COMPLEX AND REAL PARA-KÄHLER EINSTEIN SPACES*

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Complex and real neutral, 4-dimensional, para-Kähler Einstein spaces are considered. Metrics of all para-Kähler Einstein spaces which are algebraically degenerate are found.

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1. Introduction

Investigations of 4-dimensional neutral structures (4-dimensional manifolds equipped with a metric of the neutral signature $(++--)$) showed that such structures are interesting not only from a mathematical point of view. They also play an important role in mathematical physics. For example, they were considered in the theory of integrable systems as real spaces which admit anti-self-dual (ASD) conformal structures [1]. They played a crucial role in a very interesting problem of a geometry of two solid bodies which roll on each other without slipping or twisting [2]. Recently, a very interesting paper by Bor, Makhmali, and Nurowski has been published [3]. In this paper, the authors pointed out a relation between 4-dimensional conformal structures and twistor distributions.

More precisely, so-called *para-Kähler Einstein spaces (pKE-spaces)* were investigated in [3]. These are spaces for which the ASD part of the Weyl tensor is of the Petrov–Penrose type [D] and the self-dual (SD) part is arbitrary or *vice versa*. In what follows, we assume that the ASD part of the Weyl

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tensor is of the type [D]. PKE-spaces are also equipped with two parallel propagated, totally null, 2-dimensional, integrable ASD distributions. In our formalism sketched in Section 2.3, these are spaces of the types $[\text{any}] \otimes [\text{D}]^{nn}$. Examples of all the Petrov–Penrose types of algebraically degenerate pKE-spaces[1] were presented in [3]. Also, Bor, Makhmali, and Nurowski found all pKE-spaces of the type $[\text{D}]^{ee} \otimes [\text{D}]^{nn}$ (which depends on 5 constants) and the type $[\text{D}]^{nn} \otimes [\text{D}]^{nn}$ (which depends only on cosmological constant Λ).

This remarkable result motivated us for more comprehensive studies on algebraically degenerate pKE-spaces. We classified such spaces according to three criteria:

(i) Petrov–Penrose type of the SD part of the Weyl tensor,
(ii) properties of a congruence of SD null strings,
(iii) properties of intersections of SD and ASD congruences of null strings.

According to criterion *(ii)*, there are two "parent" types of algebraically degenerate para-Kähler Einstein spaces: types $[\text{deg}]^{n} \otimes [\text{D}]^{nn}$ (which we investigated in detail in [4, 5]) and $[\text{deg}]^{e} \otimes [\text{D}]^{nn}$ (which we investigated in [6]). Our ambitious goal was to find all metrics of such spaces with all the generality. After almost two years of intensive studies, our goal has been completely fulfilled.

To find explicit metrics of pKE-spaces, we use a special technique, namely, we consider 4-dimensional (in a complex sense) spaces which are called *hyperheavenly spaces ($\mathcal{HH}$-spaces)*. $\mathcal{HH}$-spaces are defined as 4-dimensional complex manifolds $\mathcal{M}$ equipped with a holomorphic metric and such that the SD (or ASD) part of the Weyl tensor is algebraically degenerate. A great result by Plebański and Robinson is that Einstein vacuum field equations in $\mathcal{HH}$-spaces can be reduced to a single, nonlinear, partial differential equation of the second order (so-called *hyperheavenly equation ($\mathcal{HH}$-equation)*) for a single function W (*the key function*) which completely determines the metric.

A great advantage of the complex 4-dimensional geometry is the fact that any real 4-dimensional space can be obtained from a generic complex space as a *real slice* of such a complex space [7]. Unfortunately, it is quite complicated to find real slices which lead to Lorentzian spaces. Probably this was a reason why $\mathcal{HH}$-spaces formalism was not a breakthrough in the General Theory of Relativity. However, it is much simpler to find real neutral slices. It is enough to replace all complex coordinates by real ones and all holomorphic functions by real smooth ones. Thus, $\mathcal{HH}$-spaces formalism is a perfect tool for finding real neutral 4-dimensional spaces, especially pKE-spaces.

[1] Examples of algebraically general pKE-spaces (*i.e.*, spaces of the type $[\text{I}] \otimes [\text{D}]^{nn}$) are unknown.

The paper is organized as follows. In Section 2, we remind the definition of congruences of SD and ASD null strings and congruences of null geodesics. Also, a brief review of a special formalism which we use in our paper is presented. In Section 3, the definitions of para-Hermite and para-Kähler spaces as well as hyperheavenly spaces are given. Finally, in Section 4, we arrive at the main results: explicit and general metrics of all algebraically degenerate pKE-spaces.

All considerations are purely local. All metrics presented in our paper are complex but they have neutral slices. We use spinorial formalism in the Infeld–Van der Waerden–Plebański notation (see [8, 9] for details).

2. Congruences of null strings and congruences of null geodesics

2.1. Congruences of null strings

Consider a 2-dimensional totally null distribution $\mathcal{D}_{m^A} = \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{A}} b^{\dot{B}} \neq 0$. It is completely integrable in the Frobenius sense if

$$m^A m^B \nabla_{A\dot{M}} m_B = 0 \,. \tag{2.1}$$

Equations (2.1) are called *the SD null string equations* and we say that the spinor m^A *generates the congruence of SD null strings*. Integral manifolds of the distribution $\mathcal{D}_{m^A}$ are 2-dimensional, totally null, and totally geodesic surfaces called *null strings*. A family of such surfaces constitutes *a congruence (foliation) of SD null strings*.

Equations (2.1) can be rewritten in the equivalent form

$$m^B \, \nabla_{A\dot{M}} m_B = m_A M_{\dot{M}} \,. \tag{2.2}$$

The dotted spinor field $M_{\dot{M}}$ describes the most important property of a congruence of SD null strings. If $M_{\dot{M}} = 0$, then distribution $\mathcal{D}_{m^A}$ is parallel propagated. It means that $\nabla_X V \in \mathcal{D}_{m^A}$ for every vector field $V \in \mathcal{D}_{m^A}$ and for arbitrary vector field X. Thus, following Plebański and Rózga [10], we call $M_{\dot{M}}$ *an expansion* of a congruence of SD null strings. If $M_{\dot{M}} \neq 0$ ($M_{\dot{M}} = 0$), then a congruence is *expanding* (*nonexpanding*). Note that real spaces equipped with the nonexpanding congruence of null strings are called the *Walker spaces* [11].

The definition of a congruence of ASD null strings is analogous. It is generated by 1-index dotted spinors. If a spinor $m^{\dot{A}}$ generates a congruence of ASD null strings, then it satisfies *the ASD null string equations*

$$m^{\dot{B}} \, \nabla_{A\dot{M}} m_{\dot{B}} = m_{\dot{M}} M_A \,, \tag{2.3}$$

where M_A is *an expansion of a congruence of ASD null strings.*

Remark 2.1. Note that in complex geometry a spinor m^A is in general complex. Hence, a congruence of null strings is a family of 2-dimensional holomorphic surfaces. However, if we consider real neutral case, we deal with real smooth 2-dimensional surfaces and a congruence is generated by a real spinor.

2.2. Congruences of null geodesics

In the General Theory of Relativity, a concept of a congruence of null geodesics in well known. It describes a behavior of a beam of light traveling through a spacetime. Properties of such a congruence are used as a criterion for a classification of the exact solutions to Einstein field equations. There are three *optical scalars*, namely expansion θ, twist ϱ, and shear σ defined as follows:

$$\theta := \frac{1}{2}\nabla^a K_a \,, \quad \varrho^2 := \frac{1}{2}\nabla_{[a}K_{b]}\,\nabla^a K^b \,, \quad \sigma^2 := \frac{1}{2}\nabla_{(a}K_{b)}\,\nabla^a K^b - \theta^2 \,, \tag{2.4}$$

where K_a is a null vector field which integral curves constitute a congruence of null geodesics.

In a complex geometry (and in a real neutral geometry), the concept of congruences of null geodesics can be also used. If a space admits congruences of both SD and ASD null strings, then these congruences necessarily intersect. The intersection constitutes a congruence of complex null geodesic lines. Let K_a be a null vector field along the intersection of congruences of null strings generated by spinors m_A and $m_{\dot{A}}$, respectively. Then $K_a \sim m_A m_{\dot{A}}$. One quickly finds that properties of complex null geodesics are related to the properties of congruences of null strings. Indeed, if a congruence of null geodesics is in an affine parametrization, then $\sigma = 0$ and

$$\theta \sim m_A M^A + m_{\dot{A}} M^{\dot{A}} \,, \qquad \varrho \sim m_A M^A - m_{\dot{A}} M^{\dot{A}} \,. \tag{2.5}$$

However, it must be pointed out that the optical scalars in complex and real neutral geometries do not have such a transparent geometrical interpretation as their counterparts in the Lorentzian case. Also, an expansion of a congruence of SD (ASD) null strings $M^{\dot{A}}$ (M^A) and an expansion of a congruence of null geodesics θ are different concepts. We refer to both these quantities as "expansions" but we believe it will not be misleading.

There are four subtypes of intersections of congruences of null strings for which we propose the following symbols:

$$\begin{aligned}
[++] &: \ \theta \neq 0\,, \quad \varrho \neq 0\,, \\
[+-] &: \ \theta \neq 0\,, \quad \varrho = 0\,, \\
[-+] &: \ \theta = 0\,, \quad \varrho \neq 0\,, \\
[--] &: \ \theta = 0\,, \quad \varrho = 0\,.
\end{aligned} \tag{2.6}$$

2.3. Formalism

In complex and real neutral spaces SD and ASD, parts of the Weyl tensor are independent. Thus, the following symbol is usually used:

$$[\mathrm{SD}_{\mathrm{type}}] \otimes [\mathrm{ASD}_{\mathrm{type}}] \,, \tag{2.7}$$

where

$\mathrm{SD}_{\mathrm{type}}, \mathrm{ASD}_{\mathrm{type}} = \{\mathrm{I}, \mathrm{II}, \mathrm{D}, \mathrm{III}, \mathrm{N}, \mathrm{O}\}$ in complex spaces,
$\mathrm{SD}_{\mathrm{type}}, \mathrm{ASD}_{\mathrm{type}} = \{\mathrm{I}_r, \mathrm{I}_{rc}, \mathrm{I}_c, \mathrm{II}_r, \mathrm{II}_{rc}, \mathrm{D}_r, \mathrm{D}_c, \mathrm{III}_r, \mathrm{N}_r, \mathrm{O}_r\}$ in neutral spaces.

An abbreviation *any* (*deg*) means that the type is arbitrary (algebraically degenerate). For a more detailed discussion about Petrov–Penrose types in complex and real neutral geometries see, *e.g.*, [3, 4].

If a space is additionally equipped with SD and ASD congruences of null strings, symbol (2.7) is not sufficient. Hence, we propose the following extension:

$$\{[\mathrm{SD}_{\mathrm{type}}]^{i_1 i_2 \cdots} \otimes [\mathrm{ASD}_{\mathrm{type}}]^{j_1 j_2 \cdots}, [k_{11}, k_{12}, \ldots, k_{21}, k_{22}, \ldots]\} \,. \tag{2.8}$$

The number of superscripts $i(j)$ says about the number of SD (ASD) congruences of null strings. $i_1, i_2, \ldots, j_1, j_2, \cdots = \{n, e\}$ where n stands for nonexpanding congruences, while e stands for expanding congruences. Symbols $k_{11}, k_{12}, \ldots, k_{21}, k_{22}, \cdots = \{--, -+, +-, ++\}$ tell us about the properties of intersections of congruences of null strings. Properties of the intersection of i_m-congruence of SD null strings with j_n-congruence of ASD null strings are gathered in the symbol k_{mn}. For a more detailed explanation of the meaning of the symbol (2.8), see [4].

For further purposes, it is necessary to specialize symbol (2.8) to the case of pKE-spaces. In the next section, we point out that para-Kähler spaces are equipped with two different nonexpanding congruences of null strings of the same duality. We chose an orientation in such a manner that these congruences are ASD. Thus, the ASD part of the Weyl tensor is of the type $[\mathrm{D}]^{nn}$. Additionally, we assume that the SD part of the Weyl tensor is algebraically degenerate what is equivalent to the fact that there exists also a congruence of SD null strings. Hence, there are two intersections of SD and ASD congruences of null strings. Detailed considerations prove that there are only three different subtypes of algebraically degenerate pKE-spaces. Namely,

$$\{[\mathrm{deg}]^n \otimes [\mathrm{D}]^{nn}, [--, --]\} \,, \tag{2.9a}$$
$$\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [++, --]\} \,, \tag{2.9b}$$
$$\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [++, ++]\} \,. \tag{2.9c}$$

In what follows, we focus on the types in (2.9b)–(2.9c).

3. Para-Kähler spaces and hyperheavenly spaces

3.1. Para-Kähler spaces

Let us remind the basic definitions of *para-Kähler spaces* and *para-Hermite spaces* (for a brief review of the topic, see Section 2.2 of [3]).

Definition 3.1. *An almost para-Hermitian structure* $(\mathcal{M}, g, K)$ *is a 4-dimensional manifold* $\mathcal{M}$ *equipped with a metric* g *of signature* $(++--)$ *and an endomorphism* $K : T\mathcal{M} \to T\mathcal{M}$ *such that* $K^2 = id_{T\mathcal{M}}$ *(*K *is para-complex) whose* ± 1*-eigenvalues have rank 2 and it satisfies the compatibility condition* $g(KX, KY) = -g(X, Y)$ *for all* $X, Y \in T\mathcal{M}$*.*

Eigenspaces which correspond to ± 1-eigenvalues of K define rank 2 distributions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$

$$\mathcal{D} = (K + id_{T\mathcal{M}})\, T\mathcal{M}\,, \qquad \widetilde{\mathcal{D}} = (K - id_{T\mathcal{M}})\, T\mathcal{M}\,. \tag{3.1}$$

Hence,

$$T\mathcal{M} = \mathcal{D} \bigoplus \widetilde{\mathcal{D}}\,. \tag{3.2}$$

Both $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are null with respect to g and both are of the same duality.

Definition 3.2. *An almost para-Hermitian structure* $(\mathcal{M}, g, K)$ *is called para-Hermitian if both* $\mathcal{D}$ *and* $\widetilde{\mathcal{D}}$ *are integrable.*

Define 2-form ρ

$$\rho(X, Y) := g(KX, Y)\,. \tag{3.3}$$

2-form ρ defined by (3.3) is called *para-Kähler 2-form.*

Definition 3.3. *A para-Hermitian structure is called para-Kähler if and only if 2-form* ρ *is closed,* $d\rho = 0$*.*

In Plebański's terminology, an integrable, totally null distribution is exactly a congruence of null strings. In other words, para-Hermite spaces are equipped with two different congruences of null strings of the same duality. Moreover, both congruences are nonexpanding if and only if the para-Kähler 2-form ρ is closed. Hence, para-Kähler spaces are spaces equipped with two different nonexpanding congruences of null strings of the same duality.

Note, that the terms *para-Hermite spaces* and *para-Kähler spaces* refer to real neutral geometries (in such a case, null strings are real smooth surfaces, see Remark 2.1). In a complex case, the terms *complex para-Hermite spaces* and *complex para-Kähler spaces* are usually used (in this case, null strings are holomorphic surfaces). Hence, para-Hermite spaces (para-Kähler spaces) appear as real neutral slices of complex para-Hermite spaces (complex para-Kähler spaces).

3.2. Hyperheavenly spaces

Definition 3.4. *A hyperheavenly space ($\mathcal{HH}$-space) is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a 4-dimensional, complex analytic differential manifold and g is a holomorphic metric which satisfies vacuum Einstein field equations with cosmological constant and such that the self-dual or anti-self-dual part of the Weyl tensor is algebraically degenerate.*

The fact that the SD (or ASD) part of the Weyl tensor is algebraically degenerate is equivalent to the fact that a space admits a congruence of SD (or ASD) null strings. It is a statement of the *complex Goldberg–Sachs theorem* [12]. In what follows, we chose an orientation in such a manner that the congruence is SD and we assume that it is necessary expanding (a nonexpanding case was considered in [4, 5]).

It was proved in [13] that the metric of any (expanding) $\mathcal{HH}$-space can be brought to the form

$$\frac{1}{2}\mathrm{d}s^2 = x^{-2}\left(\mathrm{d}q\,\mathrm{d}y - \mathrm{d}p\,\mathrm{d}x + \mathcal{A}\,\mathrm{d}p^2 - 2\mathcal{Q}\,\mathrm{d}q\,\mathrm{d}p + \mathcal{B}\,\mathrm{d}q^2\right), \tag{3.4}$$

where (q, p, x, y) is a local coordinate system, and

$$\mathcal{A} := -xW_{yy} + \mu x^3 + \frac{\Lambda}{6}, \quad \mathcal{Q} := xW_{xy} - W_y, \quad \mathcal{B} := -xW_{xx} + 2W_x. \tag{3.5}$$

Einstein field equations with cosmological constant reduce to a single $\mathcal{HH}$-*equation*

$$\begin{aligned} &W_{xx}W_{yy} - W_{xy}^2 + \frac{2}{x}(W_yW_{xy} - W_xW_{yy}) + \frac{1}{x}(W_{qy} - W_{px}) \\ &-\mu(x^2W_{xx} - 3xW_x + 3W) - \frac{\Lambda}{6x}W_{xx} + \frac{1}{2}y(\mu_py - \mu_qx) = \frac{1}{2}\varkappa x - \frac{1}{2}\nu y + \gamma, \end{aligned} \tag{3.6}$$

where μ, ν, $\varkappa$, and γ are arbitrary functions of (q, p) only and $W = W(q, p, x, y)$ is called *the key function.*

Metric (3.4) is a general metric of the Einstein space of the types $[\mathrm{deg}]^e \otimes [\mathrm{any}]$. If we equip such a space with two different congruences of ASD null strings, we arrive at the complex para-Hermite Einstein space. If we demand additionally that both these ASD congruences are nonexpanding, we obtain the complex para-Kähler Einstein space. The existence of two different congruences of ASD null strings is equivalent to the fact that the ASD part of the Weyl tensor is of the type [D].

Let "the first" congruence be generated by a spinor $m_{\dot{A}}$ which can be always re-scaled to the form $m_{\dot{A}} \sim [z, 1]$, $z = z(q, p, x, y)$. Equations (2.3) written explicitly yield

$$z_x - zz_y = 0, \tag{3.7a}$$

$$z_q - zz_p - z_y\mathcal{Z} + z\frac{\partial\mathcal{Z}}{\partial y} - \frac{\partial\mathcal{Z}}{\partial x} = 0, \qquad \mathcal{Z} := \mathcal{B} + 2z\mathcal{Q} + z^2\mathcal{A}, \tag{3.7b}$$

and the expansion of this congruence is given by the formulas

$$\frac{x}{\sqrt{2}}M_1 = -xz_y - 1\,, \tag{3.8a}$$

$$\frac{1}{\sqrt{2}x}M_2 = -xz_p + xz\frac{\partial}{\partial y}(\mathcal{Q}+z\mathcal{A}) - x\frac{\partial}{\partial x}(\mathcal{Q}+z\mathcal{A}) + (\mathcal{Q}+z\mathcal{A})(1-xz_y)\,. \tag{3.8b}$$

The expansion and twist of the intersection of the congruence of SD null strings and "the first" congruence of ASD null strings read

$$\theta_1 \sim xz_y + 2\,, \qquad \varrho_1 \sim z_y\,. \tag{3.9}$$

"The second" ASD congruence must be generated by a spinor $n^{\dot{A}}$ such that $n^{\dot{A}} m_{\dot{A}} \neq 0$. The spinor $n^{\dot{A}}$ can be always brought to the form $n_{\dot{A}} \sim [1, w]$, $w = w(q,p,x,y)$. In this case we have equations

$$w_y - ww_x = 0\,, \tag{3.10a}$$

$$w_p - ww_q + w_x\mathcal{W} - w\frac{\partial \mathcal{W}}{\partial x} + \frac{\partial \mathcal{W}}{\partial y} = 0\,, \qquad \mathcal{W} := \mathcal{A} + 2w\mathcal{Q} + w^2\mathcal{B}\,. \tag{3.10b}$$

The expansion reads

$$\frac{x}{\sqrt{2}}N_1 = xw_x - w\,, \tag{3.11a}$$

$$\frac{1}{\sqrt{2}x}N_2 = xw_q + xw\frac{\partial}{\partial x}(\mathcal{Q}+w\mathcal{B}) - x\frac{\partial}{\partial y}(\mathcal{Q}+w\mathcal{B}) - xw_x(\mathcal{Q}+w\mathcal{B}) + (\mathcal{A}+w\mathcal{Q})\,, \tag{3.11b}$$

and optical scalars take the form

$$\theta_2 \sim xw_x - 2w\,, \qquad \varrho_2 \sim w_x\,. \tag{3.12}$$

If both ASD congruences are nonexpanding, $M_A = N_A = 0$, we obtain the set of 8 equations supplemented by $\mathcal{HH}$-equation (3.6). A procedure of integration of these equations is quite complicated and tedious. Thus, we omit all the details. In Section 4, we present only final results. For a more detailed discussion, see [6].

4. The metrics

4.1. Type $\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [++,--]\}$ para-Kähler Einstein spaces

For the subtypes which are characterized by $\theta_1 \neq 0$, $\varrho_1 \neq 0$, $\theta_2 = \varrho_2 = 0$, one finds that $z = -y/x$, $w = 0$, and the key function takes the form

$$W = \frac{1}{2}\left(Ax^2 - \frac{\Lambda}{6x}\right)y^2 + \frac{1}{2}\left(\frac{M_p}{\Lambda}x^2 - Mx\right)y + \frac{1}{2}Nx^2\,, \tag{4.1}$$

where A, M, and N are arbitrary functions of (q,p) only. Also, the functions μ, ν, $\varkappa$, and γ read

$$\mu = 2A\,, \quad \nu = \frac{2}{\Lambda} M_{pp}\,, \quad \varkappa = \frac{1}{\Lambda}(M_{pq} - M M_p)\,, \quad \gamma = -N_p - \frac{M_q}{2} + \frac{1}{4} M^2\,. \tag{4.2}$$

Finally, one formulates

Theorem 4.1. *Let $(\mathcal{M}, ds^2)$ be an Einstein complex (neutral) space of the type $\{[deg]^e \otimes [D]^{nn}, [++,--]\}$ ($\{[deg]^e \otimes [D_r]^{nn}, [++,--]\}$). Then there exists a local coordinate system (q,p,x,y) such that the metric takes the form*

$$\begin{aligned} \frac{1}{2} ds^2 \;=\; x^{-2} \Big\{ & dq\,dy - dp\,dx + \left(A x^3 + \frac{\Lambda}{3} \right) dp^2 \\ & -2 \left(A y x^2 + \frac{\Lambda}{3} \frac{y}{x} + \frac{M_p}{2\Lambda} x^2 \right) dq\,dp \\ & + \left(A y^2 x + \frac{\Lambda}{3} \frac{y^2}{x^2} + \frac{M_p}{\Lambda} x y + N x - M y \right) dq^2 \Big\}\,, \end{aligned} \tag{4.3}$$

where $\Lambda \neq 0$ is a cosmological constant, $A = A(q,p)$, $M = M(q,p)$, and $N = N(q,p)$ are arbitrary holomorphic (real smooth) functions.

4.2. Type $\{[deg]^e \otimes [D]^{nn}, [++,++]\}$ para-Kähler Einstein spaces

For more general subtypes with $\theta_1 \neq 0$, $\varrho_1 \neq 0$, $\theta_2 \neq 0$, $\varrho_2 \neq 0$, one finds that $z = -y/x$, $w = x/(1-y)$, and the key function reads

$$\begin{aligned} W \;=\; & -\frac{\Lambda}{12} \frac{y^4}{x} + \left(\frac{\Lambda}{x} - 2B \right) \frac{y^3}{6} + \frac{1}{2} \left(A x^2 + C x + B - \frac{\Lambda}{6x} \right) y^2 \\ & - \left(M x^2 + \frac{C}{2} x \right) y + \frac{1}{2} \left(M + \frac{2B_q + C_p}{2\Lambda} \right) x^2\,, \end{aligned} \tag{4.4}$$

where A, B, C, and M are arbitrary functions of (q,p). The functions μ, ν, $\varkappa$, and γ take the form

$$\begin{aligned} & \mu = 2A\,, \quad \nu = -4M_p - 2C_q - \frac{2B}{\Lambda}(2B_q + C_p)\,, \quad \varkappa = -2M_q - 2C\,\frac{2B_q + C_p}{2\Lambda}\,, \\ & \gamma = \frac{C^2}{4} - \frac{C_q}{2} - M_p - \frac{2B_{qp} + C_{pp}}{2\Lambda} - B \left(M + \frac{2B_q + C_p}{2\Lambda} \right)\,. \end{aligned} \tag{4.5}$$

Eventually, one arrives at

Theorem 4.2. *Let $(\mathcal{M}, ds^2)$ be an Einstein complex (neutral) space of the type $\{[deg]^e \otimes [D]^{nn}, [++,++]\}$ ($\{[deg]^e \otimes [D_r]^{nn}, [++,++]\}$). Then there* *exists a local coordinate system* (q,p,x,y) *such that the metric takes the form*

$$\frac{1}{2}ds^2 = x^{-2}\Bigg\{dq\,dy - dp\,dx + \left(Ax^3 - Cx^2 - Bx(1-2y) + \frac{\Lambda}{3}(1-3y+3y^2)\right)dp^2 \\ -2\left(Ax^2y - By(1-y) - Mx^2 + \frac{\Lambda}{3}\frac{y(1-y)(1-2y)}{x}\right)dq\,dp \\ + \left(Axy^2 - Cy(1-y) + Mx(1-2y) + \frac{\Lambda}{3}\frac{y^2(1-y)^2}{x^2} + \frac{2B_q + C_p}{2\Lambda}x\right)dq^2\Bigg\} \tag{4.6}$$

where $\Lambda \neq 0$ *is a cosmological constant,* $A = A(q,p)$*,* $B = B(q,p)$*,* $C = C(q,p)$*, and* $M = M(q,p)$ *are arbitrary holomorphic (real smooth) functions.*

It is worth to note that metric (4.6) is the general metric of the algebraically degenerate para-Kähler Einstein spaces. It depends on 4 functions of two variables, as it was proved in [3].

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