THE RELATIVISTIC GAS IN THE GRAVITATIONAL FIELD*

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The Boltzmann kinetic equation is established for the rarefied relativistic gas in the gravitational field with smallest restrictions on the topological structure of the space-time manifold. The equation of change of molecular properties is obtained. It is proved that the divergences of the numerical flux vector and the energy-momentum tensor equal zero. It is also proved that the divergence of the flux vector of the entropy is nonnegative (H-theorem). The principle of the detailed balancing for the relativistic gas in the equilibrium state is considered.

1. Introduction

The Maxwell-Boltzmann kinetic theory of the rarefied gas makes it possible to construct its complete relativistic analogue. In the present paper basic foundations are stated for the kinetic theory of the relativistic gas taking into account the gravitation. The two following phenomena characterize the rarefied gas: the collision of two particles and the motion of one particle in the external field in the intervals between collisions. They account for all phenomena occurring in the rarefied gas, what just ensured success in the construction of a relativistic analogue of the Maxwell- Boltzmann kinetic theory.

To explain the last idea we note that any physical theory is based in any case on the concept of the velocity-space of the massive point [1—3]. This space is three-dimensional one. It has at any rate an absolute geometry. Usually one calls so the science of space independent of the postulate on the parallels. The postulate on the parallels in the velocity-space distinguishes between the non-relativistic and relativistic physics. In the non-relativistic case the velocity- space is Euclidean one, while in the relativistic case it is the Lobachevsky space radius k of curvature of which equals the light velocity c. The properties of both the non-relativistic and analogous relativistic theory can be connected only with those properties of the velocity- space which are the object of the absolute geometry. In this sense, in any physical theory an "absolute content" can be singled out. The latter is related to phenomena which are not connected with the factor of independence of the light velocity

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on the source velocity. Basing on this idea, going back to Lobachevsky and Bolyai, in constructing the relativistic analogue of the non-relativistic physical theory we should first single out an absolute part which is transferred to the relativistic case without change.

This can be made without difficulty in the mechanics of a single particle in the external field. However the theory of two and more particles met difficulties yet unavoided. The only completed result of the consideration of many particles is a so-called kinematics of decay or fusion of particles. From the point of view of kinematics to the two last phenomena reduce their arbitrary combinations so, for example, the particle collision. The kinematics of these phenomena is considered in [4] starting from the absolute mechanics. The success in the kinematics is due to the fact that it deals only with momentum-velocity characteristics of particles involved in the reaction and does not touch upon a subject on the particle coordinates during the reaction.

To single out the absolute content of the Maxwell-Boltzmann kinetic theory a special method is needed which has been suggested and developed in [5—7].

The Boltzmann relativistic kinetic equation was obtained there under the assumption that the space-time manifold is a Galileian one. Any external forces were taken into account for the exception of the gravitational ones. In a subsequent paper [8] the relativistic collision integral obtained in [6—7] is represented in the form of the Boltzmann five-fold integral. In all these papers the problems have been considered from the absolute point of view.

The developed method [5—7] allowed one to take into account the Einstein gravitational phenomenon. In [9] the Boltzmann kinetic equation for the relativistic gas in the static spherical — symmetrical gravitational field is derived and the solution of this equation is found which corresponds to the equilibrium state of the gas. A kinetic equation for the relativistic gas in the Einstein arbitrary gravitational field is derived in [10]. The flux vector and the mass tensor of the relativistic gas in the gravitational field are investigated in [11] on the basis of the above equation. The problem of a gravitating relativistic gas is raised there. In [12] a close connection is established between the Maxwell-Boltzmann relativistic distribution and the integral form of laws of conservation.

In the present paper these results are systematized and supplemented with new results, to which belongs the equation of transfer for the relativistic gas in the arbitrary gravitational field and, in conformity with this case, H-theorem and the principle of the detailed balancing.

In the theory of the relativistic gas formulated here we digress in two respect from the absolute point of view. First, particles with zero proper mass are considered, what is surely absent in the Maxwell-Boltzmann non- relativistic theory. Secondly, the Einstein gravitation effects are taken into account, while in the non- relativistic case the theory of space- time with curvature is not constructed yet. From the consistent absolute point of view the corresponding gap in the non-relativistic theory should be make up.

For methodical reasons we restrict ourselves to a case when external forces (of a non-gravitational character) are absent although the account of such forces in a kinetic equation is not difficult. For the same reasons we restrict ourselves to the consideration of only elastic collisions of particles in a gas.

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2. *Space of the states of a particle*

The pair (x, p) of the space- time position x and momentum p of the particle is referred to as the particle state. The proper mass m of the particle is fixed so that dimension of the space F of states of the particle equals seven. As coordinates in the state- space we can choose seven quantities $x^0, x^1, x^2, x^3, p^1, p^2, p^3$. The component p^0 is found from the conditions

$$(p, p) = g_{\alpha\beta}(x)p^\alpha p^\beta = m^2c^2, \quad p_0>0. \tag{2.1}$$

Hence,

$$p^0 = \frac{p_0-g_{0k}p^k}{g_{00}} \tag{2.2}$$

where

$$p_0 = \sqrt{g_{00}m^2c^2+(g_{0i}g_{0k}-g_{00}g_{ik})p^ip^k}. \tag{2.3}$$

The coordinate p^k change in infinite limits. For $m = 0$ the point $p^1 = p^2 = p^3 = 0$ is omitted owing to the fact that the zero momentum characterizes the *vacuum* but not the particle.

If we did not considered particles with zero proper mass then instead of the momentum it would be convenient to use the particle velocity.

If we did not considered the Einstein gravitational phenomenon then we might restrict ourselves to the choice of the Galileian coordinates t, x, y, z in the space-time.

The element of the volume in the state-space F is chosen to be equal to

$$dF = -\frac{g(x)}{p_0}\, dx^0dx^1dx^2dx^3dp^1dp^2dp^3, \tag{2.4}$$

where $g(x) = |g_{\alpha\beta}|$ is the determinant of the matrix $(g_{\alpha\beta})$.

The set of possible momenta of a particle in a given space-time position x is called a particle momentum space $P(x)$. Thus the state-space F is a bundle $P(X)$ of momenta applied to various points of the space-time X:

$$F = P(X) = \bigcup_{x\epsilon X} P(x). \tag{2.5}$$

This bundle is a fibre bundle [13] over the base space X with a fibre of the type $P(x)$ and with a group of motions of the three- dimensional Lobachevsky space as a structure group. This group is isomorphous to the Lorentz orthochronous group. The element of the volume in $P(x)$ is chosen to be equal to

$$dP = \frac{\sqrt{-g}}{p_0}\, dp^1dp^2dp^3, \tag{2.6}$$

so that

$$dF = dXdP, \tag{2.7}$$

where dX is the element of the volume in the space-time X:

$$dX = \sqrt{-g}\, dx^0\, dx^1\, dx^2\, dx^3. \tag{2.8}$$

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In principle, the element of the volume dF is a skew- symmetrical 7-linear form [14]

$$dF = \varepsilon(x, p; d_0, d_1, \ldots, d_6), \tag{2.8}$$

assuming the value (2.4) on the vectors of elementary displacements along the coordinate lines x^ν, p_k. We show why such a choice of the element of the volume is convenient. Let us go over to new variables

$$x^{\nu'} = \varphi^\nu(x^0, x^1, x^2, x^3), \quad \nu = 0, 1, 2, 3,$$

$$p^{k'} = \sum_{\alpha=0}^{3} p^\alpha \frac{\partial}{\partial x^\alpha} \varphi^k(x^0, x^1, x^2, x^3), \quad k = 1, 2, 3. \tag{2.10}$$

The Jacobian of this transformation is

$$J = \frac{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'}, p^{1'}, p^{2'}, p^{3'})}{\partial(x^0, x^1, x^2, x^3, p^1, p^2, p^3)} = \frac{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'})}{\partial(x^0, x^1, x^2, x^3)} \cdot \frac{\partial(p^{1'}, p^{2'}, p^{3'})}{\partial(p^1, p^2, p^3)}. \tag{2.11}$$

But since

$$\frac{\partial p^{k'}}{\partial p^i} = \frac{\partial \varphi^k}{\partial x^i} - \frac{\partial \varphi^k}{\partial x^0} \frac{p_i}{p_0}, \tag{2.12}$$

then

$$\frac{\partial(p^{1'}, p^{2'}, p^{3'})}{\partial(p^1, p^2, p^3)} = \frac{1}{p_0} \begin{vmatrix} p_0 & p_1 & p_2 & p_3 \\ \frac{\partial \varphi^1}{\partial x^0} & \frac{\partial \varphi^1}{\partial x^1} & \frac{\partial \varphi^1}{\partial x^2} & \frac{\partial \varphi^1}{\partial x^3} \\ \frac{\partial \varphi^2}{\partial x^0} & \frac{\partial \varphi^2}{\partial x^1} & \frac{\partial \varphi^2}{\partial x^2} & \frac{\partial \varphi^2}{\partial x^3} \\ \frac{\partial \varphi^3}{\partial x^0} & \frac{\partial \varphi^3}{\partial x^1} & \frac{\partial \varphi^3}{\partial x^2} & \frac{\partial \varphi^3}{\partial x^3} \end{vmatrix}. \tag{2.13}$$

Taking into account that

$$p_\nu = p'_\alpha \frac{\partial \varphi^\alpha}{\partial x^\nu}, \tag{2.14}$$

we obtain

$$\frac{\partial(p^{1'}, p^{2'}, p^{3'})}{\partial(p^1, p^2, p^3)} = \frac{p'_0}{p_0} \left|\frac{\partial \varphi^\alpha}{\partial x^\beta}\right|, \tag{2.15}$$

Hence

$$J = \frac{p'_0}{p_0} \left[\frac{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'})}{\partial(x^0, x^1, x^2, x^3)}\right]^2 = \frac{p'_0 g}{p_0 g'}. \tag{2.16}$$

Thus, the element dF (see (2.4)) has an invariant form under the transformations of coordinates (2.10).

Since the Jacobian (2.16) is positive then the state-space F is orientable independently of the fact if it is possible to orientate the space- time X or not. Indeed, we assume that the metric tensor is continuous in the all space- time X. Then in the X there exists a continuous field $\omega(x)$ of non- zero vectors directed into the isotropical (light) cone [15]. On the coordinates x^0, x^1, x^2, x^3 we impose the condition that the vector of displacement along the coordinate line x^0 in direction of positive values of x^0 will be directed at each point x into the same part of the isotropic cone as the vector $\omega(x)$. If regions covered by some pair of the coordinate systems are overlaped then at the intersection of these regions the Jacobian (2.16) is positive. Hence, it follows that the state- space F is orientable.

In the intervals between collisions the particle of a gas moves according to the law

$$\frac{dx^\nu}{d\tau} = p^\nu, \qquad \frac{dp^k}{d\tau} = -\Gamma^k_{\alpha\beta}(x)p^\alpha p^\beta, \tag{2.17}$$

where τ is the proper time of the particle connected with the ordinary proper time τ_0 by a simple relation $\tau_0 = m\tau$. The proper time τ is determined for particles with zero proper mass as well. $\Gamma^\nu_{\alpha\beta}$ are the Christoffel symbols

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right). \tag{2.18}$$

We write the solution of the set of equations (2.17) in the form

$$\begin{aligned} x &= x(x^*, p^*, \tau), \qquad & p &= p(x^*, p^*, \tau), \\ x(x^*, p^*, 0) &= x^*, & p(x^*, p^*, 0) &= p^*. \end{aligned} \tag{2.19}$$

The first of Eqs. (2.19) determined the world line of the particle and the second one which is obtained from the first of Eqs. (2.19) by differentiating with respect to τ the change of the momenta along the world line. The existence and the uniqueness of the solution are ensured if $\Gamma^\nu_{\alpha\beta}(x)$ and their first derivatives with respect to x^γ are continuous.

The set of Eqs. (2.17) specifies in the state- space F the vector field $f(x, p)$. For coordinates x^ν, p^k this field has the following components:

$$f^\nu = p^\nu, \qquad f^{3+k} = -\Gamma^{\alpha\beta}_k(x)p^\alpha p^\beta, \qquad \nu = 0, 1, 2, 3, \quad k = 1, 2, 3, \tag{2.20}$$

It does not vanish anywhere since if $f = 0$ then $p = 0$ as well, and the momentum of particle differs from zero. The solution (2.19) determines in the state- space the vector line of the field $f(x, p)$ passing *via* the point (x^*, p^*) and the law of change of the state along this line depending on τ.

The vector field $f(x, p)$ defines the area on any hypersurface S in the state-space F. The element of the area is

$$dS = \varepsilon(x, p; f(x, p), d_1, \ldots, d_6) = \sigma(x, p; d_1, \ldots, d_6), \tag{2.21}$$

where $d_1, \ldots, d_6$ are the vectors of infinitesimal displacements on S, ε is the 7—linear

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form determined above (see(2.9)). We write in detail

$$dS = -\frac{g}{p_0}\begin{vmatrix} p^0 & p^1 & p^2 & p^3 & -\Gamma^1_{\alpha\beta}\,p^\alpha p^\beta & -\Gamma^2_{\alpha\beta}\,p^\alpha p^\beta & -\Gamma^3_{\alpha\beta}\,p^\alpha p^\beta \\ d_1x^0 & d_1x^1 & d_1x^2 & d_1x^3 & d_1p^1 & d_1p^2 & d_1p^3 \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_6x^0 & d_6x^1 & d_6x^2 & d_6x^3 & d_6p^1 & d_6p^2 & d_6p^3 \end{vmatrix} \tag{2.22}$$

In particular, on the hypersurface

$$S = P(U) = \bigcup_{x\epsilon U} P(x), \tag{2.23}$$

representing itself the bundle of momenta over the hypersurface $U \subset X$ in the space- time the element of the area dS is

$$dS = \sqrt{-g}\begin{vmatrix} p^0 & p^1 & p^2 & p^3 \\ d_1x^0 & d_1x^1 & d_1x^2 & d_1x^3 \\ d_2x^0 & d_2x^1 & d_2x^2 & d_2x^3 \\ d_3x^0 & d_3x^1 & d_3x^2 & d_3x^3 \end{vmatrix} dP \tag{2.24}$$

The exterior derivative $\sigma'(x, p; d_0, d_1,...,d_6)$ of a 6-linear skew form (2.21) vanishes. This statement is analogous to the Liouville's theorem. In the coordinate form the equality $\sigma' = 0$ means

$$\frac{\partial}{\partial x^\nu}\left(-\frac{gp^\nu}{p_0}\right) + \frac{\partial}{\partial p^k}\left(\frac{g}{p_0}\Gamma^k_{\alpha\beta}p^\alpha p^\beta\right) = 0. \tag{2.25}$$

The above mentioned vector field $\omega(x)$ over space-time X allows one to define the notions of the future and past at each point x of X. We call the future for a particle at the arbitrary point x that part of the tangent isotropic cone with the center at the point x into which the vector $\omega(x)$ is directed. The condition $p_0 > 0$ means that the momentum of particle is represented by a vector directed to the future. Without this condition the momentum would be represented by the pair of opposite vectors. If $d\tau > 0$ then the vector $dx = pd\tau$ would be directed to the future as well, *i. e.* to the positive course of the proper time there corresponds the motion of particle in the direction of the future. So far we do not exclude the case when the solution (2.19) represents a closed curve in the space- time. In this case the particle moving to the direction of the future would return from the past. Excluding this possibility we shall assume that the set of the curves (2.19) form a six-dimensional manifold S since the state- space F turns out to be a fibre bundle over S a type fibre of which is the Euclidean line I, and a structure group is the group of parallel transfer of the line I.

3. *System of kinetic equations*

Consider here the mixture of N gases. The distribution function of an i-th component of this mixture is denoted by $A_i(x, p)$. $A_i(x, p)$ is a scalar function in the state-space F_i of the particle of the kind i. Let the hypersurface $S \subset F_i$ be such that the particle, moving along

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any trajection in the state-space F_i specified by (2.19) meet with S not more than one time. The number of gas particles of the kind i crossing such a hypersurface is

$$n_i(S) = \int\limits_S A_i(x, p)\, dS = \int\limits_S A_i(x, p)\, \sigma(x, p; d_1, \ldots, d_6). \tag{3.1}$$

During the proper time $d\tau$ the i-kind particle starting from the state $(x, p) \in F_i$ collides with a certain probability with the particle of the kind j and ceases to interact with the latter being in the element of the volume dF' of the state-space F_i with the center at the point (x', p'). We calcualte this probability neglecting the change in the function $A_j(y, q)$, $(y, q) \in F_j$ in changing the argument y in the neighbourhood of x where collision takes place. The collision is assumed to be instantaneous. In this approximation the desired probability is

$$\delta(x'-x) w_{ij}(x, p, p')\, dX'\, dP'\, d\tau, \tag{3.2}$$

where the quantity $w_{ij}\, dP' d\tau$ equals the number of the j-kind particles crossing the six-dimensional surface ΔS in the state- space F_j formed by the three-dimensional fibre $P_j(x)$ at each point of which a three-dimensional parallelepiped is constructed with edges $(p^\nu, \Gamma^k_{\alpha\beta} q^\alpha p^\beta)\, d\tau$, $(d^\nu, \Gamma^k_{\alpha\beta} q^\alpha d^\beta)$, $(\delta^\nu, \Gamma^k_{\alpha\beta} q^\alpha \delta^\beta)$. The vectors d and δ of the space- time X are applied to the point x and form an area orthogonal to and p, q and equal the differential cross section

$$d\sigma = H_{ij}(p, q, p') dP' \tag{3.3}$$

of scattering of the i-kind particle with momentum p on the j-kind particle with momentum q. The element of the area of ΔS according to (2.22) is

$$dS = -\frac{g(x)}{q_0} \begin{vmatrix} q^0 & q^1 & q^2 & q^3 \\ p^0 & p^1 & p^2 & p^3 \\ d^0 & d^1 & d^2 & d^3 \\ \delta^0 & \delta^1 & \delta^2 & \delta^3 \end{vmatrix} dq^1\, dq^2\, dq^3\, d\tau. \tag{3.4}$$

The determinant of this expression is

$$D = \frac{\langle p, q\rangle\, d\sigma}{\sqrt{-g}}, \tag{3.5}$$

where

$$\langle p, q\rangle = \sqrt{(p, q)^2 - (p, p)\,(q, q)} \tag{3.6}$$

characterizes the relative motion of particles, Hence,

$$w_{ij}(x, p, p') = \int\limits_{P_j(x)} A_j(x, q) \langle p, q\rangle H_{ij}(p, q, p')\, dQ, \tag{3.7}$$

where dQ is the element of the volume in the momentum space $P_j(x)$.

The i-kind particle can collide with the j-kind particle being in the element $dF \subset F_i$ and cease to interact with it being in the element $dF' \subset F_i$. The number od such collisions is

$$A_i(x, p) \delta\, (x'-x) w_{ij}(x, p, p')\, dF\, dF'. \tag{3.8}$$

Indeed, we consider the infinitesimal six-dimensional parallelepiped in F_i with edges $d_1,\ldots,d_6$ and the seven-dimensional parallelepiped with edges $d_0, d_1,\ldots, d_6$, where $d_0 = f(x, p)d\tau$. The above mentioned number is obviously equal to

$$A_i(x, p)\varepsilon(x, p; f(x, p), d_1,\ldots,d_6)\delta(x'-x)w_{ij}(x, p, p')\, dX'dP'd\tau \tag{3.9}$$

and consequantly it is equal to (3.8) as well, since

$$\varepsilon(x, p; f(x, p), d_1,\ldots, d_6)d\tau = \varepsilon(x, p; d_0, d_1,\ldots, d_6) = dF. \tag{3.10}$$

Consider then the region D in the state-space F_i. The numbers $R_{ij}^{(1)}(D)$ and $R_{ij}^{(2)}(D)$ of the i-kind particles incoming to this region and outgoing from it due to collisions with the j-kind particles, as is easily seen from (3.8) and (3.7) are

$$R_{ij}^{(1,2)}(D) = \int\!\!\int_D \ldots \int I_{ij}^{(1,2)}(x, p)dF, \tag{3.11}$$

where

$$I_{ij}^{(1)}(x, p) = \int_{P_i(x)} \int_{P_j(x)} A_i(x, p')A_j(x, q')\langle p', q'\rangle H_{ij}(p', q', p)dP'dQ' \tag{3.12a}$$

$$I_{ij}^{(2)}(x, p) = \int_{P_i(x)} \int_{P_j(x)} A_i(x, p)A_j(x, q)\langle p, q\rangle H_{ij}(p, q, p')dP'dQ. \tag{3.12b}$$

Let the boudary Γ of the region D consist of two hypersurfaces S_2 and S_1 for which (3.1) is valid, S_2 being placed in the future with respect to S_1. Obviously,

$$n_i(S_2)-n_i(S_1) = \sum_{j=1}^{N} [R_{ij}^{(1)}(D)-R_{ij}^{(2)}(D)]. \tag{3.13}$$

If an external orientation is chosen at the boundary Γ then

$$n_i(S_2)-n_i(S_1) = \int_\Gamma \ldots \int A_i(x, p)\sigma(x, p; d_1,\ldots, d_6) = \int\!\!\int_D \ldots \int \hat{f}(x, p)A_i(x, p)\, dF, \tag{3.14}$$

where $\hat{f}(x, p)$ is the differential operator

$$\hat{f}(x, p) = p^\nu \frac{\partial}{\partial x^\nu} - \Gamma_{\alpha\beta}^k(x)p^\alpha p^\beta \frac{\partial}{\partial p^k}. \tag{3.15}$$

In (3.14) we have transformed the integral over boundary Γ to the integral over the region D and taken into account (2.25). Since the region D is arbitrary, then from (3.13) and (3.14) it follows:

$$\hat{f}(x, p)A_i(x, p) = I_i(x, p), \quad i = 1, 2,\ldots, N, \tag{3.16}$$

where

$$I_i(x, p) = \sum_{j=1}^{N} I_{ij}(x, p), \tag{3.17}$$

$$I_{ij}(x, p) = I_{ij}^{(1)}(x, p)-I_{ij}^{(2)}(x, p). \tag{3.18}$$

The set of the integro-differential equations (3.16) is called a system of kinetic equations for the mixture of N gases.

For more care account the diagonal elements I_{ii} should be replaced by $\frac{N_i - 1}{N_i} I_{ii}$, where N_i is the number of the i-kind particles. From the practical point of view this replacement is not essential since the numbers N_i are usually large. Besides, the multiplier $\frac{N_i - 1}{N_i}$ can be included into H_{ii}.

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4. The equation of transfer

Let a certain characteristic of the i-th component of the gas be represented by the scalar function $\psi_i(x, p)$. The mean value $\bar{\psi}_i$ of this characteristic transferred by particles of the kind i through the hypersurface $S \subset F_i$ is

$$\bar{\psi}_i = \int_S \psi_i(x, p) A_i(x, p)\, dS. \tag{4.1}$$

Let further the hypersurface S represent itself the bundle $P_i(U)$ of momenta over the hypersurface U in the spacetime. Owing to (2.24) the mean value $\bar{\psi}$ is

$$\int_S \psi_i(x, p) A_i(x, p)\, dS = \int_U \sqrt{-g} \begin{vmatrix} \psi_i^0(x) & \psi_i^1(x) & \psi_i^2(x) & \psi_i^3(x) \\ d_1x^0 & d_1x^1 & d_1x^2 & d_1x^3 \\ d_2x^0 & d_2x^1 & d_2x^2 & d_2x^3 \\ d_3x^0 & d_3x^1 & d_3x^2 & d_3x^3 \end{vmatrix}, \tag{4.2}$$

where

$$\psi_i^\nu(x) = \int_{P_i(x)} p^\nu \psi_i(x, p) A_i(x, p)\, dP. \tag{4.3}$$

The vector $\psi_i^\nu(x)$ represents itself the flux vector of the quantity $\psi_i(x, p)$ at the point $x \epsilon X$. More exactly it is c times smaller than the flux-vector; this is connected with the fact that we have represented the element of the volume of the state- space in the form $dF = dXdP$ but not in the form $dF = (c^{-1}dX)(cdP)$.

We find the divergence of the vector $\psi_i^\nu(x)$. For this we consider the bundle $S^* = P_i(U^*)$ of momenta over the closed hypersurface U^* in the space-time. Let D is the region of the state-space F_i restricted to the hypersurface S^* and R is the region of the space-time restricted to the hypersurface U^*. First of all we have the equality similar to (4.2) (with the replacement of S by S^* and U by U^*). According to the integral theorem connecting the integral over the closed hypersurface with the integral over the region restricted to this hypersurface the above equality can be written in the form:

$$\int_D \hat{f}(x, p)[\psi_i(x, p) A_i(x, p)] dF = \int_R \nabla_\alpha \psi_i^\alpha(x) dX, \tag{4.4}$$

where

$$\nabla_\alpha \psi_i^\alpha = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g}\psi_i^\alpha) = \frac{\partial \psi_i^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\beta\alpha}\psi_i^\beta. \tag{4.5}$$

In deriving the left- hand side of (4.4) we take into account the identity (2.25). Since the region R is arbitrary then in virtue of the kinetic equation (3.16) from (4.4) it follows

$$\nabla_\alpha \psi_i^\alpha(x) = \int_{P_i(x)} I_i(x, p)\psi_i(x, p)dP + \int_{P_i(x)} A_i(x, p)\hat{f}(x, p)\psi_i(x, p)dP. \tag{4.6}$$

The obtained equality (4.6) is a relativistic analogue of the Maxwell-Enskog equation of transfer. For the sum

$$\psi^\alpha(x) = \sum_{i=1}^{N} \psi_i^\alpha(x) \tag{4.7}$$

we have

$$\nabla_\alpha \psi^\alpha(x) = \sum_{i=1}^{N} \int_{P_i(x)} I_i(x, p)\psi_i(x, p)dP + \sum_{i=1}^{N} \int_{P_i(x)} A_i(x, p)\hat{f}(x, p)\psi_i(x, p)dP. \tag{4.8}$$

Assuming that $\psi_i(x, p) = 1$ we obtain the numerical flux-vector of i-kind particles (see (4.3))

$$n_i^\alpha(x) = \int_{P_i(x)} p^\alpha A_i(x, p)dP. \tag{4.9}$$

Its covariant divergence is (see(4.6))

$$\nabla_\alpha n_i^\alpha = \int_{P_i(x)} I_i(x, p)dP. \tag{4.10}$$

Assuming $\psi_i(x, p)$ to be equal to the scalar product $(\xi(x), p)$, where $\xi(x)$ is a vector field over the space- time we are led to a mass tensor (energy-momentum tensor) of particles of the kind i:

$$T_i^{\alpha\beta}(x) = \int_{P_i(x)} p^\alpha p^\beta A_i(x, p)dP. \tag{4.11}$$

Its trace is

$$T_i(x) = T^{\alpha}_{i,\alpha}(x) = m_i^2 c^2 \int_{P_i(x)} A_i(x, p)dP. \tag{4.12}$$

The covariant divergence of the mass tensor is

$$\nabla_\alpha T_i^{\alpha\beta} = \int_{P_i(x)} p^\beta I_i(x, p)dP. \tag{4.13}$$

In deriving (4.13) the equation of transfer (4.6) and the identity

$$\hat{f}(x, p)\,(\xi(x), p) = \frac{1}{2}(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha)p^\alpha p^\beta \tag{4.14}$$

are taken into account.

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Finally we are led to the entropy and the flux vector of entropy of the i-th component of the gas putting in (4.1), (4.2) and (4.3) that $\psi_i(x, p)$ equals to $-\ln A_i(x, p)$:

$$s_i^\alpha(x) = -\int\limits_{P_i(x)} p^\alpha A_i(x, p) \ln A_i(x, p) dP. \tag{4.15}$$

Owing to (4.6) the covariant divergence of this vector is

$$\nabla_\alpha s_i^\alpha(x) = -\int\limits_{P_i(x)} I_i(x, p)[\ln A_i(x, p) + 1] dP. \tag{4.16}$$

In the following paragraph we shall prove that the right- hand side of (4.10) vanishes, the right- hand side of (4.13), summed over i from 1 to N, vanishes and that the right- hand side of (4.16), summed over i from 1 to N, is non-negative.

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5. *Some properties of the collision integral*

We specify the form of the function (3.3). It is supposed that the scattering is elastic. Therefore, if the scattering differential cross section in the c.m.s. is

$$\Delta\sigma = h_{ij}(\langle p, q\rangle, \cos\vartheta) \sin\vartheta d\vartheta d\varphi \tag{5.1}$$

then

$$H_{ij}(p, q, p') = \frac{(p+q, p+q)}{\langle p, q\rangle} h_{ij}\left(\langle p, q\rangle, 1 + \frac{(p+q, p+q)\,(p, p-p')}{\langle p, q\rangle^2}\right)\delta((p-p', p+q)). \tag{5.2}$$

The presence of the δ-function in (5.2) allows one to replace the integrals (3.12) by five-fold integrals.

All quantities we are considering in this paragraph are relative to the same point x of the space- time. The argument x will therefore be omitted.

We prove that for any function $\psi(p', q', p)$ the equality

$$\int\limits_{P_j} \psi(p', q', p)\delta((p'-p, p'+q'))\, dQ' = \int\limits_{P_j} \psi(p', p+q-p', p)\delta((p-p', p+q)) dQ. \tag{5.3}$$

is valid.

To the left- hand side of (5.3) we introduce the momentum $q \in P_j$ so that the equality

$$p+q = \lambda(p'+q') \tag{5.4}$$

holds, and replace the integration variable q' by q. From (5.4) it follows that

$$\lambda = \frac{(p, u') + \sqrt{m_j^2 - m_i^2 + (p, u')^2}}{\sqrt{(p'+q', p'+q')}}, \tag{5.5}$$

where

$$u' = \frac{p'+q'}{\sqrt{(p'+q', p'+q')}}. \tag{5.6}$$

In deriving (5.5) it is taken into account that $(q, q) = m_j^2$, $(p, p) = m_i^2$ and the scalar product (q, u') is non-negative. We write q in the form

$$q = -p + u' \{(p, u') + \sqrt{m_j^2 c^2 - m_i^2 c^2 + (p, u')^2}\}. \tag{5.7}$$

It is not difficult to calculate the Jacobians

$$\left|\frac{\partial q^i}{\partial u'^k}\right| = \frac{q_0}{u_0'} \frac{(p+q, u)^3}{(p, u')}, \qquad \left|\frac{\partial u'^k}{\partial q'^l}\right| = \frac{u_0'}{q_0'} \frac{(q', p'+q')}{(p'+q', p'+q')^2}. \tag{5.8}$$

Hence

$$dQ = \frac{(q', p+q)}{(q, p+q)} \left[\frac{(p+q, p+q)}{(p'+q', p+q)}\right]^3 dQ'. \tag{5.9}$$

Then

$$\delta((p'-p, p'+q')) = \lambda \delta((p-p', p+q)). \tag{5.10}$$

If $(p-p', p+q) = 0$ then $\lambda = 1$, $q' = p+q-p'$, $(q', p+q) = (q, p+q)$. Taking into account that all multipliers of the δ-function are to be considered only when its argument vanishes hence we obtain (5.3).

We consider then the integral

$$\int_{P_i} \psi(p, q, p' q') \delta((p-p', p+q)) dP', \quad \text{where } q' = p+q-p'. \tag{5.11}$$

We shall consider the vector

$$u = \frac{p+q}{\sqrt{(p+q, p+q)}}, \qquad \zeta = \frac{(u, q)p - (u, p)q}{\langle p, q \rangle} \tag{5.12}$$

and two other vectors ξ and η satisfying the conditions

$$\begin{aligned} &(\xi, \eta) = 0, \quad (\xi, \zeta) = 0, \quad (\eta, \zeta) = 0, \\ &(\xi, u) = (\eta, u) = (\zeta, u) = 0, \\ &(\xi, \xi) = (\eta, \eta) = (\zeta, \zeta) = -(u, u) = -1. \end{aligned} \tag{5.13}$$

From the integration variables p'^k in the integral (5.11) we go over to the variables r, ϑ, φ by the formula

$$p' = u\sqrt{m_i^2 c^2 + r^2} + \xi r \sin\vartheta \cos\varphi + \eta r \sin\vartheta \sin\varphi + \zeta r \cos\vartheta. \tag{5.14}$$

It is not difficult to see that

$$dP' = \frac{r^2 \sin\vartheta}{\sqrt{m_i^2 c^2 + r^2}} dr d\vartheta d\varphi \tag{5.15}$$

and that

$$\delta((p-p', p+q)) = \frac{1}{\sqrt{(p+q, p+q)}} \delta\left(\sqrt{m_i^2 c^2 + r^2} - \frac{(p, p+q)}{\sqrt{(p+q, p+q)}}\right). \tag{5.16}$$

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Carrying out the integration over r we get the integral (5.11) to be equal to

$$\int_{P_i} \psi(p, q, p', q')\, \delta((p-p', p+q))\, dP' = \frac{\langle p, q\rangle}{(p+q, p+q)} \int_0^{\pi}\int_0^{2\pi} \psi(p, q, p', q') \sin\vartheta d\vartheta d\varphi, \quad (5.17)$$

where

$$p' = \frac{(p, p+q)}{(p+q, p+q)}(p+q) \quad \frac{\langle p, q\rangle}{\sqrt{(p+q, p+q)}}\{\xi \sin\vartheta\cos\varphi + \eta \sin\vartheta \sin\varphi + \zeta\cos\vartheta\}, \quad (5.18a)$$

$$q' = \frac{(q, p+q)}{(p+q, p+q)}(p+q) - \frac{\langle p, q\rangle}{\sqrt{(p+q, p+q)}}\{\xi \sin\vartheta\cos\varphi + \eta \sin\vartheta \sin\varphi + \zeta\cos\vartheta\}. \quad (5.18b)$$

By applying the equality (5.3) to the integral (3.12a) we write the collision integral (3.18) in the form

$$I_{ij}(p) = \int_{P_i}\int_{P_j} \{A_i(p')A_j(q') - A_i(p)A_j(q)\}\langle p, q\rangle H_{ij}(p, q, p')dP'dQ, \quad (5.19)$$

where $q' = p+q-p'$. Applying to (5.19) the equality (5.17) we get

$$I_{ij}(p) = \int_{P_j} dQ \int_0^{\pi}\int_0^{2\pi} \{A_i(p')A_j(q') - A_i(p)A_j(q)\}\langle p, q\rangle h_{ij}(\langle p, q\rangle, \cos\vartheta) \sin\vartheta d\vartheta d\varphi, \quad (5.20)$$

where p' and q' are determined from the formulas (5.18).

We consider the integral

$$I_{ij}[\psi] = \int_{P_i} \psi_i(p) I_{ij}(p)\, dP = I_{ij}^{(1)}[\psi] - I_{ij}^{(2)}[\psi], \quad (5.21)$$

where in accordance with (3.18) and (3.12)

$$I_{ij}^{(1)}[\psi] = \int_{P_i}\int_{P_i}\int_{P_j} \psi_i(p)A_i(p')A_j(q')\langle p', q'\rangle H_{ij}(p', q', p)\, dPdP'dQ', \quad (5.22a)$$

$$I_{ij}^{(2)}[\psi] = \int_{P_i}\int_{P_i}\int_{P_j} \psi_i(p)A_i(p)\, A_j(q)\langle p, q\rangle H_{ij}(p, q, p')\, dPdP'dQ. \quad (5.22b)$$

This integral enters the right-hand side of (4.6).

Replacing in (5.22a) notations $p \rightleftarrows p'$, $q' \to q$, we write the integral (5.21) in the form

$$I_{ij}[\psi] = \int_{P_i}\int_{P_i}\int_{P_j} [\psi_i(p') - \psi_i(p)]A_i(p)A_j(q)\, \langle p, q\rangle\rangle\, H_{ij}(p, q, p')\, dP\, dP'dQ. \quad (5.23)$$

Hence, it follows that the divergence (4.10) of the numerical flux- vector of particles of each kind separately vanishes:

$$\nabla_\alpha n_i^\alpha = 0. \quad (5.24)$$

This equality is obtained because we considered stable particles and assumed that the collisions of particles are elastic.

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Applying to (5.23) the equality (5.17) we get

$$I_{ij}[\psi] = \int_{P_i}\int_{P_j} A_i(p)A_j(q)\langle p, q\rangle \psi_{ij}(p, q)dPdQ, \tag{5.25}$$

where

$$\psi_{ij}(p, q) = \int_0^{\pi}\int_0^{2\pi} [\psi_i(p')-\psi_i(p)]h_{ij}(\langle p, q\rangle, \cos\vartheta) \sin\vartheta\, d\vartheta d\varphi, \tag{5.26}$$

the momentum p' is determined from the formula (5.18a). Eq. (5.18b) determines the momentum q' in the expression

$$\psi_{ji}(q, p) = \int_0^{\pi}\int_0^{2\pi} [\psi_j(q')-\psi_j(q)]h_{ji}(\langle q, p\rangle, \cos\vartheta) \sin\vartheta\, d\vartheta d\varphi. \tag{5.27}$$

Since the function $\langle p, q\rangle$ is symmetrical in the arguments p, q and function h_{ij} is symmetrical in the indices i, j then

$$I_{ij}[\psi]+I_{ji}[\psi] = \int_{P_i}\int_{P_j} A_i(p)A_j(q)\langle p, q\rangle[\psi_{ij}(p, q)+\psi_{ji}(q, p)]dPdQ, \tag{5.28}$$

$$\psi_{ij}(p, q)+\psi_{ji}(q, p) = \int_0^{\pi}\int_0^{2\pi} [\psi_i(p')+\psi_j(q')-\psi_i(p)-\psi_j(q)]h_{ji}(\langle p, q\rangle, \cos\vartheta) \sin\vartheta\, d\vartheta d\varphi. \tag{5.29}$$

Hence, it follows that the mass tensor divergence

$$T^{\alpha\beta}(x) = \sum_{i=1}^{N} T_i^{\alpha\beta}(x) \tag{5.30}$$

(see(4.13)) of all gas is zero

$$\nabla_\alpha T^{\alpha\beta} = \sum_{ij=1}^{N}\int_{P_i} p^\beta I_{ij}(p)dP = 0. \tag{5.31}$$

We obtain one more important expression for (5.21) which enables us to prove the H-theorem.

Substituting into (5.12) the expression (5.19) for the collision integral we find

$$I_{ij}[\psi] = \int_{P_i}\int_{P_i}\int_{P_j} \{A_i(p')A_j(q')-A_i(p)A_j(q)\}\, \psi_i(p)\langle p, q\rangle H_{ij}(p, q, p')dPdP'dQ. \tag{5.32}$$

By replacing in (5.32) notations $p\rightleftarrows p'$, $q\rightleftarrows q'$ and taking into account the equality (5.3) we get

$$I_{ij}[\psi] = \int_{P_i}\int_{P_i}\int_{P_j} \{A_i(p)A_j(q)-A_i(p')A_j(q')\}\psi_i(p')\langle p, q\rangle H_{ij}(p, q, p')dPdP'dQ. \tag{5.33}$$

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Summing up (5.32) and (5.33) we obtain

$$I_{ij}[\psi] = \frac{1}{2}\int_{P_i}\int_{P_i}\int_{P_j} \{A_i(p)A_j(q)-A_i(p')A_j(q')\}\,[\psi_i(p')-\psi_i(p)]\,\langle p, q\rangle H_{ij}(p, q, p')dPdP'dQ. \tag{5.34}$$

Due to Eq. (5.17) we have

$$I_{ij}[\psi] = \frac{1}{2}\int_{P_i}\int_{P_j}\langle p, q\rangle dPdQ\int_0^{\pi}\int_0^{2\pi}\{A_i(p)A_j(q)-A_i(p')A_j(q')\}\times$$
$$\times[\psi_i(p')-\psi_i(p)]h_{ij}(\langle p, q\rangle, \cos\vartheta)\sin\vartheta\, d\vartheta d\varphi. \tag{5.35}$$

Hence

$$I_{ij}[\psi]+I_{ji}[\psi] = \frac{1}{2}\int_{P_i}\int_{P_j}\langle p, q\rangle dPdQ\int_0^{\pi}\int_0^{2\pi}\{A_i(p)A_j(q)-A_i(p')A_j(q')\}\times$$
$$\times[\psi_i(p')+\psi_j(q')-\psi_i(p)-\psi_j(q)]h_{ij}(\langle p, q\rangle, \cos\vartheta)\sin\vartheta\, d\vartheta d\varphi. \tag{5.36}$$

According to (5.36) the divergence of the flux- vector of the entropy of all gas

$$s^{\alpha}(x) = \sum_{i=1}^{N} s_i^{\alpha}(x) \tag{5.37}$$

(see(4.16)) is

$$\nabla_{\alpha}s^{\alpha} = \frac{1}{4}\sum_{ij=1}^{N}\int_{P_i}\int_{P_j}\langle p, q\rangle\, dPdQ\int_0^{\pi}\int_0^{2\pi}\{A_i(p')\,A_j(q')-A_i(p)\,A_j(q)\}\times$$
$$\times\ln\frac{A_i(p')\,A_j(q')}{A_i(p)\,A_j(q)}\cdot h_{ij}(\langle p, q\rangle), \cos\vartheta)\sin\vartheta\, d\vartheta\, d\varphi. \tag{5.38}$$

Since the quantity $(x-y)\ln\frac{x}{y}$ is non-negative for any positive x and y then

$$\nabla_{\alpha}s^{\alpha}(x) \geqslant 0. \tag{5.39}$$

This is just the H-theorem in the relativistic case when the gravitational field is present.

If at a certain point x of the space- time $\nabla_{\alpha}s^{\alpha} = 0$ then owing to the positive definiteness of the integrand in (5.38) for any i, j and any p, q the equality

$$h_{ij}(\langle p, q\rangle, \cos\vartheta)\{A_i(p')A_j(q')-A_i(p)A_j(q)\} = 0. \tag{5.40}$$

is fulfilled. In this case according to (5.20) the collision integral at the point x vanishes for all values of the momentum p according to (4.16) the divergence of the flux-vector of the entropy of each gas component vanishes, according to (4.13) the divergence of the mass

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tensor of each gas component vanishes. In general, in this case (4.6) assumes the form

$$\nabla_\alpha \psi_i^\alpha(x) = \int\limits_{P_i(x)} A_i(x, p) \hat{f}(x, p)\, \psi_i(x, p)\, dP. \tag{5.41}$$

Eq. (5.40) can be written in the form

$$A_i(x, p')A_j(x, q')\langle p', q'\rangle\, h_{ij}(\langle p', q'\rangle, -(e', e))\, dP'dQ'de'dX$$
$$= A_i(x, p)\, A_j(x, q)\langle p, q\rangle\, h_{ij}(\langle p, q\rangle, -(e, e'))\, dPdQdedX, \tag{5.42}$$

where e is the vector at the point $x \in X$ satisfying conditions

$$(e, p+q) = 0, \qquad (e, e) = -1, \tag{5.43}$$

de is the element of the area on the two-dimensional sphere (5.43) in the space tangent to the space- time. The quantities p', q', e' are expressed in terms of the quantities p, q, e as follows:

$$p' = \frac{(p, p+q)}{(p+q, p+q)}(p+q) + \frac{\langle p, q\rangle}{\sqrt{(p+q, p+q)}}\, e$$
$$q' = \frac{(q, p+q)}{(p+q, p+q)}(p+q) - \frac{\langle p, q\rangle}{\sqrt{(p+q, p+q)}}\, e$$
$$e' = \frac{(p+q, q)\, p - (p+q, p)\, q}{\langle p, q\rangle \sqrt{(p+q, p+q)}}. \tag{5.44}$$

This transition is involutive, *i, e.* unprimed quantities are expressed in terms of primed quantities in the same way as primed ones in terms of unprimed one. The transition (5.44) conserves the volume element (comp. [16]).

$$dPdQde = dP'dQ'de'. \tag{5.45}$$

To prove this we go over from the variables p, q to u, v

$$u = \frac{p+q}{\sqrt{(p+q, p+q)}}, \qquad v = \frac{(p+q, q)\, p - (p+q, p)\, q}{(p+q, p+q)}. \tag{5.46}$$

The Jacobian of this transformation is

$$\frac{\partial(u^1, u^2, u^3, v^1, v^2, v^3)}{\partial(p^1, p^2, p^3, q^1, q^2, q^3)} = -\frac{u_0^2 (p, u)\, (q, u)}{p_0 q_0 (p+q, p+q)^{3/2}}. \tag{5.47}$$

Since $u' = u$, $(p+q, p+q) = (p'+q', p'+q')$ then from (5.47) it follows that Eq. (5.45) is equivalent to

$$dv^1 dv^2 dv^3 de = dv'^1 dv'^2 dv'^3 de' \tag{5.48}$$

The latter is obvious since

$$v' = \sqrt{-(v, v)}\, e, \qquad e' = \frac{v}{\sqrt{-(v, v)}}. \tag{5.49}$$

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According to (3.8) and (3.7) in the left- and right- hand sides of (5.42) are written the numbers of collisions of i-kind particles with j-kind particles in the elementary region dX of the space- time. On the right we have the momenta of particles before collision p, q and after collision p', q' while on the left, on contrary, the momenta before collision p', q' and after collision p, q. Thus, Eq. (5.42) or (5.40) is a particular case of the general principle of the detailed balancing in the equilibrium state.

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