **265**, 463 (1962).

[3] H. Stephani *et al.*, «Exact Solutions of Einstein's Field Equations. 2$^{\rm nd}$ ed.», *Cambridge University Press*, Cambridge 2003.

[4] J.B. Griffiths, J. Podolský, «Exact Space-Times in Einstein's General Relativity», *Cambridge University Press*, Cambridge 2009.

[5] I. Robinson, A. Trautman, «Conformal geometry of flows in dimensions», *J. Math. Phys.* **24**, 1425 (1983).

[6] J. Podolský, M. Ortaggio, «Robinson–Trautman spacetimes in higher dimensions», *Class. Quantum Grav.* **23**, 5785 (2006).

[7] M. Ortaggio, J. Podolský, M. Žofka, «Robinson–Trautman spacetimes with an electromagnetic field in higher dimensions», *Class. Quantum Grav.* **25**, 025006 (2008).

[8] M. Ortaggio, J. Podolský, M. Žofka, «Static and radiating p-form black holes in the higher dimensional Robinson–Trautman class», *J. High Energy Phys.* **1502**, 045 (2015).

[9] M. Hassaïne, C. Martínez, «Higher-dimensional black holes with a conformally invariant Maxwell source», *Phys. Rev. D* **75**, 027502 (2007).

[10] D. Kokoška, M. Ortaggio, «Static and radiating dyonic black holes coupled to conformally invariant electrodynamics in higher dimensions», *Phys. Rev. D* **104**, 124051 (2021).

[11] K. Yano, «Differential Geometry on Complex and Almost Complex Spaces», *Pergamon Press*, Oxford 1965.

[12] A. Sheykhi, «Higher-dimensional charged black holes», *Phys. Rev. D* **86**, 024013 (2012).

[13] S. Hervik, M. Ortaggio, «Universal p-form black holes in generalized theories of gravity», *Eur. Phys. J. C* **80**, 1020 (2020).

[14] E.F. Eiroa, G. Figueroa-Aguirre, «Black hole solutions and thin shells in N-dimensional $F(R)$ gravity with a conformally invariant Maxwell field», arXiv:2110.07562 [gr-qc].

[15] S.H. Hendi, «The relation between $F(R)$ gravity and Einstein-conformally invariant Maxwell source», *Phys. Lett. B* **690**, 220 (2010).

[16] S.H. Hendi, B. Eslam Panah, S.M. Mousavi, «Some exact solutions of $F(R)$ gravity with charged (a)dS black hole interpretation», *Gen. Relativ. Gravit.* **44**, 835 (2012).

[17] S. Hervik, M. Ortaggio, «Universal black holes», *J. High Energy Phys.* **2002**, 047 (2020).

[18] S.H. Hendi, «Topological black holes in Gauss–Bonnet gravity with conformally invariant Maxwell source», *Phys. Lett. B* **677**, 123 (2009).

[19] R. Penrose, «Naked Singularities», *Ann. N.Y. Acad. Sci.* **224**, 125 (1973).

[20] A. Ashtekar, B. Krishnan, «Dynamical Horizons: Energy, Angular Momentum, Fluxes, and Balance Laws», *Phys. Rev. Lett.* **89**, 261101 (2002).

[21] K.P. Tod, «Analogue of the past horizon in the Robinson–Trautman metrics», *Class. Quantum Grav.* **6**, 1159 (1989).

[22] A.W.C. Lun, E.W.M. Chow, «The role of the apparent horizon in the evolution of Robinson–Trautman Einstein–Maxwell space-times», arXiv:gr-qc/9409024.

[23] E.W.M. Chow, A.W.C. Lun, «Apparent horizons in vacuum Robinson–Trautman spacetimes», *J. Aust. Math. Soc. Ser. B* **41**, 217 (1999).

[24] W. Natorf, J. Tafel, «Horizons in Robinson–Trautman spacetimes», *Class. Quantum Grav.* **25**, 195012 (2008).

[25] J. Podolský, O. Svítek, «Past horizons in Robinson–Trautman spacetimes with a cosmological constant», *Phys. Rev. D* **80**, 124042 (2009).

[26] O. Svítek, «Past horizons in D-dimensional Robinson–Trautman spacetimes», *Phys. Rev. D* **84**, 044027 (2011).

[27] T. Tahamtan, O. Svítek, «Robinson–Trautman solution with nonlinear electrodynamics», *Eur. Phys. J. C* **76**, 335 (2016).