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ON LAGRANGIANS WITH HIGH DERIVATIVES

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(Received April 8, 1963)

Starting from a Lagrangian with high order derivatives, a "canocical conjugate momentum" is defined. With this, certain commutation rules are written. The field momentum and the energy, satisfying conservation laws are introduced. Finally two examples are given.

In two preceding papers [1, 2] we have presented the fundamental equations which can be obtained from the variational principle, if Lagrangians with high derivatives are introduced. Now we shall show some prospects originated by the results obtained so far.

We have formerly indicated [1] that, in the frame of a generalized mechanics, based upon a Lagrangian function of the form

$$L = L(t, \dots q_k^{(n)}, \dots) \quad (n = 0, 1, 2, \dots s), \tag{1}$$

where t is the time, and $q_k^{(n)}$ the n-th order derivative of the generalized coordinates, we can obtain the generalized Hamiltonian function

$$H = \sum_{l=0}^{s-1} \sum_{n=l+1}^{s} \sum_{k} (-1)^l (d^l/dt^l)\,(\partial L/\partial q_k^{(n)})\, q_k^{(n-l)} - L \tag{2}$$

which may be written also

$$H = \sum_{k} \sum_{j=1}^{s} \sum_{l=0}^{s-j} (-1)^l (d^l/dt^l)\,(\partial L/\partial q_k^{(l+j)})\, q_k^{(j)} - L \tag{2'}$$

that is, in the form

$$H = \sum_{k} \sum_{j=1}^{s} p_{k/j} q_k^{(j)} - L. \tag{2''}$$

Here the magnitude $p_{k/j}$, the expression of which results from the comparison of (2''), with (2'), is equivalent to the canonical conjugate momentum of the usual Hamiltonian. Starting from the generalized Lagrangian of the form

$$L = L(\dots x_\lambda, \dots q^{(n)}_{k,\lambda\dots}, \dots) \tag{3}$$

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(here x_λ are the space-time coordinates in number of σ, and $q^{(n)}_{k'\lambda\ldots} = \partial^n q_k/\partial x_\lambda \partial x_\mu \ldots$), we have obtained [2] the tensor

$$N_{\pi\varrho} = \sum_{l=0}^{s-1} \sum_{n=l+1}^{s} \sum_{\underset{(n-1)}{\underline{|\lambda+l|}}} \sum_{k} (-1)^l L^{(l;\lambda\ldots)}_{k(n[\ldots\underline{\lambda+l=\varrho}\ldots])} q^{(n-l)}_{k'\pi\alpha\ldots} - L\delta_{\pi\varrho} \tag{4}$$

which represents a generalization with high order derivatives of the energy-momentum density. (We have noted as in [2]: $(\partial^l/\partial x_\lambda \partial x_\mu \ldots)(\partial L/\partial q^{(n)}_{k'\lambda\mu\ldots\underline{\lambda+l=\varrho}\ldots}) = L^{(l;\lambda\ldots)}_{k(n[\ldots\underline{\lambda+l=\varrho}\ldots])}$. The notation $\ldots \underline{\lambda+l} = \varrho \ldots$ shows that the l-th index after λ has the value ϱ. $\Sigma_{\underset{(n-1)}{\underline{|\lambda+l|}}}$ mean the sums $\underset{\underset{(n-1)}{\underline{|\lambda\ \mu\ \ \ |}}}{\Sigma\Sigma} \ldots$ lacking the l-th after the first. π is the index of a certain variable x_π, and α is the first index after ϱ *etc.*).

In order to develop a field theory it is useful to define a "canonical conjugate momentum". We shall show that this is possible.

We observe that the passage from (2) to (2′) is made formally by the change of index n into $l+j$ and the limitting of summation over l up to $s-j$, because from $n \leqslant s$ results $l+j \leqslant s$, therefore $l \leqslant s-j$.

We make in (4) the same change and we obtain

$$N_{\pi\varrho} = \sum_{k} \sum_{j=1}^{s} \sum_{l=0}^{s-j} \sum_{\underset{(l+j-1)}{\underline{|\lambda+l|}}} (-1)^l L^{(l;\lambda\ldots)}_{k(l+j[\ldots\underline{\lambda+l=\varrho}\ldots])} q^{(j)}_{k'\pi\alpha\ldots} - L\delta_{\pi\varrho}. \tag{4'}$$

This expression has the form

$$N_{\pi\varrho} = \sum_{k} \sum_{j=1}^{s} R_{k/j\varrho} q^{(j)}_{k'\pi\alpha\ldots} - L\delta_{\pi\varrho}. \tag{4''}$$

We consider the magnitude

$$R_{k/j\varrho} = \sum_{l=0}^{s-j} \sum_{\underset{(l+j-1)}{\underline{|\lambda+l|}}} (-1)^l L^{(l;\lambda\ldots)}_{k(l+j[\ldots\underline{\lambda+l=\varrho}\ldots])} \tag{5}$$

as generalized "canonical conjugate momentum" with high derivatives. Its knowledge allows to write certain (anti-)commutation relations of equal time, analogous to those known in the literature [3]:

$$[R_k(x),\ q_{k'}(x')]_\pm = -i\delta_{kk'}\delta(\bar{x}-\bar{x}') \tag{6}$$

$$[R_k(x), R_{k'}(x')]_\pm = [q_k(x), q_{k'}(x')]_\pm = 0. \tag{7}$$

(We have noted with x all variables, and with $\bar{x}$ the hipervectors in the f-dimensional hiperspace, if $x_\sigma = ict$ and $f = \sigma - 1$.)

Here we must take into account that every R_k has $j\varrho$ components, therefore (6) and (7) are referring to all these components.

On the other hand we can define the field momentum. The tensor $N_{\pi\varrho}$ gives by its component $N_{\sigma\sigma}$ the energy density, and $(i/c)N_{a\sigma}$ $(a = 1, 2, \ldots f)$ represent the components of the f-dimensional field momentum density with high derivatives.

If L does not depend explicitly on the coordinates, $N_{\pi\varrho}$ has the divergence zero, and by a well-known way one can reach

$$\int\limits_f d^f x \, \frac{1}{ic} \frac{\partial N_{\pi\sigma}}{\partial t} = 0 \tag{8}$$

($d^f x$ is the f-dimensional volume element). This is the conservation law of the energy-momentum σ-vector P_π of the field, with its components constituing the f-dimensional momentum with high derivatives

$$P_a = \frac{i}{c} \int\limits_f d^f x \sum_{()} (-1)^l L^{(l;\lambda\ldots)}_{,k(n[\ldots\underline{\lambda+l}=\sigma\ldots])} q^{(j)}_{k,a\alpha\ldots} \tag{9}$$

and the component representing the energy

$$E = \frac{c}{i} P_\sigma = \int\limits_f d^f x \left\{ \sum_{()} (-1)^l L^{(l;\lambda\ldots)}_{,k(n[\ldots\underline{\lambda+l}=\sigma\ldots])} q^{(j)}_{k,\sigma\alpha\ldots} - L \right\}. \tag{10}$$

(We have noted with $\sum\limits_{()}$ all sums).

In a very interesting paper Fried and Plebanski [4] have adopted the Lagrangian

$$L = -\bar{\psi}[\Pi(\hat{\partial} + m)] \psi \tag{11}$$

($\hat{\partial} = \gamma_\lambda \partial_\lambda$, γ_λ being the well known matrices, and m mass parameters) [5].

The Euler-Lagrange equations are [2]:

$$\sum_{n=0}^{s} \sum_{(n)} {}_{\lambda\ldots} (-1)^n L^{(n;\lambda\ldots)}_{,k(n[\lambda\ldots])} = 0. \tag{12}$$

If we partially derive L, given by (11), with respect to $\bar{\psi}^{(n)}_{,\lambda\ldots}$ we receive zero, except the case $n = 0$, when we receive

$$\partial L / \partial \bar{\psi} = -[\Pi(\hat{\partial} + m)] \psi. \tag{13}$$

Replacing into Eq. (12) we have

$$[\Pi(\hat{\partial} + m)] \psi = 0. \tag{14}$$

If we partially derive L with respect to $\psi^{(n)}_{,\lambda\ldots}$ we obtain every time an expression without ψ, multiplied on the left with $-\bar{\psi}$, that is, of the form

$$\partial L / \partial \psi^{(n)}_{,\lambda\ldots} = -\bar{\psi} M_{s-n} \underbrace{\gamma_\lambda \cdots}_{(n)} \tag{15}$$

(s being the maximum value of n).

Acting with the operator $\partial^n / \partial x_\lambda \ldots$ on every expression, and summing accordingly to (12), we see that one receives the equation

$$\sum_{n=0}^{s} \sum_{(n)} {}_{\lambda\ldots} (-1)^n (\partial^n / \partial x_\lambda \ldots) (-\bar{\psi} M_{s-n} \underbrace{\gamma_\lambda \cdots}_{(n)}) \tag{16}$$

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which, with the above adopted notation, and with $\partial_\lambda \bar{\psi}\gamma_\lambda = \bar{\psi}\hat{\partial}$ yields

$$\bar{\psi}[\Pi(\hat{\partial}-m)] = 0 \tag{17}$$

Eqs (14) and (17) are the field equations obtained also in [4] and [5].

The "canonical conjugate momentum" for our case, $R_{j\varrho}$ can be obtained by introducing in (5) the partial derivatives (15). Taking into account the indices, we have

$$R_{j\varrho} = \sum_{l=0}^{s-j} \sum_{(l+j-1)} |\lambda+l| (-1)^l (\partial^l/\partial x_\lambda \ldots)(-\bar{\psi} M_{s-l-j} \underbrace{\gamma_\lambda \cdots \gamma_{\lambda+l=\varrho} \cdots}_{(l+j)}). \tag{18}$$

Consider

$$\begin{aligned} R_{14} &= \sum_{l=0}^{s-1} \sum_{(l)} |\lambda+l| (-1)^l (\partial^l/\partial x_\lambda \ldots)\,(-\bar{\psi} M_{s-l-1} \underbrace{\gamma_\lambda \cdots \gamma_4}_{(l+1)}) \\ &= \sum_{l=0}^{s-1} \sum_{\underset{(l)}{\lambda}} \ldots (-1)^l (\partial^l/\partial x_\lambda \ldots)\,(-\bar{\psi} M_{s-l-1} \underbrace{\gamma_\lambda \cdots}_{(l)})\, \gamma_4 \\ &= (-1)^{s-1} \bar{\psi}(\hat{\partial}-m_1) \ldots (\hat{\partial}-m_{s-1})\, \gamma_4. \end{aligned} \tag{19}$$

Condition (6) leads in this case to

$$[\psi, (-1)^{s-1} \bar{\psi}(\hat{\partial}'-m_1) \ldots (\hat{\partial}'-m_2)\gamma_4]_\pm = i\delta(\bar{x}-\bar{x}') \tag{20}$$

which is identical with the first anticommutation relation from [4]. Other relations too are deduceable.

The introduction of high derivatives into Lagrangian functions is specially adequate in the field theory, but we remark that — as we have previously asserted [1] — even in a mechanical theory it may lead to some interesting aspects.

For instance take an unrelativistic Lagrange function

$$L = \alpha - \beta \ddot{x}^2 \dot{x}^{-4} - U. \tag{21}$$

Here we have noted with dots the time-derivatives of the coordinates, U is the potential energy, α and β nonnegative constants.

Consider that a particle moves through a region I, in which $U = 0$, and enters a region II, in which $U = U_b = \text{const} \neq 0$. In both regions the equations of motion established in [1] lead in this case to

$$\ddddot{x}\, \dot{x}^{-4} - 8 \dddot{x}\, \ddot{x} \dot{x}^{-5} + 10 \ddot{x}^3 \dot{x}^{-6} = 0. \tag{22}$$

It has the general solution

$$t = C_2 x + \frac{C_2}{C_1} \frac{1}{2i} \left(e^{i(C_1 x + C_3)} - e^{-i(C_1 x + C_3)}\right) + C_4. \tag{23}$$

On the other part the equation of energy established in the general case [1] leads here to

$$\dddot{x}\, \dot{x}^{-3} - (5/2)\, \ddot{x}^2 \dot{x}^{-4} - C = 0, \tag{24}$$

where $C = (1/2\beta)(\alpha - U + E)$, E being the total energy.

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The introduction of solution (23) into (24) leads to the following relation between the constants

$$C_1 = \pm [(1/\beta)(\alpha - U + E)]^{1/2}. \tag{25}$$

In region I, $U = 0$, consequently C_1 is real. It follows that solution (23) may be put the form

$$t = C_2 x + \frac{C_2}{C_1} \sin (C_1 x + C_3) + C_4. \tag{26}$$

In particular, for $C_1 = C_3 = 0$ there is the classic uniform motion.

In region II $U = U_b \neq 0$. If $U_b \leqslant E$, C_1 is real and the solution is (26).

One notices that different solutions of forward motion of the particle may be obtained in region II also in the following conditions: $E < U_b < E + \alpha$, $U_b = E + \alpha$, $U_b > E + \alpha$, with adequate values for the constants.

Thus, unlike in classical mechanics, the particle may enter the potential barrier and move further on, even in the case $E < U_b$.

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