PRECISION CALCULATIONS IN $\text{BR}(\bar{B} \to X_s \gamma)^*$

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We briefly summarize the current status of perturbative calculations at next-to-next-to-leading-order (NNLO) accuracy in the $\bar{B} \to X_s \gamma$ decay rate as well as that of non-perturbative power-corrections.

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1. Introduction

Corrections to the $\bar{B} \to X_s \gamma$ decay are usually described in the framework of an effective theory\(^1\),

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}} \times \mathcal{L}_{\text{QED}}(u, d, s, c, b) + \frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^* \sum_{i=1}^8 C_i(\mu)O_i. \quad (1)$$

Here, $C_i$ are renormalization scale dependent effective couplings, the so-called Wilson coefficients, which encode the heavy gauge boson and the heavy top quark effects. The $b$-quark scale contributions, on the other hand, are seen as matrix elements of flavor changing operators,

$$O_1 = (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L), \quad O_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L),$$

$$O_{3,5} = (\bar{s}_L \Gamma_{3,5} b_L) \sum_q (\bar{q} \Gamma'_{3,5} q), \quad O_{4,6} = (\bar{s}_L \Gamma_{3,5} T^a b_L) \sum_q (\bar{q} \Gamma'_{3,5} T^a q),$$

$$O_7 = \frac{\alpha_{\text{em}}}{4\pi} m_b (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}, \quad O_8 = \frac{\alpha_s}{4\pi} m_b (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a. \quad (2)$$


\(^1\) In writing (1) we discarded terms proportional to $V_{ub}V_{us}^*$ since they give only small contributions to the branching ratio that start at next-to-leading-order (NLO). Similar NNLO corrections can therefore be safely neglected.
where $\Gamma_3 = \gamma_\mu$, $\Gamma'_3 = \gamma^\mu$, $\Gamma_5 = \gamma_\mu \gamma_\nu \gamma_\lambda$ and $\Gamma'_5 = \gamma^\mu \gamma^\nu \gamma^\lambda$. Using (1), the differential decay rate for $\bar{B} \to X_s \gamma$ can be written as follows,

$$d\Gamma = \frac{G_F^2 \alpha_{em} m_b^2}{256 \pi^6 m_B} \left| V_{tb} V_{ts}^* \right|^2 \frac{d^3 q}{E_\gamma} \sum_{i,j} C_i^{\text{eff}} (\mu) C_j^{\text{eff}} (\mu) W_{ij} (\mu).$$

In the equation displayed above, $q$ denotes the momentum of the photon, $C_i^{\text{eff}}$ are certain linear combinations of $C_i$ (see e.g. [1]), and the $W_{ij}$ describe the hadronic dynamics. For $(ij) = (77)$ the latter can be written as imaginary part of a forward scattering amplitude,

$$W_{77}(\mu) = 2 \text{Im} \left( i \int d^4 x e^{-iq \cdot x} \langle \bar{B} | T \left\{ O_7^\dagger (x) O_7 (0) \right\} | \bar{B} \rangle \right).$$

Since the mass of the $b$-quark is much larger than the binding energy of the $B$-meson, which is of the order of $\Lambda \equiv \Lambda_{\text{QCD}}$, we can perform an operator product expansion (OPE) of this time ordered product. Doing so, one finds that the leading term is the partonic decay rate which gives the dominant contribution, while the non-leading terms, the so-called power-corrections, are suppressed by powers of $\Lambda/m_b$ and give non-vanishing contributions starting from $O(\Lambda^2/m_b^2)$\footnote{We stress that equation (4) and its OPE hold only for $W_{77}$. In all other cases the $W_{ij}$ defined in (3) contain contributions in which the photon couples to light quarks ($u, d, s, c$), and this leads to non-perturbative effects different from that mentioned above (see Section 3).}. In what follows we describe the state-of-the-art of perturbative and non-perturbative corrections in the $\bar{B} \to X_s \gamma$ decay.

### 2. Perturbative corrections

The calculation of the perturbative corrections can be divided into three steps. In the first step one has to evaluate the effective couplings $C_i^{\text{eff}}$ at the high-energy scale $\mu \sim M_W$ by requiring equality of the Standard Model and the effective theory Green functions. Defining $\tilde{\alpha}_s(\mu) = \alpha_s(\mu)/(4\pi)$, the effective couplings can be expanded as follows,

$$C_i^{\text{eff}} (\mu) = C_i^{(0)\text{eff}} (\mu) + \tilde{\alpha}_s(\mu) C_i^{(1)\text{eff}} (\mu) + \tilde{\alpha}_s^2(\mu) C_i^{(2)\text{eff}} (\mu) + \ldots .$$

At NNLO accuracy one has to determine the coefficients $C_i^{(2)\text{eff}} (\mu)$. For $i = 7, 8$ it required performing a three-loop calculation [2] whereas for the remaining cases $i = 1, \ldots, 6$ a two-loop calculation was sufficient [3].

The second step involves the calculation of the anomalous dimension matrix $\gamma^{\text{eff}}$ which describes the mixing of the operators under renormalization. Its knowledge is necessary to solve the effective theory renormalization group equations for the effective couplings,
$$\mu \frac{d}{d \mu} C^\text{eff}_i (\mu) = \sum_j \gamma^\text{eff}_{ij} C^\text{eff}_j (\mu), \quad (6)$$

and to evolve the latter down to the low-energy scale $\mu \sim m_b$. Performing a perturbative expansion in the strong coupling constant, the anomalous dimension matrix takes the following form,

$$\gamma^\text{eff} = \tilde{\alpha}_s(\mu) \gamma^{(0)}^\text{eff} + \tilde{\alpha}_s^2(\mu) \gamma^{(1)}^\text{eff} + \tilde{\alpha}_s^3(\mu) \gamma^{(2)}^\text{eff} + \ldots. \quad (7)$$

At NNLO one has to determine $\gamma^{(2)}^\text{eff}$ which is actually a $8 \times 8$ matrix,

$$\gamma^{(2)}^\text{eff} = \begin{pmatrix} A^{(2)}_{6 \times 6} & B^{(2)}_{6 \times 2} \\ 0_{2 \times 6} & C^{(2)}_{2 \times 2} \end{pmatrix}. \quad (8)$$

The block matrices $A$ and $C$ describing the self-mixing of the four-quark operators and the self-mixing of the dipole operators at three loops, respectively, have been calculated in [4]. The block matrix $B$ describing the mixing of the four-quark operators into the dipole operators at four loops has been determined in [1]. After this calculation the first two steps of the perturbative calculation were completed, that is the effective couplings at the low-energy scale $\mu \sim m_b$ with resummed logarithms are now known at NNLO accuracy\(^3\).

In the last step one has to calculate on-shell amplitudes of the operators at the low-energy scale. This is the most difficult part of the NNLO enterprise and it is still under investigation. In order to see what has been done so far, and what still has to be done, we write the decay rate for the partonic decay $b \rightarrow X_s^{\text{partonic}} \gamma$ as follows,

$$\Gamma^{\text{partonic}}|_{E_\gamma > E_0} = \frac{G_F^2 \alpha_\text{em} m_b^5}{32 \pi^4} |V_{tb} V^*_t s|^2 \sum_{i,j} C^\text{eff}_i (\mu) C^\text{eff}_j (\mu) G_{ij}(E_0, \mu), \quad (9)$$

where $G_{ij}(E_0, \mu)$ can again be expanded in terms of $\tilde{\alpha}_s$,

$$G_{ij}(E_0, \mu) = \delta_i^7 \delta_j^7 + \tilde{\alpha}_s(\mu) Y^{(1)}_{ij}(E_0, \mu) + \tilde{\alpha}_s^2(\mu) Y^{(2)}_{ij}(E_0, \mu) + \ldots. \quad (10)$$

At NNLO one has to determine the coefficients of $\tilde{\alpha}_s^2(\mu)$ which, however, has only been done in a complete manner for $i = j = 7$ [5,6]. Once we neglect on-shell amplitudes that are proportional to the small Wilson coefficients of the four-quark penguin operators $O_3-O_6$, the remaining cases to be considered are $(ij) = (11), (12), (22), (17), (18), (27), (28), (78)$, and $(88)$. The large-$\beta_0$

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\(^3\) This means large logarithms have been resummed up to $O(\alpha_s^{n+2} \ln^n (m_b/M_W))$. 
corrections are known in all these cases except for (18) and (28) [7–9]. In addition, effects of the charm and bottom quark masses on the gluon lines are known in all the cases [10,11]. The other beyond-large-\(\beta_0\) corrections have been found only in the limit \(m_c \gg m_b/2\), except for the (78) and (88) cases [12]. This limit has been used to interpolate the unknown beyond-large-\(\beta_0\) corrections at \(O(\alpha_s^2)\) to the measured value of \(m_c \approx m_b/4\) [12].

The result for the branching ratio, for \(E_0 = 1.6\) GeV, is given by [13]\(^4\)

\[
\text{BR} \left( \bar{B} \to X_s \gamma \right)_{\text{SM}} = (3.15 \pm 0.23) \times 10^{-4} \quad (\text{1S scheme}).
\]

The theoretical uncertainty of this NNLO estimate is at the same level as the uncertainty of the current world average reported by HFAG [14]\(^5\),

\[
\text{BR} \left( \bar{B} \to X_s \gamma \right)_{\text{exp}} = (3.52 \pm 0.23 \pm 0.09) \times 10^{-4},
\]

which is furthermore expected to come down to the 5% level at the end of the \(B\)-factory era.

Here a remark concerning the overall normalization of the theoretical prediction is in order. To reduce parametric uncertainties stemming from the CKM angles as well as from the \(c\)- and \(b\)-quark masses, the partonic decay rate given in (9) is usually normalized using a combination of the \(\bar{B} \to X_c l \bar{\nu}\) and \(\bar{B} \to X_u l \bar{\nu}\) decay rates which is reflected by the appearance of the semileptonic phase-space factor

\[
C = \left| \frac{V_{ub}}{V_{cb}} \right|^2 \frac{\Gamma (\bar{B} \to X_c l \bar{\nu})}{\Gamma (\bar{B} \to X_u l \bar{\nu})}
\]

in the analytical expressions [17]\(^6\). Unfortunately, the determination of \(m_c\) and \(C\) from a fit to the measured spectrum of the \(\bar{B} \to X_c l \bar{\nu}\) decay in the 1S scheme [19] differs from that in the kinetic scheme [20]\(^7\). Using the values for \(m_c\) and \(C\) of the latter determination results in a higher central value for the \(\bar{B} \to X_s \gamma\) decay rate [21],

\[
\text{BR} \left( \bar{B} \to X_s \gamma \right)_{\text{SM}} = (3.25 \pm 0.24) \times 10^{-4} \quad (\text{kinetic scheme}).
\]

The difference of \(m_c\) and \(C\) in the 1S and kinetic scheme is likely to be due to different input data, differences in the fit method, and treatment

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\(^4\) For a discussion of the residual renormalization scale dependence of the branching ratio at NNLO we refer the reader to [13].

\(^5\) This average includes the measurements from CLEO and BaBar and Belle [15]. The recently published update by Belle [16] has not been taken into account.

\(^6\) The denominator \(\Gamma (b \to ul\bar{\nu})\) is already known at NNLO accuracy [18].

\(^7\) In [13] the values for \(m_c\) and \(C\) from [19] were adopted.
of theory errors. In this respect, supplementing the fit, for example, by the
determination of the $c$- and $b$-quark masses from sumrules [22] could possibly
be helpful to reduce the discrepancy of $C$ in both schemes.

We should also remark that not all of the aforementioned contributions
to the function $G_{ij}$ entered the analysis of [13]. These are the massive
fermionic corrections presented in [6, 10, 11] and the large-$\beta_0$ contributions
for $(ij) \neq (77)$ from [8, 9]. These contributions will be included in a future
update together with so far unknown contributions, which is, for example,
the complete knowledge of $G_{78}$ [23]. Also the complete calculation of $G_{27}$
for $m_c = 0$ is underway [24]. Especially the latter will prove useful in
the reduction of the uncertainty stemming from the interpolation in $m_c$.

Apart from the NNLO corrections also tree-level diagrams with the $u$-quark
analogues of $O_{1,2}$ and the four-quark operators $O_{3-6}$ have been neglected so
far [25]. The numerical effect of all of these contributions on the branching
ratio is or is expected to remain within the uncertainty of the NNLO estimate
given in (11).

Finally, we should mention that there are also cutoff-enhanced correc-
tions which matter close to the endpoint [26]. However, as demonstrated
in [21], the resummation of the cutoff-enhanced logarithms overestimates
the effect of the $O(\alpha_s^3)$-terms for $E_0 \lesssim 1.6$ GeV. Therefore, the prediction
for the branching ratio given in (11) should, at present, be considered as
more reliable.

3. Non-perturbative corrections

Since the perturbative calculations in $\bar{B} \to X_s\gamma$ are now performed at
NNLO, the non-perturbative corrections become more important. In general
well under control are the power-corrections stemming from the OPE of the
time ordered product contained in (4); they are known at $O(\Lambda^2/m_b^2)$ [27]
and $O(\Lambda^3/m_b^3)$ [28]. All the other $W_{ij}$ with $(ij) \neq (77)$ contain contributions
in which the photon couples to light quarks, and that causes the breakdown
of an analogous OPE. In this case non-perturbative collinear effects [29] as
well as power-corrections at $O(\Lambda^2/m_c^2)$ show up [30]. The combined effect of
all of the aforementioned non-perturbative corrections is of around 3% in the
branching ratio. Besides, non-perturbative effects appearing at $O(\alpha_s\Lambda/m_b)$
show up when the photon couples to light quarks. Their size is not known
at present, and hence a 5% uncertainty related to all the unknown non-
perturbative effects has been included in (11). The size of this uncertainty is
supported by the estimate of the $O(\alpha_s\Lambda/m_b)$-corrections in the interference
of the electro- and chromomagnetic dipole operators performed in [31]. As

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8 Also the mixing of the four-quark operators $O_{1-6}$ into the chromomagnetic dipole
operator $O_8$ [1] was not included in [13].
pointed out in references [31, 32], the magnitude of the effect considered in [31] could be probed by an improved measurement [33] of the isospin asymmetry

\[
\Delta_{0-} = \frac{\Gamma (\bar{B}^0 \to X_s \gamma) - \Gamma (\bar{B}^- \to X_s \gamma)}{\Gamma (\bar{B}^0 \to X_s \gamma) + \Gamma (\bar{B}^- \to X_s \gamma)}.
\] (15)

Finally, we note that a non-perturbative uncertainty appears also when extrapolating the three different measurements performed at CLEO, BaBar and Belle down to the common lower cut \(E_0 = 1.6\) GeV in the photon energy. It is accounted for in the error of the world average given in (12).

4. Conclusions

At present the uncertainties in the branching ratio of \(\bar{B} \to X_s \gamma\) are on the same level on both the theoretical and experimental side. Thanks to the ongoing calculations of the perturbative corrections, the uncertainty stemming from this part will further reduce. However, to reach the 5% level or even less on the theoretical side, a better understanding of the non-perturbative power-corrections at \(O(\alpha_s \Lambda/m_b)\) is required.

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