RELATIVISTIC HYDRODYNAMICS OF ROTATING FLUID MASSES MOVING WITH THE VELOCITY OF LIGHT

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(Received 13, October, 1959)

Wir verallgemeinern eine ursprünglich von Weyssenhoff vorgeschlagene Methode und beobachten das Verhalten von relativistischen flüssigen Tropfen bei Geschwindigkeiten, die derjenigen des Lichtes gleich sind (oder nahe kommen). Es folgt daraus, dass für $v = c$ neue qualitative Eigentümlichkeiten auftauchen und dass Tensor- Grössen durch Pseudotensoren, im Sinne Weyssenhoffs ersetzt werden. Als Beispiel geben wir die vollkommene, aus Tröpfchen bestehende Flüssigkeit mit einer nicht nennenswerten Restmasse und zeigen dass, wenn wir sie mit einer einfachen Verbindung zusammenbringen, die daraus entstehende Flüssigkeit genau der vor kurzem von Lee und Yang vorgeschlagenen, zweikomponenten Neutrinotheorie entspricht.

Introduction

In two recent papers the general theory of relativistic fluid masses was treated from two different points of view. In the first, Bohm[1] and one of us (J.-P. V.) have established directly the physical signification of the well-known Weyssenhoff equations of motion for such bodies. With the help of the new concept of center of matter density (which corresponds to the classical notion of geometrical body center in non relativistic mechanics) it can be shown that these equations represent the relative behaviour of this new center with respect to the usual center of mass. In the second paper[2] we gave (in collaboration with F. Halbwachs) a Lagrangian formulation of these equations with the help of a set of kinematical variables describing the rotational orientation of the element of the fluid which we call Einstein-Kramers variables. The utilization of these variables is valuable for various reasons which we have exposed elsewhere.[3]

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[1] D. Bohm and J.-P. Vigier, *Phys. Rev.*, **109**, 1882, (1958).

[2] P. Hillion, F. Halbwachs and J.-P. Vigier, Lagrangian formalism in relativistic hydrodynamics of rotating fluid masses published in *Nuovo Cimento,* **10**, 817 (1958).

[3] See ref. 2 and also F. Halbwachs, Thesis (Paris) to be Published at Gautier-Villars, (1959) under the title „ Recherches sur la dynamique du corpuscule tournant relativiste et l'hydrodynamique des fluides à spin".

In this paper, however, we want to study a special case of the latter formalism, namely the physical behaviour of rotating fluid masses for velocities approaching the velocity of light.

This problem is interesting for three main reasons. In the first place, the study of the classical behaviour of extended bodies at such velocities might give useful indications on the behaviour of matter in the high energy domain: in particular for temperatures corresponding to such velocities. This should eventually be applied to the theory of collective excitation in high energy plasmas, and to the so-called cosmic "jets".

Secondly, it might pave the way for a quantization of classical extended bodies when $v = c$.

Lastly, as we shall see, a perfect continuous fluid of a special type of relativistic fluid droplets with vanishing rest mass connected by simple tensions satisfies exactly the laws of motion of a two component neutrino field.

We wish to stress, however, that independently of these applications the question of the behaviour of particles extended in space for velocities approaching the velocity of light is interesting as such. So far as we know it has never been treated except in a qualitative way.[4] This is only natural as long as one remains attached to the basis idea that particles are point-like and devoid of internal structure.

Evidently in that case all that is needed classically is the description of the particle world-lines. On the contrary if the particles can be described by an energy momentum density $T_{\mu\nu}$, and current density j_μ enclosed within a time-like tube it is clear that they will be modified by external motion. For example, the particles shape flattens perpendicularly to the velocity as a consequence of the Lorentz contraction and, at the limit $v = c$, is reduced to a plane section. For high acceleration its various points contract at different rates thus modifying the internal distribution of matter and energy.

In this paper, we shall show that the kinematical variables introduced by Einstein and Kramers (which we have used to describe the internal states of the relativistic fluid masses) provide a convenient way to treat this problem mathematically. In section 1 we shall give a brief summary of the Lagrangian formalism in terms of the E. K. variables. In section 2 we shall see what happens to these variables when the velocity is increased up to the velocity of light. In section 3 finally, according to the preceding discussion, we shall study a particular case of relativistic fluid droplets which when accelerated up to the velocity of light and coupled in a simple way, behave like neutrinos satisfying the two component spinor theory recently proposed by Lee and Yang.[5]

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[4] With the exception of an attempt by Weyssenhoff, *Acta phys. Polon.*, **9**, 7 (1947).

[5] T. D. Lee and C. N. Yang, *Phys. Rev.*, **105**, 1671, (1957).
A. Salam, *Nuovo Cimento*, **5**, 299 (1957).
L. Landau, *Nuclear Physics*, **3**, 127 (1957).

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§ 1.

As is well known the study of extended relativistic fluid droplets rests on the idea that one can introduce integral quantities which correspond to average or global properties of the body. In this way the study of the internal complex motions can be left aside and replaced in the first approximation, by the laws of motion which govern these global quantities.

We thus start[6] from the basic assumption that the particle is comparable to a fluid mass with conserved energy momentum density $T_{\mu\nu}$ and current density j_μ which vanish outside of a time-like tube defining the physical limit of the droplet. One then defines immediately six physically important global quantities, namely:
— The total energy momentum vector

$$G_\mu = \frac{1}{ic}\int_{\Sigma} T_{\mu 4}\, dv$$

(where dv denotes a three dimensional volume element of a plane section Σ) which determines the inertial Lorentz frame Π_0 (in which $G_i = 0$).
— The centers X and Y of mass and matter density defined in Π_0 by the relations

$$X_k^o \{\int_{\Sigma_0} T_{44}^0\, dv_0\} = \int_{\Sigma_0} x_k^0\, T_{44}^0\, dv^0;$$

and

$$Y_k^0 \{\int_{\Sigma_0} j_4^0\, dv^0\} = \int_{\Sigma_0} x_k^0 j_4^0\, dv^0;$$

the latter moving along a world-line with the unitary tangent four velocity v_μ. The index $_0$ denotes the fact that the quantities are evaluated in the frame Π_0.
— The angular momentum $M_{\mu\nu}$ with respect to the center of matter density

$$M_{\mu\nu} = \frac{1}{ic}\int_{\Sigma_0} [(x_\mu - Y_\mu)\, T_{\nu_4} - (x_\nu - Y_\nu)\, T_{\mu_4}]\, dv$$

Now, if we denote by $\dot{A}$ the derivation of a quantity A along the world-line followed by the center of matter density ($\dot{A} = v_\lambda\, \partial_\lambda\, A$) we can establish the fundamental laws of motion

$$\dot{G}_\mu = 0$$

$$\dot{M}_{\mu\nu} = G_\mu\, v_\nu - G_\nu\, v_\mu \tag{1}$$

[6] See references (1). From now on we shall use the same notations. Greek indices μ vary from one to four and denote the usual tensor indices — latin indices vary from one to three and denote space — like components — indices repeated twice are summed over all possible values — We work in Minkowski's space time.

which combined with a Weyssenhoff-like condition (such as $M_{\mu\nu} v_\nu = 0$) determine completely the behaviour of the global quantities attached to the droplet. If instead of a single droplet one considers a perfect macroscopic fluid[7] made of such droplets one can define, instead of the preceding tensors, density relations obtained by multiplying (1) by the scalar density ϱ and adding to them the relation

$$\dot{\varrho} = \partial_\mu (\varrho v_\mu) = 0$$

As was pointed out elsewhere, this procedure has the advantage of introducing into the formalism the mathematical methods of field theory and involves no disadvantage since the droplets are assumed not to interact.

The passage to Lagrangian formalism rests on the introduction of the Einstein-Kramers variables. They consist of an orthogonal set of four unitary vectors $a_\mu^{(\xi)}$ (The index ξ varies also from one to four though it does not mean a vector index) one of which $a_\mu^{(4)}$ is identical to v_μ. This frame is attached to the center of matter density of each droplet which constitutes the macroscopic fluid, and its rotation represents the instantaneous rotation of the droplet through the relations

$$\omega_{\mu\nu} = \tfrac{1}{2} (\dot{a}_\mu^{(\xi)} a_\nu^{(\xi)} - a_\mu^{(\xi)} \dot{a}_\nu^{(\xi)})$$

which determine the angular four velocity.

We are now in a position to demonstrate the following general theorem[8]. If in the first approximation $M_{\mu\nu}$ is a general function of the $a_\mu^{(\xi)}$ and not of their derivatives, the equations of motion (1) are just the conservation equations associated with the Lagrangian

$$L = \tfrac{1}{2} \varrho M_{\alpha\beta} \omega_{\alpha\beta} + \varrho m_0 c^2 + ic\varrho a_\mu^4 \partial_\mu S + \lambda_{\mu\nu} (a_\mu^{(\xi)} a_\nu^{(\xi)} - \delta_{\mu\nu})$$

The physical signification of each term is clear. the first term $\frac{1}{2} \varrho M_{\alpha\beta} \omega_{\alpha\beta}$ corresponds to the rotation energy, the second $\varrho m_0 c^2$ to the rest mass energy, the third to the conservation of current and the last to the orthogonolity and unitary conditions of the $a_\mu^{(\xi)}$ variables.

The demonstration of this theorem is simple. It has been established by Belinfante and Rosenfeld that the canonical energy momentum tensor deduced from a Lagrangian L of variables q_λ namely: $t_{\mu\nu} = \dfrac{\partial L}{\partial(\partial_\nu q_\lambda)} \partial_\mu q_\nu - \delta_{\mu\nu} L$

is conserved as a consequence of the field equations, namely

$$\partial_\nu t_{\mu\nu} = 0 \tag{2}$$

If we can write $\varrho M_{\alpha\beta}$ as $m_{\alpha\beta}$ where $m_{\alpha\beta} (a_\mu^{(\xi)})$ is the average angular momentum of the droplets which constitute the macroscopic fluid at a given point (so that the

[7] In hydrodynamics this means a fluid without sources or sinks made of such elements without interaction; the center of matter density of each droplet following a given line of flow.

[8] Which generalizes the results of reference (2).

total Lagrangian is just the Lagrangian of the individual droplets $\frac{1}{2} M_{\alpha\beta}\,\omega_{\alpha\beta} + m_0 c^2 + \partial_{\mu\nu}\,(a_\mu^{(\xi)}\,a_\nu^{(\xi)} - \delta_{\mu\nu})$ multiplied by ϱ plus the Lagrange conservation condition $ic\varrho\, a_\mu^{(4)}\,\partial_\mu S$ we get by variation with respect to ϱ and using $a_\mu^{(\xi)}\,a_\nu^{(\xi)} = \delta_{\mu\nu}$:

$$L = 0\,.$$

As a consequence the energy momentum tensor $t_{\mu\nu}$ becomes:

$$t_{\mu\nu} = g_\mu\, v_\nu$$

with $g_\mu = \varrho\,(\partial_\mu S + \frac{1}{2}\, m_{\mu\nu}\, a_\alpha^{(\xi)}\,\partial_\mu\, a_\beta^{(\xi)}) = \varrho\, G_\mu$ and the corresponding conservation relation $\partial_\nu\, t_{\mu\nu} = 0$ can be written:

$$\dot{g}_\mu = 0 \tag{3}$$

In the case of density quantities, the dot implies, as it is well-known, the derivative along the local current line, that is: $\dot{g}_\mu = \partial_\nu\,(v_\nu\, g_\mu)$

The next step is to write the second Rosenfeld-Belinfante conservation relation:

$$\partial_\lambda f_{\mu\nu\lambda} = \tfrac{1}{2}\,(t_{\mu\nu} - t_{\nu\mu}) \tag{4}$$

where

$$f_{[\mu\nu]\lambda} = \mathfrak{T}^{rs}_{[\mu\nu]}\,\frac{\partial L}{\partial\,(\partial_\lambda\, q_r)}\, q^s$$

$\mathfrak{T}$ representing the infinitesimal Lorentz transform on the variables $a_\mu^{(\xi)}$ that is

$$\mathfrak{T}^{\alpha\beta}_{\mu\nu} = \tfrac{1}{2}\,(\delta_{\alpha\mu}\,\delta_{\beta\nu} - \delta_{\alpha\nu}\,\delta_{\beta\mu})$$

This gives immediately as a consequence of our assumptions

$$f_{\mu\nu\lambda} = \frac{ic}{2}\, m_{\mu\nu}\, a_\lambda^{(4)}$$

and relation (4) becomes

$$\varrho\,\dot{M}_{\mu\nu} \equiv \dot{m}_{\nu\mu} = g_\mu\, v_\nu - g_\nu\, v_\mu \tag{5}$$

(with $v_\nu = ica_\nu^{(4)}$, $h_0^2 = |M_{\alpha\beta}|$ and $\varrho\, M_{\alpha\beta} = m_{\alpha\beta}$). This demonstrates our theorem since relations (2) and (3) are evidently identical with equations (1).

We notice further that any given form of $m_{\alpha\beta}$ will then correspond to a particular type of internal motion. For example one could start from a Lagrangian satisfying Weyssenhoff's condition

$$L = \varrho\, m_0 c^2 + ic\varrho\, a_\mu^{(4)}\partial_\mu S + ic\varrho\, h_0\, a_\mu^{(4)}\, a_\lambda^{(1)}\,\partial_\mu\, a_\lambda^{(2)} + \varrho\,\lambda_{\mu\nu}\,(a_\mu^{(\xi)} a_\nu^{(\xi)} - \delta_{\mu\nu}) \tag{6}$$

which we shall utilize later. In this Lagrangian:

$$a_\mu^{(4)} = v_{\mu/ic}; \qquad a_\mu^{(3)} = \frac{i}{2c\varphi h^0}\,\varepsilon_{\mu\nu\alpha\beta}\, v_\nu\, m_{\alpha\beta} \tag{7}$$

$$a_\mu^{(1)} = \varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^{(3)}\, a_\beta^{(4)}\, a_\nu^{(2)}; \qquad a_\mu^{(2)} = -\,\varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^{(3)}\, a_\beta^{(4)}\, a_\nu^{(1)}$$

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§ 2

According to our program the next step is to see what happens to this formalism for velocities approaching the velocity of light. This could be attempted in various ways. The first and most straight forward is just to accelerate the particle in a given direction and apply the usual corresponding Lorentz formulas to all tensor quantities. We have attempted that but it leads to very complex calculations which finally give the same results as the more simple and powerful method which we shall now develop; so we shall not discuss it in this paper.

This other method rests on a very simple remark made by Weyssenhoff. If one follows the world-line of any given particle (in our case the motion of the center of matter density), the most natural parameter is the propertime τ on that line ($-c^2 d\tau^2 = dx_\alpha\, dx_\alpha$). Now, if that line becomes tangent to the light cone $d\tau = 0$ and τ loses all physical significance. This shows that relations (1) which depend on τ must be transformed for $v = c$ and that τ is not a suitable parameter for such velocities. Accordingly Weyssenhoff proposed to drop τ as parameter and to replace it by another parameter p on the world-line, such that p will still flow even when it becomes tangent to the light cone. The only restriction on p is to impose upon it the restriction

$$\tau' \equiv \frac{d\tau}{dp} \geqslant 0$$

so that the condition $\tau' \to 0$ implies $d\tau = 0$ which means the particle moves with the velocity c. One can then define the world-line followed by the center of matter density not by the unit vector:

$$ic\, a_\alpha^{(4)} = \frac{dx_\alpha}{d\tau},$$

but by another vector:

$$w_\alpha = \frac{dx_\alpha}{dp}$$

so that:

$$w_\alpha = ic\,\tau'\, a_\alpha^{(4)}$$

and

$$w_\alpha\, w_\alpha = -\, c^2\, \tau'^2$$

If $\tau' \to 0$ we then get

$$w_\alpha\, w_\alpha = 0$$

and the world-line becomes tangent to the light cone. In order to avoid confusion it is necessary to stress the fact that w_α is not an ordinary four vector but a four vector

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depending on the parametrization; its four components transform like components of a four vector when the coordinates are transformed without change of parametrization, but they are all multiplied by a common factor $dp/d\bar{p}$ when the parametrization is changed from p to $\bar{p}$.

As Weyssenhoff remarked it is even possible in a given frame to take as parameter the ordinary time $t = x_4/c$. In this case we change the parametrization for every change of coordinate system in such a way that p remains equal to t in the coordinate system of interest. Then w_α becomes $u_\alpha = x_\alpha/dt$ with the components[9] $(\vec{u}, ic)$. It is then evident that the four quantities u_α do not form a four vector neither in the ordinary sense, nor a four vector depending on the parametrization (this would only be the case if p was a scalar parameter) but form instead a new geometrical object which we can call "pseudo-vector". The components of such a pseudo-vector transform like the components of vector but with an additional multiplication factor $dt/d\bar{t} = dx^4/d\bar{x}_4$ when we pass from the frame Σ to a frame $\overline{\Sigma}$. The magnitudes of a pseudo-vector $(u_\alpha u_\alpha)^{1/2}$, or any scalar product such as $(G_\alpha u_\alpha)^{1/2}$ behave like "pseudo-scalars" and are also multiplied by $dt/d\bar{t}$ in the case of a Lorentz transformation. As a consequence, the pseudo-vector u_α has, in all possible frames, the ordinary velocity $\vec{v}$ as space components and c as time-component. In the limit $v = c$ one gets evidently:

$$u_\alpha u_\alpha = 0.$$

Weyssenhoff's method for the passage to velocities approaching the velocity of light is now clear. In any equation one replaces derivatives with respect to the proper time τ by the expression:

$$\frac{d}{d\tau} = \frac{dt}{d\tau} \cdot \frac{d}{dt}$$

and pass to the limit by pushing τ' to zero. More generally one can write laws of motion in terms of any parameter p, satisfying the condition $\tau' \equiv d\tau/dp \geqslant 0$ and pass to the limit $\tau' = 0$.

To illustrate this general method, let us treat briefly the problem of the limit of Weyssenhoff's equations for a particle moving with the velocity of light.

Let us choose as parameter the ordinary time t $\left(\tau' = \dfrac{d\tau}{dt}\right)$ and denote by f' the time derivative of a given quantity f so that:

$$f' = \frac{df}{d\tau} = \partial_4 f + u_k \partial_k f \quad (k = 1,\ 2,\ 3)$$

If we now take for example the relation

$$\dot{G}_\mu = 0$$

[9] The $\vec{u}$ designs a three vector.

it can be written $\frac{dt}{d\tau}\frac{d}{dt}G_\mu = \frac{1}{\tau'}\frac{d}{dt}G_\mu = 0$

When $\tau' \to 0$ this implies

$$G'_\mu = 0.$$

Applying the same reasoning to the other equations (1) one obtains the following laws of motion:

$$G'_\alpha = 0$$
$$M'_{\alpha\beta} = G_\alpha u_\beta - G_\beta u_\alpha$$
$$M_{\alpha\beta} u_\beta = 0$$

If we start instead from $G_\mu = m_o v_\mu + \frac{1}{c^2} M_{\mu\nu} \dot{v}$ and multiply both members by τ'^2 we get:

$$G_\mu \tau'^2 = m_0 u_\mu \tau' + \frac{1}{c^2} M_{\mu\nu} u'_\nu .$$

The left hand side goes to zero with τ' and multiplying by the matter density ϱ we find for $v \to c$

$$\mu'_0 u_\mu + \frac{1}{c^2} m_{\mu\nu} u'_\nu = 0$$

with

$$\mu'_0 = \frac{1}{c^2} g_\mu u_\mu = \varrho\, m_0 \tau' .$$

This implies that we introduce a density μ'_0 of proper mass which goes to zero with τ' since it would take an infinite energy to push a non zero rest mass particle to the velocity of light.

As a consequence the equations of motion for a fluid of Weyssenhoff particles can be written:

$$\varrho' = 0 \qquad g'_\mu = 0 \qquad m'_{\mu\nu} = g_\mu u_\nu - g_\nu u_\mu$$
$$m_{\mu\nu} u_\nu = 0 \qquad g_\mu u_\mu = 0 \qquad m_{\mu\nu} u'_\nu = 0.$$

Of course, these relations are not independent and reduce to the set:

$$\begin{array}{llll} G'_\mu = 0 & \text{(8a)} & M'_{\mu\nu} = G_\mu u_\nu - G_\nu u_\mu & \text{(8d)} \\ G_\mu u_\mu = 0 & \text{(8b)} & M_{\mu\nu} u_\mu = 0 & \text{(8e)} \\ \varrho' = 0 & \text{(8c)} & & \end{array} \qquad (8)$$

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However, this procedure of Weyssenhoff is rather cumbersome since it implies going to the limit of every equation of motion separately and in more complex cases, generalizing his simple model[10], it proves often very difficult.

In order to avoid any trouble we therefore propose to generalize Weyssenhoff's idea in the following way:

If we utilize a Lagrangian formalism in terms of the E. K. variables the behaviour of particles for $v \neq c$ is clearly determined by the corresponding Euler equations. As a consequence if we can determine the limit of these variables we can determine the corresponding limit of L for $v \to c$ and deduce therefrom the corresponding laws of motion, which are just the limit of Euler's equations.

This reduces our problem to two steps:

a. Determine the limit of the E. K. variables for $v \to c$;

b. Write the corresponding Lagrangian.

The determination of the limit of the E. K. variables can be worked out with the help of Weyssenhoff's pseudo-vectors when $p = t$. If we recall the definitions (7) we can write instead of $a_\mu^{(4)}$ the vector $\alpha_\mu^{(4)} = u_\mu/ic$ that is

$$\alpha_\mu^{(4)} = \tau'\, a_\mu^{(4)}$$

so that $\alpha_\mu^{(4)}\,\alpha_\mu^{(4)} = 0$ for an isotropic world-line ($\tau = 0$).

This relation can be written:

$$\alpha_\mu^{(4)}\,\alpha_\mu^{(4)} = \alpha_k^{(4)}\,\alpha_k^{(4)} + \alpha_4^{(4)}\,\alpha_4^{(4)} = 0 \qquad (k = 1, 2, 3) \tag{9}$$

Let us now put $p = t$ and work out the limit in that case.

We have $u_\mu = \{\vec{u}, ic\}$ $\alpha_4^{(4)} = 1$ and we can replace $\alpha_\mu^{(4)}$ by $b_\mu^{(4)} = \{b_k^{(4)}, 1\}$ with the relation $b_i^{(4)}\,b_i^{(4)} = -1$

In the same way, instead of $a_\mu^{(3)} = \dfrac{i}{2c\,\varrho\, h_0}\, t_{\mu\nu\alpha\beta}\, v_\nu\, m_{\alpha\beta}$

we can write:

$$\alpha_\mu^{(3)} = \frac{i}{2\varrho\, ch_0}\, t_{\mu\nu\alpha\beta}\, u_\nu\, m_{\alpha\beta}$$

that is:

$$\alpha_\mu^{(3)} = \tau'\, a_\mu^{(3)} \quad \text{with } \alpha_\mu^{(3)}\,\alpha_\mu^{(3)} = 0. \tag{10}$$

As we have:

$$a_\mu^{(3)}\,a_\mu^{(4)} = 0$$

we can also write:

$$\alpha_\mu^{(3)}\,\alpha_\mu^{(4)} = 0. \tag{11}$$

[10] Considered for example in ref. 2 and 3.

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The three relations (9) (10) (11) evidently imply that

$$\alpha_\mu^{(3)} = \lambda\, \alpha_\mu^{(4)} \tag{12}$$

where λ is a constant.

This results immediately from the fact that

$$i\,\alpha_4^{(4)} = \sqrt{\sum_i (\alpha_i^{(4)})^2}\,, \quad i\,\alpha_4^{(3)} = \sqrt{\sum_j (\alpha_j^{(3)})^2}$$

and

$$-\,\alpha_\mu^{(4)}\,\alpha_\mu^{(3)} = \sqrt{\sum_i (\alpha_i^{(4)})^2}\sqrt{\sum_j (\alpha_j^{(3)})^2} - \alpha_k^{(4)}\,\alpha_k^{(3)} > 0$$

which results immediately from Schwartz' inequality. As one knows the equality sign is only possible when relation (12) is satisfied.

As we shall see later this relation implies that the spin is either parallel or antiparallel to the velocity, when $v \to c$.

Let us now consider the limit of $\alpha_\mu^{(1)}$ and $\alpha_\mu^{(2)}$. These limits, which we shall call $b_\mu^{(1)}$ and $b_\mu^{(2)}$ satisfy evidently the relations

$$b_\mu^{(1)}\,b_\mu^{(4)} = 0 \qquad b_\mu^{(1)}\,b_\mu^{(1)} = 1 \tag{13}$$

$$b_\mu^{(1)}\,b_\mu^{(4)} = 0 \qquad b_\mu^{(2)}\,b_\mu^{(2)} = 1 \tag{14}$$

$$b_\mu^{(1)}\,b_\mu^{(2)} = 0 \qquad b_\mu^{(4)}\,b_\mu^{(4)} = 0. \tag{15}$$

Let us denote by β_k^r $(r = 1, 2, 4)$ $(k = 1, 2, 3)$ the space components of the quantities $\alpha_\mu^{(r)}$. The relations (13), (14), (15) become for $v \to c$

$$\begin{aligned}
\beta_k^{(1)}\,\beta_k^{(4)} &= b_4^{(1)}\,b_4^{(4)} = b_4^{(1)} \\
\beta_k^{(2)}\,\beta_k^{(4)} &= b_4^{(2)}\,b_4^{(4)} = b_4^{(2)} \\
\beta_k^{(1)}\,\beta_k^{(2)} &= b_4^{(1)}\,b_4^{(2)} \\
\beta_k^{(1)}\,\beta_k^{(1)} &= 1 - (b_4^{(1)})^2 \\
\beta_k^{(2)}\,\beta_k^{(2)} &= 1 - (b_4^{(2)})^2
\end{aligned} \tag{16}$$

where the first member is the three dimensional scalar product. These five relations are equivalent to four independent relations. We still need four relations to determine $b_\mu^{(1)}$ and $b_\mu^{(1)}$ completely. To get them let us pass to the limit for $v \to c$ of the definition of $a_\mu^{(3)}$. We have:

$$a_\mu^{(3)} = \varepsilon_{\mu\nu\alpha\beta}\, a_\nu^{(4)}\, a_\alpha^{(1)}\, a_\beta^{(2)}.$$

The time component of $a_\mu^{(3)}$ becomes:

$$a_4^{(3)} = \varepsilon_{4ijk}\, a_i^{(4)}\, a_j^{(1)}\, a_k^{(2)}.$$

Multiplying both membere by τ' and taking the limit we get

$$b_4^{(3)} = \varepsilon_{4ijk}\, b_i^{(4)}\, b_j^{(1)}\, b_k^{(2)}$$

$$\beta_4^{(3)} = \vec{\beta}^{(4)} \cdot (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)})$$

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where the symbol $\wedge$ denotes the usual three dimensional vector product[11]. As $b_4^{(3)} = \lambda\, b_4^{(4)}$ with $\lambda =$ constant.

Then:

$$\lambda = \vec{\beta}^{(4)} (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)}) \tag{17}$$

The space components of $a^{(3)}$ become

$$\beta_i^{(3)} = b_i^{(3)} = \varepsilon_{i4jk}\, b_4^{(4)}\, b_j^{(1)}\, b_k^{(2)} + \varepsilon_{ij4k}\, b_j^{(4)}\, b_4^{(1)}\, b_k^{(2)} + \varepsilon_{ijk4}\, b_j^{(4)}\, b_k^{(1)}\, b_4^{(2)}$$

$$= b_4^{(4)} (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)}) + b_4^{(2)} (\vec{\beta}^{(4)} \wedge \vec{\beta}^{(1)}) + b_4^{(1)} (\vec{\beta}^{(2)} \wedge \vec{\beta}^{(4)}).$$

Multiplying these relations by τ' and going to the limit we get the vector relation (where $b_4^{(r)}$ are numbers):

$$\vec{\beta}^{(3)} = b_4^{(4)} (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)}) + b_4^{(2)} (\vec{\beta}^{(4)} \wedge \vec{\beta}^{(1)}) + b_4^{(1)} (\vec{\beta}^{(2)} \wedge \vec{\beta}^{(4)})$$

replacing $\vec{\beta}^{(3)}$ by $\lambda\vec{\beta}^{(4)}$ and noting $\beta_4^{(4)} = 1$, we find:

$$\lambda\vec{\beta}^{(4)} = \vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)} + \vec{\beta}^{(4)} \wedge (b_4^{(2)}\, \vec{\beta}^{(1)} - b_4^{(1)}\, \vec{\beta}^{(2)}).$$

If we calculate now the product $\lambda\, (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(4)})$ we find

$$\lambda\, (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(4)}) = \vec{\beta}^{(1)} \wedge \vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)} + \vec{\beta}^{(1)} \wedge [\vec{\beta}^{(4)} \wedge (b_4^{(2)}\, \vec{\beta}^{(1)} - b_4^{(1)}\, \vec{\beta}^{(2)})]$$

taking into account the well-known vector relation

$$\vec{A} \wedge \vec{B} \wedge \vec{C} = (\bar{A} \cdot \bar{C})\, \vec{B} - (\bar{A} \cdot \bar{B})\, \vec{C}$$

the preceding relation becomes

$$\lambda\, (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(4)}) = (\vec{\beta}^{(1)} \cdot \vec{\beta}^{(2)})\, \vec{\beta}^{(1)} - (\vec{\beta}^{(1)} \cdot \vec{\beta}^{(1)})\, \vec{\beta}^{(2)} +$$
$$+ [b_4^{(2)}\, (\vec{\beta}^{(1)} \cdot \vec{\beta}^{(1)}) - b_4^{(1)}\, (\vec{\beta}^{(2)} \cdot \vec{\beta}^{(1)})]\, \vec{\beta}^{(4)} -$$
$$- (\vec{\beta}^{(1)} \cdot \vec{\beta}^{(4)})\, (b_4^{(2)}\, \vec{\beta}^{(1)} - b_4^{(1)}\, \vec{\beta}^{(2)}).$$

Taking into account relations (16) this equation can be written

$$\lambda\, (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(4)}) = b_4^{(1)}\, b_4^{(2)}\, \vec{\beta}^{(1)} - \vec{\beta}^{(2)} - (b_4^{(1)})^2\, \vec{\beta}^{(2)} + [b_4^{(2)} + (b_4^{(1)})^2\, b_4^{(2)} -$$
$$- (b_4^{(1)})^2\, b_4^{(2)}]\, \vec{\beta}^{(4)} - b_4^{(1)}\, (b_4^{(2)}\, \vec{\beta}^{(1)} - b_4^{(1)}\, \vec{\beta}^{(2)}) \qquad \lambda\, (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(4)}) = - \vec{\beta}^{(2)} + b_4^{(2)}\, \vec{\beta}^{(4)}$$

so that

$$b_4^{(2)}\, \vec{\beta}^{(4)} = \vec{\beta}^{(2)} - \lambda\, (\vec{\beta}^{(4)} \wedge \vec{\beta}^{(1)}) \tag{18}$$

If we eliminate $\beta^{(4)}$ between (17) and (15) we get

$$b_4^{(2)}\, \lambda = [\vec{\beta}^{(2)} - \lambda\, (\vec{\beta}^{(4)} \wedge \vec{\beta}^{(1)})]\, (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(2)})$$

as

$$\vec{\beta}^{(2)} \cdot (\vec{\beta}^{(1)} \wedge \vec{\beta}^{(1)}) = 0$$

[11] The dot denotes the three dimensional scalar product.

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then

$$b_4^{(2)} = -(\vec{\beta^{(4)}} \wedge \vec{\beta^{(1)}}) \cdot (\vec{\beta^{(1)}} \wedge \vec{\beta^{(2)}}).$$

Utilizing:

$$\vec{A} \cdot (\vec{B} \wedge \vec{C}) = \vec{C} \cdot (\vec{A} \wedge \vec{B})$$

we get

$$b_4^{(2)} = -\vec{\beta^{(2)}} (\vec{\beta^{(4)}} \wedge \vec{\beta^{(1)}} \wedge \vec{\beta^{(1)}}) = 0$$

and a similar calculation gives $b_4^{(1)} = 0$.

This essential result shows that for $v \to c$ the vectors go to system of 3 pseudo-vectors in the sense of Weyssenhoff. The relations

$$\begin{aligned} a_\mu^{(\xi)} a_\nu^{(\xi)} &= \delta_{\mu\nu} & \xi,\eta &= 1,\ 2,\ 3,\ 4 \\ a_\mu^{(\eta)} a_\mu^{(\xi)} &= \delta^{\eta\xi} & \mu,\nu &= 0,\ 1,\ 2,\ 3 \end{aligned}$$

valid for $v \neq c$ become for $v \to c$[12]:

$$\begin{aligned} b_i^{(r)} b_j^{(r)} &= \delta_{ij} & r,s &= 1,\ 2,\ 4 \\ b_i^{(r)} b_i^{(s)} &= \delta^{rs} & i,j &= 1,\ 2,\ 3 \end{aligned}$$

and the system $a_\mu^{(\xi)}$ tends towards the pseudo-vector system:

$$\begin{aligned} a_\mu^{(4)} &\to b_\mu^{(4)} = (\vec{\beta^{(4)}},\ 1) \\ a_\mu^{(3)} &\to \lambda\, b_\mu^{(4)} \\ a_\mu^{(1)} &\to b_\mu^{(1)} = (\vec{\beta^{(1)}},\ 0) \\ a_\mu^{(2)} &\to b_\mu^{(2)} = (\vec{\beta^{(2)}},\ 0). \end{aligned} \tag{19}$$

The next step before we determine the limit of L is to replace the fluid density ϱ which is equal to the ordinary density in the local rest frame of the fluid at a given point by a new quantity for the velocity of light. The new quantity must evidently also satisfy the limiting form of the conservation relations

$$\partial_\mu \varrho\,(a_\mu^4) = 0 \tag{20}$$

as

$$a_\mu^{(4)} = \frac{1}{\tau'}\, a_\mu^{(4)}$$

this becomes

$$\partial_\mu \left(\frac{\varrho}{\tau'}\, a_\mu^{(4)} \right) = 0$$

[12] These relations are similar to the equations which define a unitary orthogonal system of ordinary vectors in three dimensions; but here they apply to pseudo-vectors at the velocity of light which we can now treat as ordinary vector variables in our Lagrangian formalism.

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and if $\tau' \to 0$ we can assume that $\varrho/\tau' \to S_0 \neq 0$ so that the limit form of (20) is just

$$\partial_\mu (S_0 b_\mu^4) = S_0' = 0 \tag{21}$$

As S_0 depends of $t = x_4/ic$ the relation (21) must be a consequence of the equations of motion and cannot be assumed a priori as in the usual case with $v \neq c$. As we shall see this follows from our treatment where there will be no Lagrange supplementary condition implying the conservation of S_0.

As an example of step b. we are now in a position to pass to the limit of Weyssenhoff's Lagrangian (6). We define this limit as the sum of the limit of various terms of the initial Lagrangian and shall justify this procedure by showing that we get correctly the relations (8) which describe Weyssenhoff particles moving with the velocity of light.

Starting from (6)

$$L = \varrho m_0 c^2 + icp\, a_\mu^{(4)} \partial_\mu S + ic\varrho\, h_0\, a_\mu^{(4)} a_\lambda^{(1)} \partial_\mu a_\lambda^{(2)} + \lambda_{\mu\nu} (a_\mu^{(\xi)} a_\nu^{(\xi)} - \delta_{\mu\nu})$$

for $v \to c$, we have (with $m_0 = 0$):

$$\begin{aligned} \varrho &\to S_0 \tau' \\ \tau' a_\mu^{(4)} &\to b_\mu^{(4)} \\ a_\mu^{(1)} &\to b_\mu^{(1)} \\ a_\mu^{(2)} &\to b_\mu^{(2)} \\ a_\mu^{(\xi)} a_\nu^{(\xi)} - \delta_{\mu\nu} &\to b_i^{(r)} b_j^{(r)} - \delta_{ij} \end{aligned} \tag{22}$$

and, as here, according to a preceding remark concerning S_0 we do not need the Lagrange parameter $\partial_\mu S$ the equation (6) becomes at the limit $v \to c$.

$$L = ic\, S_0\, h_0\, b_\mu^4\, b_l^1\, \partial_\mu b_l^2 + \lambda_{ij} (b_i^r b_j^r - \delta_{ij}). \tag{23}$$

The deduction of relations (8) from the Lagrange function (23) is very simple. The canonical energy momentum tensor associated to L is just:

$$\begin{aligned} t_{\mu\nu} &= \frac{\partial L}{\partial (\partial_\nu b_l^r)} \partial_\mu b_l^{(r)} - \delta_{\mu\nu} L \\ &= ic\, h_0\, S_0\, b_\nu^{(4)} b_l^{(1)} \partial_\mu b_l^{(2)} \\ &= g_\mu u_\nu; \end{aligned}$$

with:

$$g_\mu = S_0 (ic\, h_0\, b_l^{(1)} \partial_\mu b_l^{(2)}).$$

The conservation equation $\partial_\nu t_{\mu\nu} = 0$ is then just (5)

$$g_\mu' = 0$$

Remarking now that Euler's equation with respect to the variable S_0 is $\partial L/\partial S_0 = 0$ we find

$$g_\mu u_\mu = 0$$

that is precisely (5).

The relation $S_0' = 0$ results immediately from a simple combination of Euler's equations, derived from the preceding Lagrangian.

Variation of (23) with respect to $b_l^{(1)}$ gives:

$$h_0 c S_0 b_l^{(r)\prime} + 2\lambda_{lj} b_j^1 = 0 \tag{a}$$

Variation of (23) with respect to $b_l^{(2)}$ gives:

$$h_0 c b_l^{(1)} \partial_\mu (S_0 b_\mu^4) + h_0 c S_0 b_l^{(1)\prime} - 2\lambda_{lj} b_j^{(2)} = 0 \tag{b}$$

So if we multiply (a) by $b_l^{(2)}$, (b) by $b_l^{(1)}$ and add the result we find:

$$2\lambda_{lj} (b_j^1 b_l^? - b_j^2 b_l^1) + h_0 c S_0 (b_l^{(2)} b_l^{(2)\prime} + b_l^{(1)} b_l^{(1)\prime}) + h_0 c b_l^{(1)} b_l^{(1)} \partial_\mu (S_0 b_\mu^{(4)}) = 0$$

that is:

$$\partial_\mu (S_0 b_\mu^{(4)}) = 0;$$

since the first and second term vanish because $b_l^{(2)} b_l^{(2)\prime} = b_l^{(1)} b_l^{(1)\prime} = 0$ and λ_{ij} is a symetric tensor.

The last two relations $m_{\mu\nu} b_\nu^4 = 0$ and $m_{\mu\nu}' = g_\mu v_\nu - g_\nu u_\mu$ will be obtained from the Belinfante-Rosenfeld tensor $f_{\mu\nu\lambda}$.

Indeed utilizing definition (4) we find:

$$f_{\mu\nu\lambda} = ic\, h_0 S_0 (b_\mu^{(1)} b_\nu^{(2)} - b_\mu^{(2)} b_\nu^{(1)}) b_\lambda^{(4)} = m_{\mu\nu} b_\lambda^{(4)}$$

with

$$m_{\mu\nu} = ih_0 S_0 (b_\mu^{(1)} b_\nu^{(2)} - b_\mu^{(2)} b_\nu^{(1)});$$

so that relations

$$\partial_\lambda f_{\mu\nu\lambda} = t_{\mu\nu} - t_{\nu\mu}$$

give

$$m_{\mu\nu}' = g_\mu b_\nu^4 - g_\nu b_\mu^4.$$

The last condition $m_{\mu\nu} b_\nu^4 = 0$ results immediately from the orthogonality relations $\beta_i^r \beta_k^r = \delta_{ik}$ deduced from the variation of the $\lambda_{ik}'s$.

To finish this section we would like to make two remarks.

The first is that as a result of the Lorentz contraction Weyssenhoff particles moving with the velocity of light flatten into disc shaped forms in the plane orthogonal to the velocity. The spin is then either parallel or antiparallel to the velocity. Taking into account the fact that $\beta^1 \cdot \beta^2 = 0$ one sees immediately that the Lagrangian takes the simple form

$$L = -\frac{h_0}{2}\frac{S_0}{2}\left[b_k^{(1)} \frac{db_k^{(2)}}{dt} - b_k^{(2)} \frac{db_k^{(1)}}{dt}\right] + \lambda_{ij} (b_i^{(r)} b_j^{(r)} - \delta_{ij}).$$

If we remark that the first term in the Lagrangian $b_\mu^{(4)}\, b_l^{(4)}\, \partial_\mu\, b_l^{(2)}$ is just the scalar product of the angular velocity $\vec{\omega}$ on the four velocity $b_\mu^{(4)}$ we see that the equations of motion imply that this term vanishes so that the particle is reduced to a flat disc which does not rotate around its axis.

§ 3.

According to our program, the last section will include the comparison of the preceding perfect fluid constituted with Weyssenhoff particles with the hydrodynamical representation of the two component spinor theory of the neutrino[5 13] (recently proposed by Lee and Yang). As we shall see, this comparison shows that the two are identical when we introduce a simple type of tensions between those particles. We can thus demonstrate the following theorem:

Theorem

A perfect continuous fluid of Weyssenhoff particles with negligible rest mass which move with the velocity of light satisfies with suitable tensions the two component neutrino equation: the two component spinor density defining all tensor densities needed to characterize the fluids behaviour. To prove this result let us recall briefly certain mathematical elements[14] concerning two component spinors.

The first step is to establish a representation of 2 component spinors by tensor quantities. If one starts from a right handed coordinate frame a right handed neutrino field is represented by a right handed 2 component spinor φ^d (in all that follows spinors are considered as c numbers). Such a spinor transforms under a proper Lorentz transformation

$$x'_\mu = a_{\mu\nu}\, x_\nu \tag{1$_3$}$$

by

$$\varphi^d \rightarrow \varphi^{d'} = \Lambda^d\, \varphi^d \tag{2$_3$}$$

where Λ^d is a 2×2 matrix defined by the relation

$$(\Lambda^d)^+\, \sigma_\mu\, \Lambda^d = a_{\mu\nu}\, \sigma_\nu \tag{3$_3$}$$

where

$$\sigma_\mu = \{\sigma_k, \sigma_4\} \tag{4$_3$}$$

[13] Cf. Takabayasi, *Comptes-Rendus*, **246**, 1010 (1958).

[14] The transformation properties of the 2 component spinor and its associated bilinear quantities were developed in: Takabayasi, *Nucl. Physics*, **7**. 237 (1958) The general mathematical techniques to represent a two component spinor by tensor quantities is included in Takabayasi, *Prog. Th. Physics*, **14**, 283 (1955) and Takabayasi and Vigier, *Prog. Th. Physics*, **18**, 573 (1957) (especially in the Appendix).

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σ_k are just the Pauli matrices and $\sigma_4 = i\sigma_0$ is the so-called time Pauli matrix with

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

indices $* + T$ have here the usual signification and denote complex conjugation, hermitian conjugation and transposition.

In the case of the neutrino we know that φ obeys the wave equation:

$$\partial_0 \varphi^d + \sigma_k \partial_k \varphi^d = 0 \tag{5$_3$}$$

i. e.

$$\sigma_\mu \partial_\mu \varphi^d = 0.$$

The complex conjugate of (5) is:

$$\partial_\mu \varphi^{d+} \sigma_\mu = 0. \tag{6$_3$}$$

Similarly, a left-handed neutrino field φ^g is represented by a different 2 component spinor (left-handed spinor) φ^g which transforms, under the proper Lorentz transformation (1), like

$$\varphi^g \to \varphi^{g'} = \Lambda^g \varphi^g \tag{7$_3$}$$

where

$$\Lambda^{g+} \sigma'_\mu \Lambda^g = a_{\mu\nu} \sigma'_\nu$$

with

$$\sigma'_\mu = \{-\sigma_k, \sigma_4\} = -\sigma^+_\mu$$

Λ^g is related with Λ^d by:

$$\Lambda^g = (\Lambda^{d+})^{-1} \tag{8$_3$}$$

The wave equation satisfied by φ^g is:

$$\partial_0 \varphi^g - \sigma_k \partial_k \varphi^g = 0 \tag{9$_3$}$$

i. e.,

$$\sigma'_\mu \partial_\mu \varphi_g = 0. \tag{10$_3$}$$

If we consider a transformation

$$\varphi^d \to \varphi^{d\Gamma} = \Gamma \varphi^{d*} \tag{11$_3$}$$

where Γ is a 2-by-2 matrix defined by

$$\Gamma^T = -\Gamma, \quad \Gamma^+ \Gamma = 1; \quad \sigma_k^T = -\Gamma^+ \sigma_k \Gamma \tag{12$_3$}$$

$$\text{(note that } (\varphi^{d\Gamma})^+ = \varphi^{dT} \Gamma^+)$$

then we can show that $\varphi^{d\Gamma}$ is a left-handed spinor. So we call the transformation (11) a chirality-conjugation.

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Now, the only possible non-vanishing bilinear covariants formed with φ^d are the following (in so far as we do not use derivatives):

$$S_\mu^d = \varphi^{d+} \sigma_\mu \varphi^d \tag{13$_3$}$$

and

$$\xi_k = (\varphi^{d\Gamma})^+ \sigma_k \varphi^d = \varphi^{dT} \Gamma^+ \sigma_k \varphi^d \tag{14$_3$}$$

S_μ^d is a real 4-vector, while ξ_k describe 3 independent components of a self-dual tensor. In other words, writing real and imaginary parts of ξ_k as:

$$\lambda_k^{(1)} = \frac{1}{2} (\xi_k + \xi_k^*)$$

$$\lambda_k^{(2)} = \frac{1}{2i} (\xi_k - \xi_k^*) \tag{15$_3$}$$

we see that $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ constitute the space and time components of a real anti-symmetric tensor $\xi_{\mu\nu}$; i. e.,

$$\lambda_k^{(1)} = \xi_{ij}$$

$$i\,\lambda_k^{(2)} = \xi_{k4} \qquad (ijk) \sim (1,\ 2,\ 3) \tag{16$_3$}$$

Note that each of S_k^d, $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ transforms as a 3-dimensional vector for pure space rotation.

We can now prove that S_μ^d and ξ_k satisfy the following identical relations:

$$S_\mu^d S_\mu^d = 0 \tag{17$_3$}$$

$$\xi_k \xi_k = 0 \tag{18$_3$}$$

$$\xi_k S_k^d = 0. \tag{19$_3$}$$

It can immediately be shown[15] that the spinor φ^d is equivalently represented by the set of quantities

$$\{S_\mu^d,\ \xi_k\}$$

restricted by the conditions (17) — (19). Relations (17) — (19) are proved by using the formula concerning the components of Pauli matrices:

$$(\sigma_k)_{\alpha\beta} (\sigma_k)_{\alpha'\beta'} = 2\delta_{\alpha\beta'} \delta_{\alpha'\beta} - S_{\alpha\beta} S_{\alpha'\beta'}$$

(see ref. (4)), and also the properties of Γ matrice (12).

In terms of $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$, in place of ξ_k, equations (18) and (19) mean that $\lambda_k^{(1)}$, $\lambda_k^{(2)}$ and S_k^d are mutally orthogonal. Relation (17) can be rewritten as

$$(S_k^d)^2 = (S_o^d)^2 \qquad (S_0^d = S_4^d/i \geqslant 0)$$

[15] See reference (14).

Furthermore, we can prove that:

$$(\lambda_k^{(1)})^2 = (\lambda_k^{(2)})^2 = (S_0^d)^2$$

with

$$\begin{vmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} \\ \lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} \\ S_1^d & S_2^d & S_3^d \end{vmatrix} = (S_0^d)^3 \tag{20$_3$}$$

If we therefore introduce unit vectors $b_k^{(r)}$ by

$$b_k^{(r)} = \lambda_k^{(r)}/S_0^d \qquad \begin{matrix} (r = 1,\ 2,\ 3) \\ (\lambda_k^3 = S_k^d) \end{matrix} \tag{21$_3$}$$

$b_k^{(r)}$ $(r = 1,\ 2,\ 3)$ span a 3-dimensional right-handed orthogonal axes; thus:

$$b_k^r\, b_k^s = \delta_{rs} \qquad b_i^r\, b_k^r = \delta_{ik}\,. \tag{22$_3$}$$

This completes the first step of our demonstration: the original spinor φ^d is now equivalently represented by the set of variables:

$$\{\lambda_k^{(r)}\} \qquad (r = 1,\ 2,\ 3) \tag{23$_3$}$$

i. e., by

$$\{S_0^d,\ b_k^{(r)}\} \tag{24$_3$}$$

and it is now clear that any expression of φ^d must be representable explicitly in terms of (24). For instance bilinear quantities including 1-st order derivatives are represented as follows:

$$\varphi^{d+}\,\partial_\mu\,\varphi^d - \partial_\mu\,\varphi^{d+}\,\varphi^d = i\,S_o^d\,b_k^1\,\partial_\mu\,b_k^2 \tag{26$_3$}$$

and:

$$\varphi^{d+}\,\sigma_k\,\partial_\mu\,\varphi^d - \partial_\mu\,\varphi^{d+}\,\sigma_k\,\varphi^d = -\frac{i}{2}\,S_o^d\,e_{klm}\,b_l^r\,\partial_\mu\,b_m^r\,. \tag{27$_3$}$$

The second step is to show how the equation of motion $(5)_3$ is representable in terms of our variables $(24)_3$. First, we multiply $(5)_3$ by φ^{d+} from the left and multiply $(6)_3$ by φ^d from the right, then add or substract the resulting equations, then we obtain:

$$\partial_0\,(\varphi^{d+}\,\varphi^d) + \partial_k\,(\varphi^{d+}\,\sigma_k\,\varphi^d) = 0 \tag{30$_3$}$$

and

$$(\varphi^{d+}\,\partial_0\,\varphi^d - \partial_0\,\varphi^{d+}\,\varphi^d) + (\varphi^{d+}\,\sigma_k\,\partial_k\,\varphi^d - \partial_k\,\varphi^{d+}\,\sigma_k\,\varphi^d) = 0 \tag{31$_3$}$$

Equation (30) is nothing but

$$\partial_\mu\,S_\mu^d = 0$$

i. e.,

$$\partial_0\,S_o^d + \partial_k\,(S_o^d\,b_k^3) = 0. \tag{32$_3$}$$

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On the other hand, equation (31) is expressed by means of the reduction formulae (26) and (27), as

$$b_k^{(1)} \partial_0 b_k^{(2)} + \tfrac{1}{2} \varepsilon_{klm} b_k^{(r)} \partial_l b_m^{(r)} = 0 \qquad (33)_3$$

We then proceed in the following way: first we multiply (5) by σ_i to obtain

$$\sigma_i \partial_0 \varphi^d + \partial_i \varphi^d + i (\sigma_k \partial_j \varphi^d - \sigma_j \partial_k \varphi^d) = 0 \qquad (34)_3$$

Then we multiply (34) by φ^{d+} from the left, obtaining

$$\varphi^{d+} \sigma_i \partial_0 \varphi^d + \varphi^{d+} \partial_i \varphi^d + i (\varphi^{d+} \sigma_k \partial_j \varphi^d - \varphi^{d+} \sigma_j \partial_k \varphi^d) = 0 \qquad (34')_3$$

then add or substract (34′) with its complex conjugate. The results are:

$$\partial_0 (\varphi^{d+} \sigma_i \varphi^d) + \partial_i (\varphi^{d+} \varphi^d) + i \{(\varphi^{d+} \sigma_k \partial_j \varphi^d - \partial_j \varphi^{d+} \sigma_k \varphi^d) - [i,j]\} = 0 \quad (35)_3$$

and

$$(\varphi^{d+} \sigma_i \partial_0 \varphi^d - \partial_0 \varphi^{d+} \sigma_i \varphi^d) + (\varphi^{d+} \partial_i \varphi^d - \partial_i \varphi^{d+} \varphi^d) + i \{\partial_j (\varphi^{d+} \sigma_k \varphi^d) - (j,k)\} = 0 \qquad (36)_3$$

where the symbol (j, k) denotes the preceding term after permutation of the indices. By means of the reduction formulae $(26)_3$, $(27)_3$, these are re-written as

$$\partial_i S_o^d + \partial_o (S_o^d b_i^3) + S_o^d b_i^{(r)} \partial_l b_l^{(r)} = 0 \qquad (37)_3$$

and

$$b_k^{(1)} \partial_i b_k^{(2)} - \frac{1}{2} \varepsilon_{ijk} b_j^{(r)} \partial_0 b_k^{(r)} + \frac{1}{S_o^d} \varepsilon_{ijk} \partial_j (S_o^d b_k^3) = 0 \qquad (38)_3$$

The 6 real equations $(37)_3$ and $(38)_3$ are not all independent of $(32)_3$ and $(33)_3$. The relations, which are contained in $(37)_3$ and $(38)_3$ which are independent of $(32)_3$ and (33), are only the following:

$$\partial_k (S_o^d b_k^2) = S_o^d b_k^3 \partial_0 b_k^2 \qquad (39)_3$$

and

$$\partial_k (S_o^d b_k^1) = S_o^d b_k^{(3)} \partial_0 b_k^{(1)}. \qquad (40)_3$$

For instance, relation $(37)_3$ results from $(39)_3$ and $(40)_3$, by the procedure

$$(39)_3 \times b_i^{(2)} + (40)_3 \times b_i^{(1)}.$$

In this way we have obtained 4 independent real equations:

$$\{(32)_3, (33)_3, (39)_3, (40)_3\} \qquad (41)$$

which are equivalent to the original spinor equation $(5)_3$ (which also contains 4 real equations).

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The last part of our demonstration is just to remark that our equations of motion $(41)_3$ (together with the constraints $(22)_3$) result from the following Lagrangian density:

$$L = -\frac{\hbar c}{2} S_0^d (b_k^1 \partial_0 b_k^2 + \frac{1}{2} \varepsilon_{klm} b_k^{(r)} \partial_l b_m^{(r)}) + \lambda_{rs} (b_k^r b_k^s - \delta_{rs}) \qquad (42)_3$$

From this L, we can derive the energy-momentum tensor and intrinsic angular momentum tensor of the field by the normal Belinfante-Rosenfeld procedure.

In the spinor formalism, the Lagrangian is:

$$L = -\frac{\hbar c}{2i} (\varphi^{d+} \sigma_\mu \partial_\mu \varphi^d - \partial_\mu \varphi^{d+} \sigma_\mu \varphi^d) \cdot \qquad (43)_3$$

Using the reduction formulas $(26)_3$ and $(27)_3$, $(43)_3$ is rewritten, then we get just the first term in the right-hand side of (42). Adding $\partial_{rs} (b_k^r b_k^s - \delta_{rs})$, we could thus at once reach the form $(42)_3$. From this form the equations of motion $(41)_3$ can be immediately derived by variation.

The same procedure can be applied, of course, to the φ^g field[16].

The physical interpretation of the Lagrangian $(42)_3$ is clear if we take into account the vector identity:

$$b_l^1 b_k^3 \partial_k b_l^2 + \varepsilon_{klm} b_k^3 \partial_l b_m^3 = \tfrac{1}{2} \varepsilon_{klm} b_k^r \partial_l b_m^r$$

which results immediately from the orthogonality conditions on the b_l^r. This relation implies that if we denote by L_W and L_V the Lagrangians (10—2) and (42—3), we have:

$$L_V = L_W + \hbar_0 S_0^d \varepsilon_{klm} b_k^3 \partial_l b_m^3$$
$$= L_W + \Theta_{kk}$$

If we denote by $\Theta_{\mu\nu}$ the tensor $\Theta_{\mu\nu} = h_0 S_0^d \varepsilon_{k\mu l} b_k^3 \partial_r b_l^3$ that is the contribution of the Θ_{kk} term to the canonical energy momentum tensor.

We see immediately that such a term can be interpreted as a tension term binding together the Weyssenhoff droplets (moving with $v = c$), constituting the neutrino fluid. Indeed $\Theta_{\mu\nu}$ satisfies the well-known property of ordinary tensions namely

$$\Theta_{\mu\nu} b_\nu^3 = 0$$

The physical meaning of these tensions can be understood if we remark that the divergence ∂_λ of the term

$$f_{\mu\nu\lambda} = \mathfrak{T}_{\mu\nu}^{rs} \frac{\partial L}{\partial (\partial_\lambda q^r)} q^s$$

(namely the Belinfante-Rosenfeld angular momentum density) is zero.

[16] We want to remark here that it is not at all astonishing to find that our particles are described by a two component spinor with a definite chirality. As we have seen only two possibilities arise for the velocity of light: the spin is either parallel or antiparallel to the velocity corresponding to opposite chiralities. This was also pointed out by Wigner, *Rev. Mod. Phys.*, (1957).

Calling $f_{\mu\nu\lambda}$ the term corresponding to L_W we have then:

$$\tfrac{1}{2} S_0 \, m_{\mu\nu} = \partial_\lambda f_{\mu\nu\lambda} = \tfrac{1}{2} S_0 \, \varepsilon_{\mu\nu\alpha\beta} \, \partial_\alpha \, b^3_\beta = \tfrac{1}{2} (t_{\mu\nu} - t_{\nu\mu}) \qquad (44)_3$$

which is a known consequence of the neutrino equation.

Physically this means that the tension angular momentum is zero so that the orbital angular momentum $(x_\nu \, t_{\mu\lambda} - x_\mu \, t_{\nu\lambda})$ is just equal to the variation of the particles' angular momentum. Noting that $t_{\mu\nu} = g_\mu \, b^3_\nu + \Theta_{\mu\nu}$ this can also be written:

$$\dot{m}_{\mu\nu} = g_\mu \, b^3_\nu - g_\nu \, b^3_\mu + \tfrac{1}{2} (\Theta_{\mu\nu} - \Theta_{\nu\mu})$$

which means that the variation of the total angular momentum (orbital plus spin) is just equal to the torque exerted on any given droplet by the tensions of the neighbouring fluid. We do not propose to discuss here any specific physical model which would lead to such an interaction but remark, however, that it is certainly one of the simplest possible types. Any other type resulting from a general $\Theta_{\mu\nu}$ (function of the $\partial_l \, b^k_r$) would in general contribute to the $f_{\mu\nu\lambda}$ and $\partial_\nu f_{\mu\nu\lambda}$ introducing supplementary angular momentum.

As we said before, this result can be interpreted in the following way: Consider a continuous distribution of a two component spinor satisfying the neutrino's equation $(5)_3$. With the help of this spinor you can determine variables by the relations $(13)_3$ to $(15)_3$ which satisfy $(42)_3$ and therefore characterize completely a continuous fluid made of Weyssenhoff particles with negligible rest mass moving with the velocity of light. The lines of flow tangent to S^d_μ are followed by the center of matter density of the corresponding droplet and the three frame b^r_k which varies along that path determines completely the evolution of all physical properties (angular momentum, etc.), which characterize Weyssenhoff's motion for such velocities.

In other words, in this last section we have used a new set of parameters (φ^d) instead of the E. K. variables to describe the droplet's behaviour. A similar idea had already been proposed by two of us (T. T. and J.-P. V.) to describe the motion of Weyssenhoff's particles at ordinary velocities[17]. Only in that case one uses 4 component normalized Dirac spinors ψ and Dirac's matrices γ_μ. The correspondence with the E. K. variables a^ξ_μ is given by $a^4_\mu = \psi^+ \gamma_\mu \psi$; $a^3_\mu = \psi^+ \gamma_\mu \gamma_5 \psi$ and $a^1_\mu + i a^2_\mu = \tilde{\psi}^+ \gamma_\mu \psi$. This has also been proposed by Gürsey[18].

Conclusion

As we propose to discuss in detail in another paper some of the physical consequences of the preceding results we will limit here the discussion to a few remarks.

As we said before the first two solutions give a general method to treat the behaviour of any continuous energy-momentum and current distribution enclosed

[17] T. Takabayasi (unpublished). Unal and J. P. Vigier. *Comptes-Rendus*. **245**, 1787, 1891 (1957).

[18] Gürsey. *Nuovo Cimento* **5**, 784 (1957).

within a time like tube by using E. K. variables. This can be applied to many physical problems and an attempt to utilize this in the case of the beams of high energy accelerators is under progress at the Institute Henri Poincaré.

The second remark is that we have obtained in sections 2 and 3 a concrete completely clear relativistic model of the hydrodynamical representation of the neutrino equation. It includes a simple "quantum coupling" corresponding to "quantum potential" terms in the hydrodynamical representations of the other quantum equations. It shows that the neutrino equation can be interpreted 1) as governing the evolution of a probability distribution of neutrinos (with density S_0^d) as in the usual interpretation of quantum mechanics or 2) as describing a real perfect continuous fluid of Weyssenhoff with zero rest mass moving with the velocity of light connected by the tensions $\Theta_{\mu\nu}$ 3) as giving a real physical model of the behaviour of the real physical wave u associated with the neutrino in the causal interpretation of quantum mechanics proposed by de Broglie, Bohm and one of us (J.-P. Vigier)[19].

The third and last remark is that such an hydrodynamical representation of the neutrino equation paves the way to many interesting developments. For example in a subsequent paper we shall demonstrate that a mixture of two such perfect fluids with a spin-spin interaction leads to hydrodynamical equations which when expressed in spinor variables correspond exactly to the non-linear equation recently proposed by Heisenberg and Iwanenko[20].

We wish to express our gratitude to Professors L. de Broglie, D. Bohm and to Dr. F. Halbwachs for the interest shown in this work and for various valuable suggestions. Professor Bohm in particular has greatly contributed to the clarification of our model.

Appendix

To demonstrate that at the velocity of light the set of E. K. variables a_μ^ξ transform into the three orthogonal pseudo-vectors b_k^r, we can proceed in the following way: we have the following relations:

$$a_\mu^{(4)} = \frac{u_\mu}{i_c}, \quad a_\mu^{(3)} = \frac{i}{2ch_0}\,\varepsilon_{\mu\nu\alpha\beta}\,a_\nu^{(4)}\,s^{\alpha\beta}, \quad a_\mu^{(1)} = \varepsilon_{\mu\nu\alpha\beta}\,a_\nu^{(2)}\,a_{(\alpha)}^{(3)}\,a_\beta^{(4)} \tag{1}$$

and

$$b'^{(4)}\,a_\mu^\xi\,a_\nu^\xi = \delta_{\mu\nu}\;a_\mu^\xi\,a_\mu^\eta = \delta^{\xi\eta}\;(\mu,\nu,\xi,\eta \sim 1,2,3,4) \tag{2}$$

we defined the parameter b'

$$b' = \frac{db}{dt} = \sqrt{1-\frac{v^2}{c^2}} \tag{3}$$

[19] See for example: De Broglie „*Une Interpretation Causale et non-linéaire de la Mécanique*". David Bohm, *Phys. Rev.* David Bohm and J.-P. Vigier, *Phys. Rev.* The theory is also summarized in Halbwachs, see ref. 3.

[20] That is $\gamma_\mu \partial_\mu \Psi - \chi\,\{\Psi^+\Psi - (\Psi^+\gamma_5\Psi)\gamma_5\}\,\Psi = 0.$

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if we take:

$$w_\mu = b'\mu_\mu, \quad a_\mu^{(3)} \text{ and } a_\mu^{(4)} \text{ become } b'\, a_\mu^{(3)},\, b'\, a_\mu^{(4)}$$

so:

$$(b'\, a_\mu^{(3)})\,(b'\, a_\mu^{(3)}) = (b'\, a_\mu^{(4)})\,(b'\, a_\mu^{(4)}) = b'^2 \tag{4}$$

now if we define:

$$\beta_\mu^{(3)} = \lim_{b'\to 0} b'\, a_\mu^{(3)}, \qquad \beta_\mu^{(4)} = \lim_{b'\to 0} b'\, a_\mu^{(4)} \tag{5}$$

and we have:

$$\beta_\mu^{(3)}\,\beta_\mu^{(3)} = \beta_\mu^{(4)}\,\beta_\mu^{(4)} = 0 \tag{6}$$

$$\beta_\mu^{(3)}\,\beta_\mu^{(4)} = 0 \tag{7}$$

it results immediately from Schwarz inequality that:

$$\beta_\mu^{(3)} = i\lambda\beta_\mu^{(4)} \tag{8}$$

where λ is a real scalar.

Let us choose:

$$\beta_4^{(4)} = 1 \qquad b_h^{(4)} = i\beta_h^{(4)} \qquad (h \sim 1, 2, 3) \tag{9}$$

$b_h^{(4)}$ is like a two dimensional unitary vector and

$$b_h^{(4)}\, b_h^{(4)} = 1 \tag{10}$$

Let us prove that $\lambda = \pm 1$. For this, let us start with the relation:

$$a_\mu^{(2)}\, a_\nu^{(3)} - a_\mu^{(3)}\, a_\nu^{(2)} = \varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^{(4)}\, a_\beta^{(1)}$$

which multiplied by b' and going to the limit (taking into account the relation (8))

$$i\lambda\,(a_\mu^{(2)}\beta_\nu^{(4)} - a_\nu^{(2)}\,\beta_\mu^{(4)}) = \varepsilon_{\mu\nu\alpha\beta}\,\beta_\alpha^{(4)}\, a_\beta^{(1)}$$

the space components taken out of this expression are:

$$i\lambda\,(a_0^{(2)}\,\beta_j^{(4)} - a_j^{(2)}\,\beta_0^{(4)}) = \varepsilon_{ijk4}\,\beta_k^{(4)}\, a_4^{(1)}$$

and, as we have: $a_4^{(1)} = i\, a_0^{(1)} \quad \beta_j^{(4)} = -\, i\, b_j^{(4)} \quad \beta_4^{(4)} = 1$

$$\lambda\,(\vec{a}^{(2)} \wedge \vec{b}^{(4)}) = a_0^{(1)}\,\vec{b}^{(4)} - \vec{a}^{(1)} \tag{11}$$

where the symbol $\rightarrow$ denotes a three dimensional vector.

From (11), we have:

$$\lambda^2\,(\vec{a}^{(2)} \wedge \vec{b}^{(4)})\,(\vec{a}^{(2)} \wedge \vec{b}^{(4)}) = (a_0^{(1)}\,\vec{b}^{(4)} - \vec{a}^{(1)})\,(a_0^{(1)}\,\vec{b}^{(4)} - \vec{a}^{(1)})$$

$$\lambda^2\,[(\vec{a}^{(2)}\cdot\vec{a}^{(2)})\,(\vec{b}^{(4)}\cdot\vec{b}^{(4)}) - (\vec{a}^{(2)}\cdot\vec{b}^{(4)})^2] = a_0^{(1)}\,a_0^{(1)}\,\vec{b}^{(4)}\cdot\vec{b}^{(4)} + \vec{a}^{(1)}\cdot\vec{a}^{(1)} - 2a_0^{(1)}\,\vec{a}^{(1)}\cdot\vec{b}^{(4)}$$

taking into account relation (2):

$$\vec{a}^{(1)} \cdot \vec{a}^{(1)} = 1 + a_0^{(1)} a_0^{(1)} \qquad \vec{a}^{(2)} \cdot \vec{a}^{(2)} = 1 + a_0^{(2)} a_0^{(2)}$$
$$\vec{a}^{(1)} \cdot \vec{b}^{(4)} = a_0^{(1)} \qquad \vec{a}^{(2)} \cdot \vec{b}^{(4)} = a_0^{(2)}$$

and relation (10):
it comes for the preceding expression: $\lambda^2 = 1$
and finally:

$$\beta_\mu^{(3)} = \pm i \beta_\mu^{(4)}$$

Now to obtain the limits for $a_\mu^{(1)}$ and $a_\mu^{(1)}$ let us suppose b' sufficiently weak to enable us to write:

$$b' a_\mu^{(3)} = i \beta_\mu^{(4)} + b'^2 \delta_\mu^{(3)} + O_1 (b'^2) \tag{12}$$

$$b' a_\mu^{(4)} = \beta_\mu^{(4)} + i b'^2 \delta_\mu^{(4)} + O_2 (b'^2) \tag{13}$$

where $\delta_\mu^{(3)}$ and $\delta_\mu^{(4)}$ are space-like vectors, $O_1(b'^2)$ and $O_2(b'^2)$ are negligible quantities.

For $b' \neq 0$, $b' a_\mu^{(3)}$, and $b' a_\mu^{(4)}$ are not isotropic vectors and their norm must be positive; of course, they are always orthogonal so we have

$$i \beta_\mu^{(4)} \delta_\mu^{(3)} > 0 \tag{14}$$

$$i \beta_\mu^{(4)} \delta_\mu^{(4)} > 0 \tag{15}$$

$$i \beta_\mu^{(4)} (\delta_\mu^{(3)} - \delta_\mu^{(4)}) = 0 \tag{16}$$

Moreover $\delta_0^{(4)} = 0$ as $\beta_4^{(4)} = 1$ and:

$$b' a_\mu^{(4)} = \frac{b'}{\sqrt{1 - \frac{v^2}{c^2}}} = 1$$

we conclude: $b'^2 \delta_0^{(4)} 0$ or $\delta_0^{(4)} = 0$
let us apply (12) and (13) to the relation:

$$b' a_\mu^{(3)} = \varepsilon_{\mu\nu\alpha\beta} a_\alpha^{(1)} a_\beta^{(2)} (b' a_\nu^{(4)})$$

we obtain:

$$i \beta_\mu^{(4)} + b'^2 \delta_\mu^{(3)} = \varepsilon_{\mu\nu\alpha\beta} a_\alpha^{(1)} a_\beta^{(2)} (\beta_\nu^{(4)} + i b'^2 \delta_\nu^{(4)})$$

or:

$$i \beta_\mu^{(4)} = \varepsilon_{\mu\nu\alpha\beta} a_\alpha^{(1)} a_\beta^{(2)} \beta_\nu^{(4)} \tag{17}$$

$$\delta_\mu^{(3)} = i \varepsilon_{\mu\nu\alpha\beta} a_\alpha^{(1)} a_\beta^{(2)} \delta_\nu^{(4)} \tag{18}$$

The relation (18) gives:

$$- \delta_0^{(3)} = \vec{\delta}^{(4)} \cdot (\vec{a}^{(1)} \wedge \vec{a}^{(2)}) \tag{19}$$

$$\delta_i^{(3)} = - \delta_0^{(4)} (\vec{a}^{(1)} \wedge \vec{a}^{(2)})_i - a_0^{(2)} (\vec{\delta}^{(4)} \wedge \vec{a}^{(1)})_i - a_0^{(1)} (\vec{a}^{(2)} \wedge \vec{\delta}^{(4)})_i \tag{20}$$

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Now, let us start with the relation:

$$b' a_\mu^{(4)} = \varepsilon_{\mu\nu\alpha\beta}\, a_\nu^{(1)}\, a_\alpha^{(2)}\, a_\beta^{(3)}$$

The same calculation brings to:

$$-\,\delta_0^{(4)} = \vec{\delta}^{(3)} \cdot (\vec{a}^{(1)} \wedge \vec{a}^{(2)}) \tag{21}$$

$$\delta_i^{(4)} = -\,\delta_0^{(3)}\,(\vec{a}^{(1)} \wedge \vec{a}^{(2)})_i - a_0^{(2)}\,(\vec{\delta}^{(3)} \wedge \vec{a}^{(1)})_i - a_0^{(1)}\,(\vec{a}^{(2)} \wedge \vec{\delta}^{(3)})_i \tag{22}$$

As $\delta_0^{(4)} = 0$ the equation (21) has two solutions:

$$\vec{\delta}^{(3)} = \lambda \vec{a}^{(1)} + \mu\, \vec{a}^{(2)} \tag{23}$$

$$\delta^{(3)} = 0 \tag{24}$$

Let us consider the first solution. Let us multiply (20), by $\vec{a}^{(1)}$ and by $\vec{a}^{(2)}$ taking into account relation (19); we have:

$$\vec{\delta}^{(3)} \cdot \vec{a}^{(1)} = a_0^{(1)}\,\delta_0^{(3)} \qquad \vec{\delta}^{(3)} \cdot \vec{a}^{(2)} = a_0^{(2)}\,\delta_0^{(3)} \tag{25}$$

Because the orthogonality conditions, when we put (23) into (25) we obtain the system:

$$\begin{aligned} \lambda + \lambda\,(a_0^{(1)})^2 + \mu\, a_0^{(1)}\, a_0^{(2)} &= a_0^{(1)}\,\delta_0^{(3)} \\ \lambda\, a_0^{(1)}\, a_0^{(2)} + \mu + \mu\, a_0^{(2)}\, a_0^{(2)} &= a_0^{(2)}\,\delta_0^{(3)} \end{aligned} \tag{26}$$

The solution is:

$$\lambda = a_0^{(1)}, \qquad \mu = a_0^{(2)}, \qquad \delta_0^{(3)} = 1 + (a_0^{(1)})^2 + (a_0^{(2)})^2 \tag{27}$$

Now, with $\vec{\delta}^{(3)}$ it is possible to calcutate $\vec{\delta}^{(4)}$. It's sufficient to multiply successively (22) by $a^{(1)}$ and $a^{(2)}$, this gives:

$$\vec{\delta}^{(4)} \cdot \vec{a}^{(1)} = \vec{\delta}^{(4)} \cdot \vec{a}^{(2)} = 0$$

or

$$\vec{\delta}^{(4)} = \nu\,(\vec{a}^{(1)} \wedge \vec{a}^{(2)}) \tag{28}$$

If we bring (28) into (19):

$$\begin{aligned} -\,\delta_0^{(3)} &= \nu\,(\vec{a}^{(1)} \wedge \vec{a}^{(2)}) \cdot (\vec{a}^{(1)} \wedge \vec{a}^{(2)}) = \nu\,[(\vec{a}^{(1)} \cdot \vec{a}^{(1)})\,(\vec{a}^{(2)} \cdot \vec{a}^{(2)}) - (\vec{a}^{(1)} \cdot \vec{a}^{(2)})^2] = \\ &= \nu\,[1 + (a_0^{(1)})^2 + (a^{(2)})^2] = \nu \delta_0^{(3)} \end{aligned}$$

This implies: $\nu_0 = -1$

and:

$$\vec{\delta}^{(4)} = -\,(\vec{a}^{(1)} \wedge \vec{a}^{(2)}) \tag{29}$$

But the relation (14) writes:

$$\vec{b}^4 \cdot \vec{\delta}^{(4)} > 0 \tag{30}$$

and the relation (17) gives:

$$\vec{b}^4 \cdot (\vec{a}^{(1)} \wedge \vec{a}^{(2)}) = 1 \tag{31}$$

We see that (29), (30), and (31) are not compatible.
Now we have the second solution $\vec{\delta}^{(3)} = 0$
From (22):

$$\delta_i^{(4)} = -\delta_0^{(3)} (\vec{a}^{(1)} \wedge \vec{a}^{(2)})_i \tag{32}$$

which implies (16), moreover (14), (15), (31) are verified if $\delta_0^{(3)} < 0$. But bringing (12) and (13) into the expression:

$$(a_\mu^{(1)} a_\nu^{(2)} - a_\mu^{(2)} a_\nu^{(1)}) b'^2 = \varepsilon_{\mu\nu\alpha\beta} (b' a_\alpha^{(3)}) (b' a_\beta^{(4)})$$

a direct calculation gives:

$$\vec{a}^{(1)} \wedge \vec{a}^{(2)} = \vec{\delta}^{(3)} + \vec{\delta}^{(4)} - (\delta_0^{(3)} + \delta_0^{(4)}) \vec{b}^{(4)} = -\delta_0^{(3)} (\vec{a}^{(1)} \wedge \vec{a}^{(2)} + \vec{b}^{(4)})$$

let us multiply this result by $\vec{b}^{(4)}$, taking into account the relation (31) and $\vec{b}^{(4)} \cdot \vec{b}^{(4)} = 1$ we have:

$$s = -2\delta_0^{(3)} \text{ or } \delta_0^{(3)} = -\tfrac{1}{2}$$

that is what we intend. Besides putting (32) into (19), we find:

$$(\vec{a}^{(1)} \wedge \vec{a}^{(2)}) \cdot (\vec{a}^{(1)} \wedge \vec{a}^{(2)}) = 1$$

which implies

$$(a_0^{(1)})^2 + (a_0^{(2)})^2 = 0$$

or

$$a_0^{(1)} = a_0^{(2)} = 0$$

this is the resulte which we wanted to prove.

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